

## GRAPHS WITHOUT A $2C_3$ -MINOR AND BICIRCULAR MATROIDS WITHOUT A $U_{3,6}$ -MINOR

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ABSTRACT. In this note we characterize all graphs without a  $2C_3$ -minor. A consequence of this result is a characterization of the bicircular matroids with no  $U_{3,6}$ -minor.

### 1. Introduction

We assume the reader has a basic familiarity with matroid theory as in [5]; however, it isn't completely necessary to read this note. Given a fixed graph  $H$ , results characterizing the structure of graphs  $G$  without an  $H$ -minor have a well-established history going back as far as 1937 with Wagner's seminal result [7] for  $H = K_5$ . Recently Ding and Liu [3] surveyed the known results for 3-connected graphs  $H$  and an older survey by Diestel [2] lists results for some other small graphs. In all of the results listed in [2, 3], the graph  $H$  is simple. The graph  $2C_3$  is obtained from the cycle of length 3 by doubling each edge. The graph  $2C_3$  is of interest in matroid theory in that a bicircular matroid  $B(G)$  is isomorphic to  $U_{3,6}$  if and only if  $G \cong 2C_3$  up to removal of isolated vertices (see [8, Lemma 2.12] or [1, Theorem 4.11]).

The main result of this note is Theorem 1.1 which describes the very limited structure that a graph with no  $2C_3$ -minor can have. We remark that Theorem 1.1 is enough to characterize all graphs without a  $2C_3$ -minor because:  $G$  has a  $2C_3$ -minor if and only if some block of  $G$  has a  $2C_3$ -minor and if  $G$  has a vertex  $v$  of degree 2, then  $G$  has a  $2C_3$ -minor if and only if the graph obtained from  $G$  by smoothing out  $v$  has a  $2C_3$ -minor. We also prove Theorem 1.2.

Let  $G$  be an outerplanar simple graph. Thus  $G$  consists of a Hamilton cycle  $H$  along with a set of chords  $C$ . Let  $C'$  be a disjoint copy of  $C$ . Embed  $G \cup C'$  with chords  $C$  inside  $H$  and chords  $C'$

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outside  $H$ . A *doubled outerplanar embedding* is any graph  $K$  contained between  $H$  and  $G \cup C'$  with embedding inherited from  $G \cup C'$ .

**Theorem 1.1.** *If  $G$  is a connected and nonseparable graph with minimum degree 3, then  $G$  has no  $2C_3$ -minor if and only if*

- (1)  $G \cong K_4$  or
- (2)  $G$  is the topological dual graph of some doubled outerplanar embedding.

**Theorem 1.2.** *If  $G$  is 3-connected and loopless, then  $G \cong K_4$  or  $G$  contains a  $2C_3$ -minor.*

## 2. Proofs

Given a graph  $G$ , a *k-separation* is an expression  $G = G_1 \cup G_2$  in which each  $G_i$  has at least  $k$  edges and  $G_1 \cap G_2$  is a set of  $k$  vertices. A connected graph is *separable* when it has a 1-separation. A graph  $G$  is *nonseparable* when it is connected and has no 1-separation. A *link* is an edge in a graph that is not a loop. Note that every edge in a nonseparable graph is a link. A connected graph  $G$  is *k-connected* when it has at least  $k + 1$  vertices and it has no set of  $t < k$  vertices whose removal leaves a disconnected subgraph.

*Proof of Theorem 1.1.* Assume that  $G \cong K_4$  or  $G = H^*$  where  $H$  is a doubled outerplanar embedding. If  $G \cong K_4$ , then  $G$  has no  $2C_3$ -minor. If  $G = H^*$ , then  $H$  has no  $K_{2,3}$ -minor. (It is well known that a graph  $G$  is outerplanar if and only if it has no  $K_{2,3}$ - or  $K_4$ -minor.) Since any embedding of  $2C_3$  in the plane has topological dual graph isomorphic to  $K_{2,3}$ , we get that  $G$  has no  $2C_3$ -minor.

Conversely, suppose that  $G$  has no  $2C_3$ -minor. The reader can check that  $K_{3,3}$  and  $K_5$  both contain  $2C_3$ -minors and hence  $G$  is planar. Let  $H$  be the topological dual graph of some embedding of  $G$  in the plane. Note that  $H$  has no faces of length two because  $G$  has minimum degree 3. Furthermore, since  $G$  is nonseparable, so must be  $H$ . If  $|V(H)| = 2$ , then our result holds, hence we may assume that  $|V(H)| \geq 3$ . This implies that  $H$  is 2-connected. Let  $H_v$  be the graph obtained from  $H$  by adjoining an apex vertex to all other vertices of  $H$ . Thus  $H_v$  is 3-connected. If  $H_v$  is planar, then  $H$  is outerplanar and has an embedding in the plane without faces of length 2. Thus  $H$  is a doubled outerplanar embedding, a desired result. If  $H_v$  is non-planar, then by a theorem of D.W. Hall ([4] or see [5, 12.2.11]) either  $H_v \cong K_5$  along with maybe some doubled edges or  $H_v$  contains a  $K_{3,3}$ -subdivision. In the former case,  $H \cong K_4$  along with maybe some doubled edges. If an edge of  $K_4$  is doubled, however, the resulting graph has a  $2C_3$ -minor, a contradiction. Thus  $G \cong K_4$ , a desired outcome. In the latter case  $H$  contains a  $K_{2,3}$ -subdivision and so  $G$  contains a  $2C_3$ -minor, a contradiction.  $\square$

*Proof of Theorem 1.2.* Let  $\hat{G}$  be the simplification of  $G$ ; that is, for each class of parallel links, delete all but one of them. Thus  $\hat{G}$  is 3-connected and simple. By Tutte's Wheel Theorem ([6] or see [5, Theorem 8.8.4]) there is a sequence of 3-connected simple graphs  $G_1, \dots, G_t$  such that  $G_1 = \hat{G}$ ,  $G_{i+1} = G_i/e$  or  $G_i \setminus e$ , and  $G_t \cong W_n$  for  $n \geq 3$  where  $W_n$  is the  $n$ -spoked wheel. If  $n \geq 4$ , then  $G_t$  has a  $2C_3$ -minor and therefore so does  $G$ . So suppose that  $G_t \cong W_3 \cong K_4$ . If  $G = G_t$ , then we are

done. So suppose that  $G_t$  is a proper minor of  $G$ . Since there is no 3-connected simple graph  $H$  for which  $H/e$  or  $H \setminus e$  is  $K_4$ , we must have that  $\hat{G} = G_1 = G_t \cong K_4$ . Since  $G_t$  is a proper minor of  $G$ ,  $G$  contains  $K_4$  along with one doubled edge. This contains a  $2C_3$ -minor, as required.  $\square$

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