Line Graphs of Directed Graphs I

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Abstract

We determine the forbidden induced subgraphs for the intersection of the classes of chordal bipartite graphs and line graphs of acyclic directed graphs. This is a first step towards finding the forbidden induced subgraphs for the class of line graphs of directed graphs.

1 Introduction

Hereditary classes of graphs have been and continue to be a vibrant area of research. A graph class is hereditary when it is closed under taking induced subgraphs. Examples of hereditary classes of graphs include: line graphs, perfect graphs, chordal graphs, strongly chordal graphs, bipartite graphs, and many more. Given a hereditary class of graphs $\mathcal{H}$, a basic question to ask is, "What are the minimal forbidden induced subgraphs which characterize $\mathcal{H}$?" The complete list can be infinite or finite. For example, a graph G is bipartite if and only if G contains no induced cycle of odd length. Beineke [3] proved that a graph G is a line graph if and only if G does not contain one of a list of nine graphs as an induced subgraph.

Among hereditary classes of graphs, line graphs and their abundant analogues are one of the many well-studied topics. Harary and Norman [8] defined what is often called the *line digraph* of a directed graph D. Given D, construct a new directed graph $\mathcal{L}'(D)$ whose vertices are $E(D)$ where there is a directed edge in $\mathcal{L}'(D)$ from the vertex corresponding to $(a, b) \in E(D)$ to the vertex corresponding to $(c, d) \in E(D)$ if and only if $b = c$. (This definition of Harary and Norman was generalized to bidirected line graphs of bidirected graphs and applied to edge colorings of signed graphs by Behr [2, Sec.5].) Bagga and Beineke recently published [1] a survey article on line digraphs as well as a book [4] on line graphs and line digraphs. We will define the *line graph* of a directed graph D to be the underlying graph of $\mathcal{L}'(D)$. We will denote the line graph of D by $\mathcal{L}(D)$. Thus more information is lost in the transformation from D to $\mathcal{L}(D)$ compared to the transformation from D to $\mathcal{L}'(D)$. The class of line graphs of directed graphs is, however, clearly a hereditary class of graphs. We could not find any published work on this topic. (It seems that the result of Beineke and Hemminger [5] on line digraphs of infinite directed graphs has the most similarity to our work here.)

We are interested in finding the complete list of minimal forbidden induced subgraphs for the class of line graphs of directed graphs. At first, it might seem that the answer to such a question might involve similar work to the already substantial work in [3]; however, this problem for line graphs of directed graphs seems to be even more difficult. A first indication of this is the following. Aside from

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a triangle, the line graph of an ordinary graph is a path or cycle if and only if the original graph is respectively a path or cycle. In contrast to this, if the line graph of a directed graph is a path or cycle, then the directed graph has a much more varied structure. (See Section 2.)

In reaction to this difficulty, we consider subclasses of line graphs of directed graphs in hopes of building up to the full result. Our first restriction is a natural one in that we will only consider directed graphs whose underlying graphs are simple. **The reader should please note that for the remainder of the paper, it is assumed that all directed graphs under consideration are directed simple graphs unless otherwise specified.** Second, one of the simplest subclasses of line graphs of directed graphs is line graphs of directed trees. (Line graphs of directed forests would just be vertex-disjoint unions of line graphs of directed trees.) The class of line graphs of directed trees is easily shown to be contained within the class of *chordal bipartite graphs.* A bipartite graph G is chordal bipartite when it contains no cycle of length at least 6 as an induced subgraph. Thus the class of chordal bipartite graphs is defined as the class of graphs containing no induced cycles of length other than 4.

Proposition 1. *If D is a directed tree, then $\mathcal{L}(D)$ is chordal bipartite.*

Proof of Proposition 1. If D is a directed tree, then the underlying graph of $\mathcal{L}(D)$ has no induced cycles of length other than 4 by Propositions 6, 7, and 8. Thus $\mathcal{L}(D)$ is chordal bipartite. □

Consider now the Venn Diagram showing the intersections of the classes of line graphs of directed graphs, line graphs of acyclic directed graphs, line graphs of directed trees (or line graphs of forests if one does not wish to assume connectedness of the underlying graphs), and chordal bipartite graphs. It is perhaps surprising that the intersection of the class of line graphs of acyclic directed graphs with chordal bipartite graphs is actually exactly the class of line graphs of directed trees.

Theorem 2. *If G is connected and chordal bipartite, then G is the line graph of an acyclic directed graph if and only if G is the line graph of a directed tree.*

Proof. If $G = \mathcal{L}(T)$ in which T is a directed tree, then we are done because T is an acyclic directed graph. If $G = \mathcal{L}(D)$ in which D is an acyclic directed graph, then using Theorem 13 we obtain a directed tree T for which $\mathcal{L}(D) = \mathcal{L}(T)$. □

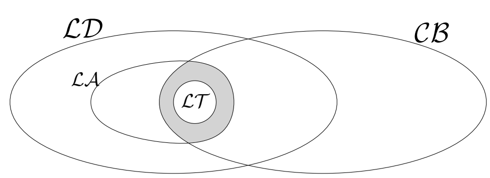

Figure 1: $\mathcal{LD}$, $\mathcal{CB}$, $\mathcal{LA}$, $\mathcal{LT}$ denote respectively the classes: line graphs of directed graphs, chordal bipartite graphs, line graphs of acyclic directed graphs, and line graphs of directed trees.

Our main result of this paper is Theorem 3 which gives us a list of three forbidden induced subgraphs which characterize membership in $\mathcal{LA} \cap \mathcal{CB}$.

Theorem 3. *If G is a chordal bipartite graph, then G is the line graph of an acyclic directed simple graph if and only if G contains none of the graphs of Figure 2 as an induced subgraph.*

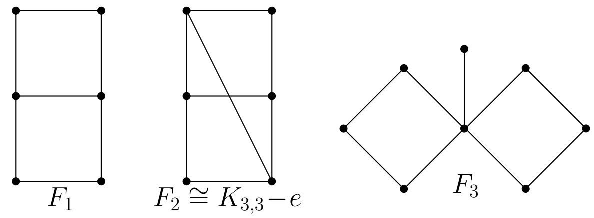

Figure 2: The three forbidden induced subgraphs for line graphs of acyclic directed graphs within the class of chordal bipartite graphs.

Chordal bipartite graphs were first introduced by Golumbic and Goss [7]. A significant result describing different ways of characterizing the class of chordal bipartite graphs as different types of intersection graphs is given by Huang [9]. In this paper we are considering a subclass of chordal bipartite graphs. The work of Dabrowski, Lozin, Zamaraev [6] also looks at subclasses of chordal bipartite graphs which are related to those addressed in this paper. (Confer Theorem 3 and [6, Lemma 4].)

2 Paths and cycles

In this section we characterize the directed graphs D for which $\mathcal{L}(D)$ is a path or a cycle. The interested reader might compare the results in this section as well as Proposition 9 to [4, Theorem 10.2].

If D is a directed graph, then the *reversal* of D is the directed graph D' obtained from D by reversing each arrow. Note that $\mathcal{L}(D) = \mathcal{L}(D')$. A vertex v in a directed graph D is called a *singular* vertex when v is either a source or a sink. Let D be a directed graph containing distinct singular vertices v and w which are of the same type; that is, both sources or both sinks. Now let D' be the directed graph obtained from D by identifying v and w. We call this operation *singular identification* and its reverse *singular splitting*. Propositions 4 is evident.

Proposition 4. *If directed graphs D_1 and D_2 are related by a sequence of singular identifications and splits, then $\mathcal{L}(D_1) = \mathcal{L}(D_2)$.*

Proposition 5. *If D is a directed graph with n edges, then $\mathcal{L}(D)$ is a path of vertices $1, \ldots, n$ in that order if and only if up to reversal and singular identifications and splits, D is constructed as follows. The underlying graph of D has max degree at most 3 and consists of a path along with an arbitrary number (including zero) of pendant edges attached to the interior vertices of the path and which is directed as indicated in Figure 3.*

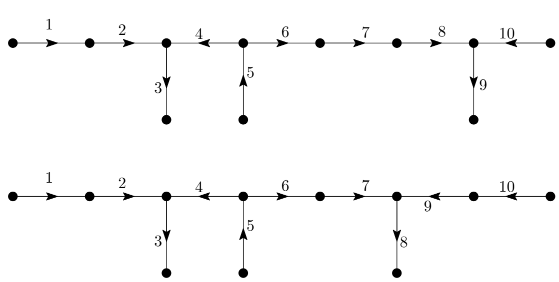

Figure 3: Two examples of a directed graphs whose line graphs are both the path on vertices $1, \ldots, 10$ in that order.

Proof of Proposition 5. If G is a 1-vertex path, then $G = \mathcal{L}(D)$ for D consisting of just one edge and its endpoints. Of course D has the desired form. Now let G be a path on vertices $a_1, \ldots, a_{n+1}$ in that order. Thus $G - a_{n+1} = \mathcal{L}(D_n)$ in which, by induction, D_n has the desired form. Any directed graph D_{n+1} for which $\mathcal{L}(D_{n+1}) = G$ must be obtained from some possible D_n by adding in the final edge a_{n+1}. By inspection, adding a_{n+1} to D_n must preserve the desired form. □

The proofs of Propositions 6 and 7 are left to the reader.

Proposition 6. *If D is a directed graph, then $\mathcal{L}(D)$ is a triangle if and only if D is a directed triangle.*

Proposition 7. *If D is a directed graph, then $\mathcal{L}(D)$ is a 4-cycle if and only if D is a directed 4-cycle or D consists of four directed edges $(v_1, w), (v_2, w), (w, u_1), (w, u_2)$.*

Proposition 8. *If D is a directed graph with $n \geq 5$ edges, then $\mathcal{L}(D)$ is the cycle with vertices $1, \ldots, n$ in that cyclic order if and only if up to singular identifications and splits, D is constructed as follows. The underlying graph of D has max degree at most 3 and consists of a cycle along with an arbitrary even number (including zero) of pendant edges and which is directed as indicated in Figure 4.*

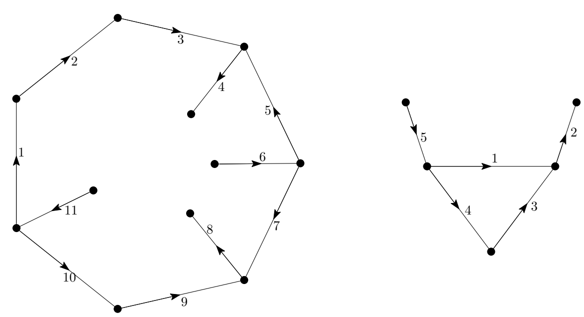

Figure 4: Directed graphs whose line graphs are cycles of length 11 and 5.

Proof of Proposition 8. Let G be a cycle on vertices $a_1, \ldots, a_{n+1}$ for $n + 1 \geq 5$ and suppose that $G = \mathcal{L}(D)$. Thus $G - a_{n+1}$ is a path of $n \geq 4$ vertices for which $G - a_{n+1} = \mathcal{L}(D - a_{n+1})$. Thus up to reversal and singular identifications and splits, $D - a_{n+1}$ has the structure indicated by Proposition 5. Let P be the path in $D - a_{n+1}$ along which a_1 and a_n are the outermost edges.

First, we claim that a_1 and a_n do not share an endpoint. Say that there are x edges on P and y pendant edges. Thus $4 \leq n = x + y$. Since the pendant edges must be on internal vertices of P, $y \leq x - 1$. Therefore $4 \leq x + y \leq 2x - 1$ which makes $x \geq 3$. Thus a_1 and a_n do not share a common endpoint.

Now, if a_1 and a_n point in the same direction along P, then the number of pendant edges on P is even; furthermore, if a_1 and a_n point in opposite directions along P, then the number of pendant edges on P is odd. In the former case, the only way to add a_{n+1} would be to connect the degree-1 endpoints of a_1 and a_n. In the latter case, the only way to add a_{n+1} would be to perform a singular identification on the degree-1 endpoints of a_1 and a_n and then add a_{n+1} as a new pendant. Each case results in the desired form for D. □

3 Proofs

Because the underlying graph of our directed graphs is assumed to be simple, we can represent edges of such directed graphs as ordered pairs of vertices.

Proposition 9. *Let m and n be integers greater than 1. If D is an acyclic directed graph, then $\mathcal{L}(D) \cong K_{m,n}$ if and only if D has underlying graph $K_{1,m+n}$ with m edges directed one way and n directed the other way relative to the common endpoint.*

Proof. The case for $m = n = 2$ follows from Proposition 7. We now proceed by induction on $m + n$ in which $m, n \geq 2$. Given $K_{m,n}$ outside of the base case $m = n = 2$, either $m > 2$ or $n > 2$. Assuming without loss of generality that $m > 2$, $\mathcal{L}(D) \cong K_{m-1,n}$ implies that D without loss of generality must consists of edges $(v_1, w), \ldots, (v_{m-1}, w), (w, u_1), \ldots, (w, u_n)$. Evidently the only way to add an edge e to D such that $\mathcal{L}(D) \cong K_{m,n}$ is $e = (v_m, w)$ in which v_m is a new vertex. □

Proposition 10. *If D is an acyclic directed graph, then $\mathcal{L}(D)$ is not one of the graphs from Figure 2.*

Proof. Given a set of edges X in a directed graph D, let $D{:}X$ denote the subgraph of D consisting of the edges in X along with the vertices the edges in X have as endpoints.

Assume that D is an acyclic directed graph for which $\mathcal{L}(D) = F_1$. Let v_1 and v_2 denote the degree-3 vertices in F_1 and w_1, w_2 and u_1, u_2 denote the pairs of adjacent degree-2 vertices in F_1. Since D is acyclic, Proposition 7 implies that $D{:}\{v_1, v_2, u_1, u_2\}$ and $D{:}\{v_1, v_2, w_1, w_2\}$ are both directed $K_{1,4}$'s with two edges in one direction and two edges in the other. Since v_1, v_2 are edges in both of these directed $K_{1,4}$'s, combining them into one directed graph yields a directed $K_{1,6}$ for which $\mathcal{L}(D) \cong K_{3,3}$, a contradiction.

Assume that D is an acyclic directed graph for which $\mathcal{L}(D) = F_2$. Consider the labeling of the vertices of F_1 in the previous paragraph. Add a u_1w_1-edge to obtain a vertex labeling of F_2. Again, $D{:}\{v_1, v_2, u_1, u_2\}$ and $D{:}\{v_1, v_2, w_1, w_2\}$ are both directed $K_{1,4}$'s with two edges in one direction and two edges in the other. Again, D is a directed $K_{1,6}$ such that $\mathcal{L}(D) \cong K_{3,3}$, a contradiction.

Assume that D is an acyclic directed graph for which $\mathcal{L}(D) = F_3$. Let x be the degree-5 vertex of F_3. Thus the vertices of $F_3 - x$ partition into three sets A, B, C in which $A \cup x$ and $B \cup x$ induce 4-cycles and $C \cup x$ in induces K_2. Hence $D{:}A \cup x$ and $D{:}B \cup x$ are directed $K_{1,4}$'s and $D{:}A \cup B \cup x$ consists of directed edge x with the three edges of A on one endpoint of x and the three edges of B at the other endpoint of x. Now there is no way to attach one more directed edge to obtain D such that $\mathcal{L}(D) = F_3$, a contradiction. □

A *biclique* is a complete bipartite graph. Recall that a *block* of a simple graph is a maximal subgraph that is either an isolated vertex, an edge with its endpoints, or a 2-connected subgraph. This latter type of block is called non-trivial while the former two are trivial.

Proposition 11. *If G is chordal bipartite and has no isolated vertices, then for any block B of G either B is a biclique or B contains F_1 or F_2 as an induced subgraph. Furthermore, if $G = \mathcal{L}(D)$ in which D is acyclic, then any block B of G is a biclique.*

Proof. Assuming that the first statement is true, the furthermore part of the proposition then follows from Proposition 10.

Now, a trivial block is a single edge which is $K_{1,1}$. So consider a non-trivial block B in $\mathcal{L}(D)$ and assume that it does not contain F_1 or F_2 as an induced subgraph. Then there must be a cycle C of even length contained in B; furthermore, because $\mathcal{L}(D)$ is chordal bipartite we may assume that C has length 4. If $B = C$, then we are done because $C \cong K_{2,2}$. If not, let K be a biclique in B on the largest possible number of vertices and which contains $K_{2,2}$. If $B = K$, then we are done. If not, then there is a path P in B where $P \cap K$ consists of the endpoints of P only. Let C' be the shortest possible cycle in $K \cup P$ which contains P. Thus C' consist of P along with exactly one or two edges of K depending on whether the endpoints of P are in the same partite set of K or not. By using chords, there is a 4-cycle C'' in B which shares exactly one or two edges with K. In Case 1, say that C'' has exactly one edge in K and in Case 2 that C'' has exactly two edges in K. In both cases denote the partite sets of K by $\{a_1, \dots, a_s\}$ and $\{b_1, \dots, b_t\}$ in which $s, t \geq 2$.

Case 1 Without loss of generality say that $e = (a_1, b_1)$ is the edge of C'' that is in K. For any other edge $f = (a_i, b_j) \in K$ for which $i, j \neq 1$, there is $H \cong F_1$ in $K \cup C''$ for which $C'' \cup f \subset H$. Since B does not have F_1 or F_2 as an induced subgraph, H is not induced and so the vertices of H induce a $K_{3,3}$-subgraph in B. Repeating this process with all such edges $f = (a_i, b_j) \in K$ will result with the vertices of $K \cup C''$ inducing a biclique in B which is strictly larger than K, a contradiction.

Case 2 Without loss of generality say that $e_1 = (a_1, b_1)$ and $e_2 = (a_2, b_1)$ are the edges of C'' that are in K. If $|\{a_1, \dots, a_s\}| = 2$, then $K \cup C'' \cong K_{2,t+1}$ and so B contains a larger biclique than K, a contradiction. If $|\{a_1, \dots, a_s\}| > 2$, then take any edge $f = (a_i, b_j) \in K$ for which $i \notin \{1, 2\}$ and $j \neq 1$. Proceeding as in Case 1, we obtain a larger biclique than K in B, a contradiction. □

Proposition 12. *If D is an acyclic directed graph for which $\mathcal{L}(D)$ is chordal bipartite and has no isolated vertices, then every block of $\mathcal{L}(D)$ is a biclique and each vertex of $\mathcal{L}(D)$ is either contained in exactly two non-trivial blocks and no other blocks or in at most one non-trivial block and k trivial blocks where k can be any non-negative integer.*

Proof. Proposition 11 implies that every block of $\mathcal{L}(D)$ is a biclique. Now let v be a vertex of $\mathcal{L}(D)$ which is contained in more than one block; that is, v is a cut vertex of $\mathcal{L}(D)$. If there are two non-trivial blocks incident to v, then any third block incident to v would yield an induced F_3-subgraph in $\mathcal{L}(D)$, a contradiction. If v is incident to at most one non-trivial block, then let $B_1, B_2, \dots, B_k$ denote the blocks incident to v in which $B_2, \dots, B_k$ are all trivial blocks. Proposition 9 implies that, up to reversal, there is only one possible directed graph D for which $\mathcal{L}(D) = B_1$ and D has underlying graph $K_{1,t}$. Consider the edge v in D and the endpoint of v having degree 1. Assume that v is directed at this endpoint. Now attach $k-1$ directed edges at this endpoint and direct away from it. This directed graph has line graph $B_1 \cup B_2 \cup \cdots \cup B_k$. □

Given a simple graph G, let $\mathcal{B}(G)$ denote the block-cutpoint-graph of G. The vertices of $\mathcal{B}(G)$ consist of the blocks of G along with the cutpoints of G. A block B is adjacent to a cutpoint c when $c \in B$. When G is connected, $\mathcal{B}(G)$ is a tree.

Theorem 13. *If G is connected, chordal bipartite, and $G = \mathcal{L}(D)$ for some acyclic directed graph D, then up to singular splitting D is a directed tree.*

Proof. Assume that $G = \mathcal{L}(D)$ for some acyclic directed graph D. Thus G is a graph having the block structure given in Proposition 12. We proceed by induction on the number of blocks. If G has one block, then we are done by Propositions 9 and 11. If G has more than one block, then consider a block B in G which is a leaf in the block-cutpoint tree $\mathcal{B}(G)$. Let c be the unique cutpoint of G which is incident to B. Removing B from $\mathcal{B}(G)$ indicates a connected and induced subgraph of $H \subseteq G$ for which $H = \mathcal{L}(D')$ in which D' is obtained from D by removing the edges corresponding to the vertices of B. The number of blocks of H is exactly one less than the number of blocks of G and by induction, we may apply singular splits to transform D' into a directed tree T'. Let $c = (x, y)$ be the directed edge in T' corresponding to the cutpoint c in G. Because G is connected, there is at least one other block $B' \neq B$ incident with c in G. In Case 1 say that B' can be chosen to be non-trivial and in Case 2 all choices for B' are trivial.

Case 1 If B' is a non-trivial block, then by Proposition 9, the vertices of $B' - c$ correspond to directed edges which up to reversal of T' are incident to x in T' with at least one such edge pointing into x and one pointing away from x. There can be no other edges in T' incident with x aside from these. Since x is non-singular in T', vertex x is not obtained from a sequence of singular splits starting with D'. In other words, x and its incident vertices in T' appear exactly as they do in D'.

Now let $\hat{y}$ be the vertex of D' from which vertex y of T' was obtained after a sequence (possibly empty) of singular splits. The edges of D corresponding to the vertices of $B - c$ must be incident with $\hat{y}$ with at least one such edge pointing away from $\hat{y}$. Thus $\hat{y}$ is nonsingular in D. In Case 1.1 say that B is a non-trivial block of G and in Case 1.2 that B is a trivial block of G.

Case 1.1 In this case, the vertices in $B - c$ correspond to edges in D which are all incident to $\hat{y}$ with at least one having $\hat{y}$ as its head and at least one having $\hat{y}$ as its tail and there can be no other edges of D incident to $\hat{y}$. Furthermore, since c is the unique cutpoint of G incident to B, the edges of D which are incident to $\hat{y}$ have second endpoints which are singular. Thus we can apply singular splitting to D to obtain directed tree T for which $G = \mathcal{L}(D) = \mathcal{L}(T)$ and which is obtained from T' by adding the same edges incident to y as we did to $\hat{y}$.

Case 1.2 In this case, B along with the remaining blocks (possibly zero) $B_1, \ldots, B_k$ of G incident to c are all trivial blocks. Since B' is a non-trivial block, the vertices $B - c, B_1 - c, \ldots, B_k - c$ all correspond to edges in D which all have tail $\hat{y}$ and heads which must be sinks. As in Case 1.1 we obtain T by singular splitting from D.

Case 2 In this case, the blocks of G incident to c are B and B' along with any number (possibly zero) of trivial blocks $B_1, \ldots, B_k$. Let $\hat{x}$ and $\hat{y}$ be the vertices of D' from which vertices x and y of T' are obtained after a sequence (possibly empty) of singular splits. As in Case 1, we may assume without loss of generality that the vertex in $B' - c$ corresponds to an edge of T' which is incident with x. Now, consider the vertices of $B - c, B' - c, B_1 - c, \ldots, B_k - c$ in G. These vertices correspond to all of the edges of D which are incident with $c = (\hat{x}, \hat{y})$. If B is a trivial block, then any such edge that is incident with $\hat{x}$ has $\hat{x}$ as its head and any such edge incident with $\hat{y}$ has $\hat{y}$ as its tail. If B is a non-trivial block, then without loss of generality, the edges corresponding to $B' - c, B_1 - c, \ldots, B_k - c$ are all incident to $\hat{x}$ and have $\hat{x}$ as their head and the edges of $B - c$ are all incident to $\hat{y}$ with at least one such edge having $\hat{y}$ as its tail. In both cases, the other endpoints of these edges in D which are incident to $\hat{x}$ and $\hat{y}$ must be singular. Also, in both cases, $\hat{x}$ and $\hat{y}$ are non-singular in D as long as they have at least one incident edge other than c itself. Thus we apply singular splits to D to obtain a tree T for which $G = \mathcal{L}(D) = \mathcal{L}(T)$ and which can also be obtained from T' by adding the same edges incident to x or y as we do to $\hat{x}$ or $\hat{y}$ in D'. □

Proof of Theorem 3. Let G be chordal bipartite. If G is the line graph of an acyclic directed graph, then by Proposition 10, G contains none of F_1, F_2, and F_3 as an induced subgraph. Conversely, assume that G is not a line graph of a directed tree and minimal with respect to this property, then G is connected and does not have the block structure described in Proposition 12. If G has a block which is not a biclique, then Proposition 11 implies that B (and hence G) has F_1 or F_2 as an induced subgraph, a desired result. If all blocks of G are bicliques, then there must be a vertex v of G which is incident with at least three blocks B_1, B_2, B_3 in which B_1 and B_2 are nontrivial. Thus B_1 and B_2 are bicliques with at least two vertices in each of their partite sets and so G has an induced F_3-subgraph, a desired result. □

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