Signings of the Platonic Solids

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Abstract

We determine all of the signings up to switching isomorphism for the tetrahedron, cube, octahedron, and dodecahedron. We determine the number of signings up to switching isomorphism of the icosahedron.

1 Introduction

A *signed graph* (G, σ) is a pair in which G is a graph and $\sigma\colon E(G) \to \{+, -\}$. A cycle is *positive* or *negative* according to the product of signs on its edges. A *switching function* is $\eta\colon V(G) \to \{+, -\}$. Given $\sigma\colon E(G) \to \{+, -\}$, define $\sigma^\eta(e) = \sigma(u)\sigma(e)\sigma(v)$ in which u and v are the endpoints of e, this includes the case where e is a loop and $u = v$. Zaslavsky [5] showed that signed graphs (G, σ) and (G, ψ) have the same positive and negative cycles if and only if $\psi = \varphi^\eta$ for some η. Thus sign switching yields an equivalence relation on all signings of a labeled graph G. Two signed graphs (G, σ) and (H, ψ) are said to be *switching isomorphic* when there is an isomorphism from G to H which preserves the signs of cycles. Finding the number of signings up to switching isomorphism of a particular graph G is an exercise which immediately presents itself. For example, the tetrahedron (i.e., K_4) has 3 signings up to switching isomorphism. They are as shown in Figure 1 where a solid edge is positive and a dashed edge is negative. We leave this easy fact as an exercise for the reader.

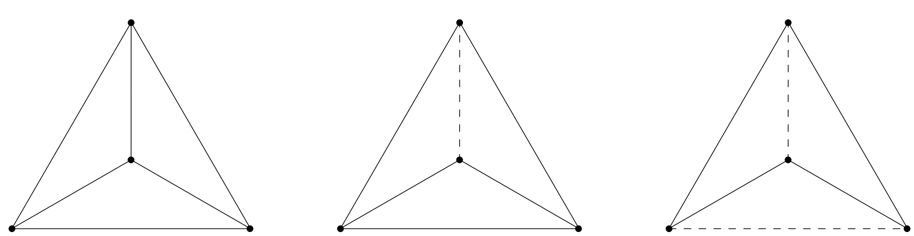

Figure 1: The signings of K_4.

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A set E may be viewed as a binary vector space in which addition is the symmetric difference operation and $0 = \emptyset$. For a graph G, its *cycle space* $Z(G)$ is the subspace of $E(G)$ spanned by the edge sets of cycles of G. Its dimension is $|E(G)| - |V(G)| + c(G)$ in which $c(G)$ is the number of connected components of G. Given a sign function σ on $E(G)$, there is an induced linear transformation $\hat{\sigma}\colon Z(G) \to \mathbb{Z}_2 \cong \{+, -\}$. Furthermore, any linear transformation $f\colon Z(G) \to \mathbb{Z}_2$ is equal to $\hat{\sigma}$ for some σ and σ is uniquely determined up to switching by the values of f on a basis of $Z(G)$. When G is a 3-connected planar graph, a canonical choice of basis for $Z(G)$ is the set of facial boundary cycles minus any one cycle; furthermore, the sum of all facial boundary cycles is 0.

Proposition 1 is another immediate result. It also suggests that this problem of determining all signings up to switching isomorphism is more interesting for graphs with a relatively high degree of symmetry.

Proposition 1. *If G is a connected graph with v vertices, e edges, and no non-trivial automorphisms, then G has 2^{e-v+1} signings up to switching isomorphism.*

Proof. A sign function is determined up to switching by it values on a basis for $Z(G)$. No two signings inequivalent by switching can be switching isomorphic because G is asymmetric. □

Zaslavsky [6] found all possible signings of the Petersen graph (there are 6), Sivaraman [4] found all possible signings of the Heawood graph (there are 7), Sehrawat and Bhattacharjya [3] found all possible signings of K_6 (there are 16), and Asiri [1] found all possible signings of $K_6 - e$, $K_6 - M_2$, and $K_6 - P_2$ (there are respectively 34, 36, and 44). All of these investigations make interesting use of different invariants of signed graphs. The graphs considered in [1, 3, 4, 6] are all non-planar. When G is a 3-connected planar graph, however, a different technique can be used. Up to switching a signing of a labeled copy of G is determined by signing the faces of the unique planar embedding of G where the number of negative faces is even. Thus the number of signings up to switching isomorphism is then exactly the number of signings of the faces of an unlabeled copy of G. This number can be determined by the familiar technique of Pólya enumeration (see, for example, [2, Ch.14]). From there, ad hoc methods may be able to determine the signings of the faces and edges.

An obvious choice of planar graphs on which to try these ideas are the 1-skeletons of the Platonic solids: the tetrahedron K_4, the cube Q_3, the octahedron O_3, the dodecahedron D_{12}, and the icosahedron I_{20}. The problem for K_4 is solved above. The problem for Q_3 is not much more difficult. There is 1 signing with 0 negative faces, 2 signings with 2 negative faces, 2 signings with 4 negative faces, and 1 signing with 6 negative faces. These are shown in Figure 2.

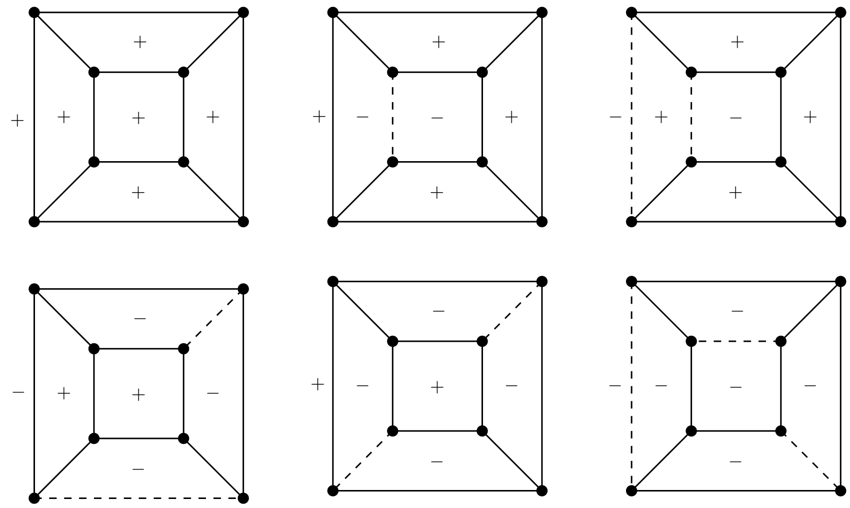

Figure 2: The 6 signings of Q_3.

In Section 2 we determine all of the signings of the O_3, in Section 3 we determine all of the signings of the D_{12}, and in Section 4 we determine only the number of signings of the I_{20} which is exactly 4,788.

1.1 Sign Symmetry

A signed graph (G, σ) is said to be *sign symmetric* when (G, σ) is switching isomorphic to $(G, -\sigma)$. Notice that a cycle C in G has different sign under σ and $-\sigma$ if and only if the length of C is odd. Thus If G is bipartite, then (G, σ) is always sign symmetric. Thus all of the signings of Q_3 (see Figure 2) are sign symmetric. For a sign-symmetric (G, σ) in which G is planar and all faces have odd length (e.g., K_4, O_3, D_{12}, and I_{20}) it is necessary that half of the faces are positive and half negative. For example, the middle signing of K_4 in Figure 1 is its only sign-symmetric one. In Section 2 we find that all signings of O_3 with 4 negative and 4 positive faces are sign-symmetric. In Section 3 we will also note which signings of D_{12} with 6 positive and 6 negative faces are sign symmetric and which are not.

It would be interesting to consider the question of whether or not any signing of the hyper-octahedron O_n with 2^{n-1} positive and 2^{n-1} negative $(n-1)$-cells must be sign symmetric.

2 The Octahedron

Theorem 2. *The number of signings of O_3 up to switching isomorphism is 14. The number of signings with*

- *0 negative faces is 1,*
- *2 negative faces is 3,*

- *4 negative faces is 6,*
- *6 negative faces is 3, and*
- *8 negative faces is 1.*

Proof. The planar dual graph of O_3 is the cube Q_3. We will determine the number of 2-colorings of the vertices of the cube with an even number of colors each. These correspond to the number of signings of the octahedron with the prescribed number of positive and negative faces. Consider the labeling of the eight vertices of the cube shown in Figure 3.

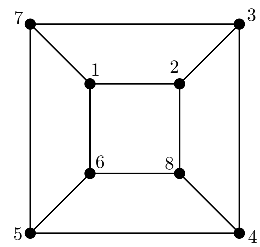

Figure 3: The cube with labeled vertices.

Because Q_3 is 3-regular and vertex transitive, its automorphism group $\mathrm{Aut}(Q_3)$ has order $8 \cdot 6 = 48$. If we represent $\mathrm{Aut}(Q_3)$ as permutations on the vertex labels given in Figure 3 we see that $(1,2,3,7)(4,5,6,8)$ and $(7,8)(1,2,3,4,5,6) \in \mathrm{Aut}(Q_3)$. Using the SageMath computational package, we calculated that

$$\left|\left\langle(1,2,3,7)(4,5,6,8),(7,8)(1,2,3,4,5,6)\right\rangle\right| = 48$$

and therefore

$$\left\langle(1,2,3,7)(4,5,6,8),(7,8)(1,2,3,4,5,6)\right\rangle = \mathrm{Aut}(Q_3).$$

We again use SageMath and calculate the associated cycle-index polynomial

$$\tfrac{1}{48}\left(x_1^8 + 6x_2^2x_1^4 + 13x_2^4 + 8x_3^2x_1^2 + 12x_4^2 + 8x_6x_2\right).$$

This allows us to obtain the following: the number of signings of the faces of O_3 with two negative faces is

$$\frac{1}{48}\left[\binom{8}{2} + 6\binom{2}{1} + 6\binom{4}{2} + 13\binom{4}{1} + 8\binom{2}{2} + 0 + 8\binom{1}{1}\right] = 3$$

and the number of signings of the faces of O_3 with four negative faces is

$$\frac{1}{48}\left[\binom{8}{4} + 6\binom{2}{2} + 6\binom{2}{1}\binom{4}{2} + 6\binom{4}{4} + 13\binom{4}{2} + 8\binom{2}{1}\binom{2}{1} + 12\binom{2}{1} + 0\right] = 6.$$

Because the total number of faces is even, the other cases follow by symmetry. $\square$

Theorem 3. *The signings of O_3 up to switching isomorphism are as shown in Figure 7. All signings with 4 negative and 4 positive faces are sign symmetric.*

Proof. Again we consider Q_3 as the topological dual graph of O_3. The signings of O_3 correspond to the 2-colorings (say black and white) of the vertices of Q_3 with 0 white vertices, 2 white vertices, and 4 white vertices. (Colors may be exchanged for 6 and 8 white vertices.) Since the number of such 2-colorings is known (Theorem 2) we need only find, in each case, the requisite number of 2-colorings which are pairwise distinct. The exercise is trivial for 0 and 2 white vertices. Figure 4 depicts the 2-colorings of Q_3 4 black and 4 white vertices. These are all pairwise distinct because the induced subgraphs on the white vertices (shown with red edges) are pairwise non-isomorphic.

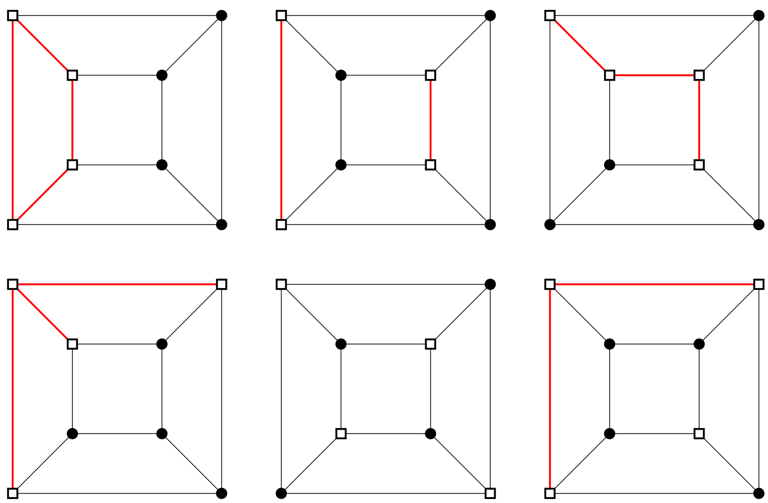

Figure 4: 2-colorings of the vertices of Q_3 with 4 white vertices and 4 black vertices.

Now, each 2-coloring of Q_3 gives one or two corresponding signings of the faces of Q_3 given in Figure 7. (The signings with 4 positive and 4 negative faces in Figure 7 are presented in the order corresponding to the 2-colorings in Figures 4.) The signing on the edges given in Figure 7 can easily be checked by the reader. Note that each of the 6 signings with 4 positive and 4 negative faces is sign symmetric because the corresponding 2-coloring of Q_3 yields isomorphic induced subgraphs on the 4 black and the 4 white vertices. □

3 The Dodecahedron

Theorem 4. *The number of signings of D_{12} up to switching isomorphism is 46. The number of signings with*

- *0 negative faces is 1,*
- *2 negative faces is 3,*
- *4 negative faces is 10,*

- *6 negative faces is 18,*
- *8 negative faces is 10,*
- *10 negative faces is 3, and*
- *12 negative faces is 1.*

Proof. Again we calculate the number of 2-colorings of I_{20} (which is the planar dual graph of D_{12}) using Pólya enumeration. So, consider the labeling of the 12 vertices of I_{20} shown in the figure.

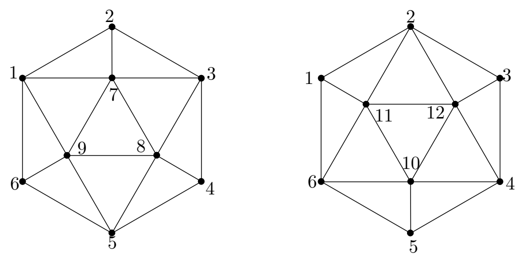

Figure 5: The Icosahedron with labeled vertices.

Because I_{20} is 3-regular and vertex transitive, its automorphism group $\mathrm{Aut}(I_{20})$ has order $12 \cdot 6 = 120$. If we represent $\mathrm{Aut}(I_{20})$ as permutations on the vertex labels given in Figure 5 we see that $(1)(2,7,9,6,11)(4)(3,8,5,10,12)$ and $(1,2,3,4,5,6)(7,12,8,10,9,11) \in \mathrm{Aut}(I_{20})$. Using SageMath we calculated that

$$\left|\left\langle (1)(2,7,9,6,11)(4)(3,8,5,10,12), (1,2,3,4,5,6)(7,12,8,10,9,11) \right\rangle\right| = 120$$

and therefore

$$\left\langle (1)(2,7,9,6,11)(4)(3,8,5,10,12), (1,2,3,4,5,6)(7,12,8,10,9,11) \right\rangle = \mathrm{Aut}(I_{20}).$$

Again using SageMath we calculate the associated cycle-index polynomial

$$\tfrac{1}{120}\left(x_1^{12} + 15x_1^4x_2^4 + 16x_2^6 + 20x_3^4 + 24x_1^2x_5^2 + 20x_6^2 + 24x_2x_{10}\right).$$

We now obtain the following: the number of signings of D_{12} with two negative faces is

$$\frac{1}{120}\left[\binom{12}{2} + 15\binom{4}{2} + 15\binom{4}{1} + 16\binom{6}{1} + 0 + 24 + 0 + 24\right] = 3,$$

the number of signings of D_{12} with four negative faces is

$$\frac{1}{120}\left[\binom{12}{4} + 15\binom{4}{4} + 15\binom{4}{2} + 15\binom{4}{2}\binom{4}{1} + 16\binom{6}{2} + 0 + 0 + 0 + 0\right] = 10,$$

and the number of signings of D_{12} with six negative faces is

$$\frac{1}{120}\left[\binom{12}{6} + 15\binom{4}{3} + 15\binom{4}{2}\binom{4}{2} + 15\binom{4}{1} + 16\binom{6}{3} + 20\binom{4}{2} + 24\binom{2}{1}\binom{2}{1} + 20\binom{2}{1} + 0\right] = 18.$$

Because the total number of faces is even, the remaining cases follow by symmetry. $\square$

Theorem 5. *The signings of D_{12} up to switching isomorphism are as shown in Figures 10–14. Of the 18 signings with 6 positive and 6 negative faces, the 8 in Figure 12 are not sign symmetric and the 10 in Figure 13 are sign symmetric.*

Proof. The strategy of this proof is again to consider 2-colorings of the vertices of the topological dual graph of I_{20}. The 2-colorings with where the number of white vertices is 0, 2, 10, and 12 are trivially determined. The 2-colorings with 4 white vertices and with 6 white vertices are shown in Figures 8 and 9. Any pair of 2-colorings shown are distinct because the induced subgraph on the white vertices (whose edges are shown in red) are non-isomorphic except in the following cases. The fourth and fifth 2-colorings of Figure 8 are distinct because the endpoints of the red 3-path have different distance in the subgraph of black edges. The sixth and seventh 2-colorings of Figure 8 are distinct because the isolated white vertex has different distances from the interior vertex of the red 2-path. The eighth and ninth 2-colorings of Figure 8 are distinct because the pair of red edges is antipodal in the ninth but not the eighth. The second and third 2-colorings in Figure 8 are distinct because the induced subgraphs on the black vertices are non-isomorphic.

Now, each 2-coloring of I_{20} gives one or two corresponding signings of the faces of Q_3 given in Figures 10–14. (The signings are presented in the corresponding order with the 2-colorings of Figures 8 and 9.) The signing on the edges given in the figures can easily be checked by the reader. Note that among the 18 signings with 6 positive and 6 negative faces, the 8 in Figure 12 not sign symmetric and the 10 shown in Figure 13 are sign symmetric. This is because in the corresponding 2-colorings of I_{20} the pair of induced subgraphs on the red and black vertices are not isomorphic in Figure 12 but are isomorphic in Figure 13. □

4 The Icosahedron

Theorem 6. *The number of signings of I_{20} up to switching isomorphism is 4,788. The number of signings with*

- *0 negative faces is 1,*
- *2 negative faces is 5,*
- *4 negative faces is 58,*
- *6 negative faces is 371,*
- *8 negative faces is 1,135,*
- *10 negative faces is 1,648,*
- *12 negative faces is 1,135,*
- *14 negative faces is 371,*
- *16 negative faces is 58,*
- *18 negative faces is 5, and*
- *20 negative faces is 1.*

Proof. Again, we calculate the number of 2-colorings of the vertices of the topological dual graph D_{12}.

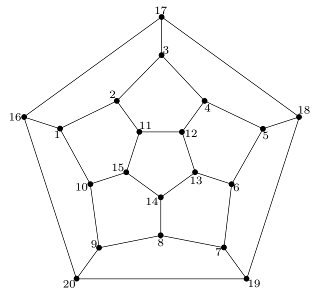

Figure 6: The Dodecahedron with labeled vertices.

According to the labeling of the vertices of D_{12} given in Figure 6 we see that

$$\alpha = (1,2,3,4,5,6,7,8,9,10)(16,11,17,12,18,13,19,14,20,15) \in \mathrm{Aut}(D_{12}) \text{ and}$$

$$\beta = (1)(6)(10,16,2)(5,13,7)(12,8,18),(19,4,14)(11,9,17)(20,3,15) \in \mathrm{Aut}(D_{12}).$$

Using SageMath we calculated that $\left|\langle\alpha,\beta\rangle\right| = 120$ and therefore $\langle\alpha,\beta\rangle = \mathrm{Aut}(I_{20})$. The cycle index polynomial is again given by SageMath.

$$\tfrac{1}{120}\left(x_1^{20} + 15x_2^8x_1^4 + 16x_2^{10} + 20x_3^6x_1^2 + 24x_5^4 + 20x_6^3x_2 + 24x_{10}^2\right)$$

We now obtain the following, the number of signings of I_{20} with two negative faces is

$$\frac{1}{120}\left[\binom{20}{2} + 15\binom{8}{1} + 15\binom{4}{2} + 16\binom{10}{1} + 20 + 0 + 20 + 0\right] = 5,$$

the number of signings of I_{20} with four negative faces is

$$\frac{1}{120}\left[\binom{20}{4} + 15\binom{8}{2} + 15\binom{8}{1}\binom{4}{2} + 15\binom{4}{4} + 16\binom{10}{2} + 20\binom{6}{1}\binom{2}{1}\right] = 58,$$

the number of signings of I_{20} with six negative faces is

$$\frac{1}{120}\left[\binom{20}{6} + 15\binom{8}{3} + 15\binom{8}{2}\binom{4}{2} + 15\binom{8}{1}\binom{4}{4} + 16\binom{10}{3} + 20\binom{6}{2} + 20\binom{3}{1}\right] = 371,$$

the number of signings of I_{20} with eight negative faces is

$$\frac{1}{120}\left[\binom{20}{8} + 15\binom{8}{4} + 15\binom{8}{3}\binom{4}{2} + 15\binom{8}{2} + 16\binom{10}{4} + 20\binom{6}{2} + 20\binom{3}{1}\right] = 1,135,$$

and the number of signings of I_{20} with ten negative faces is

$$\frac{1}{120}\left[\binom{20}{10} + 15\binom{8}{5} + 15\binom{8}{4}\binom{4}{2} + 15\binom{8}{3} + 16\binom{10}{5} + 20\binom{6}{3}\binom{2}{1} + 24\binom{4}{2} + 24\binom{2}{1}\right] = 1,648.$$

By symmetry the remaining cases follow. $\square$

References

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[6] ———, *Six signed Petersen graphs, and their automorphisms*, Discrete Math. **312** (2012), no. 9, 1558–1583. MR 2899889

5 Appendix: Large Figures

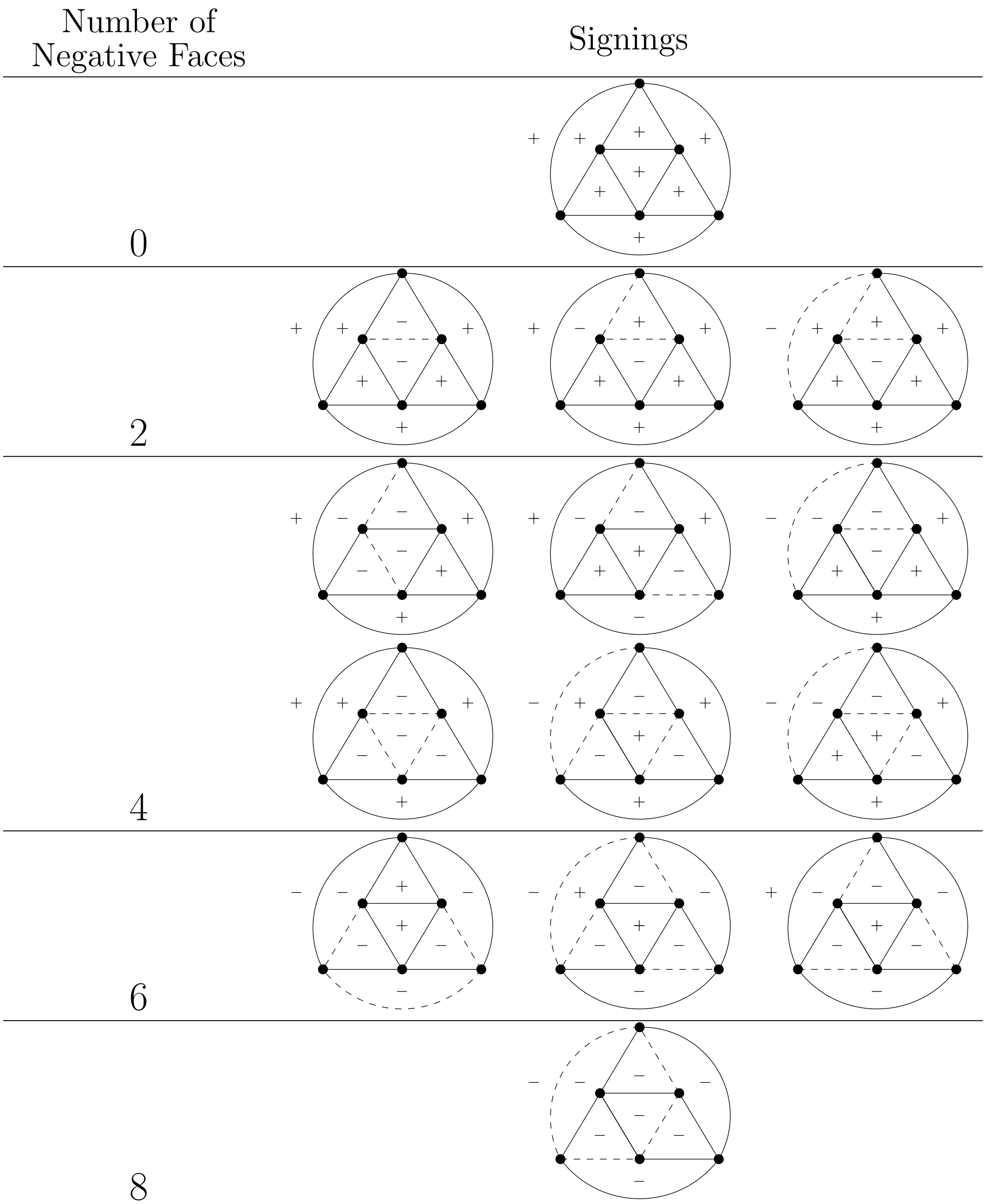

Figure 7: Signings of O_3.

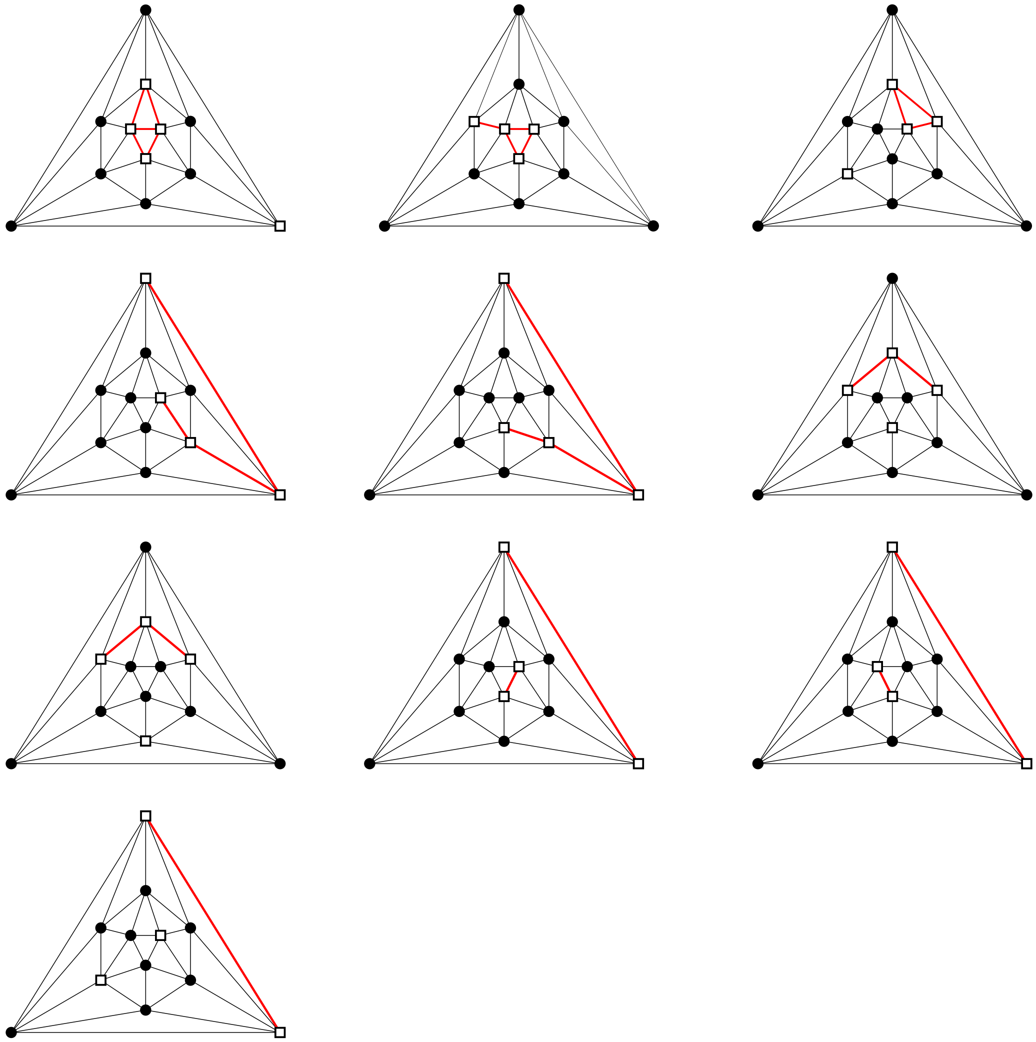

Figure 8: The 2-colorings of I_{20} with 4 white and 8 black vertices.

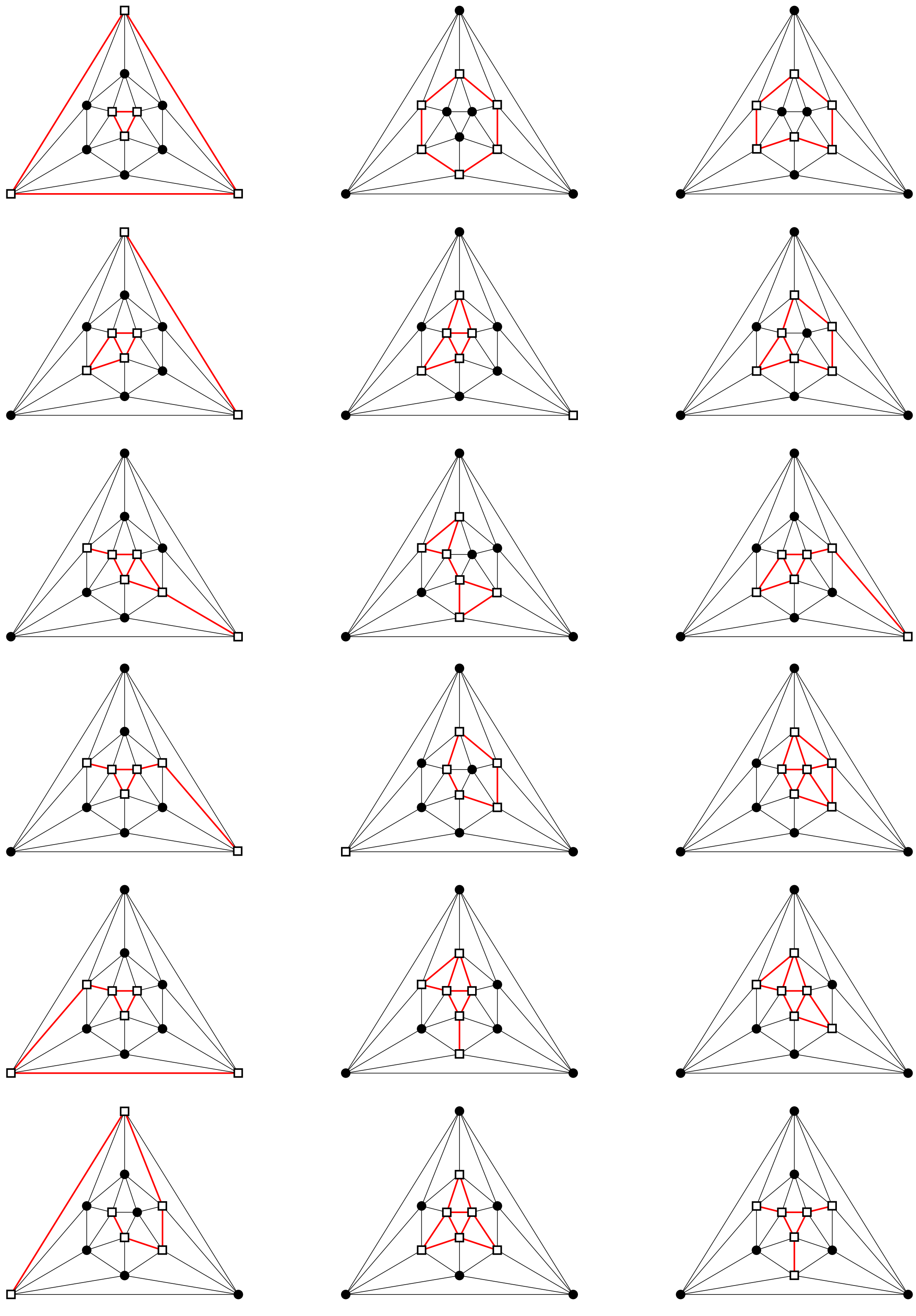

Figure 9: The 2-colorings of I_{20} with 6 white and 6 black vertices.

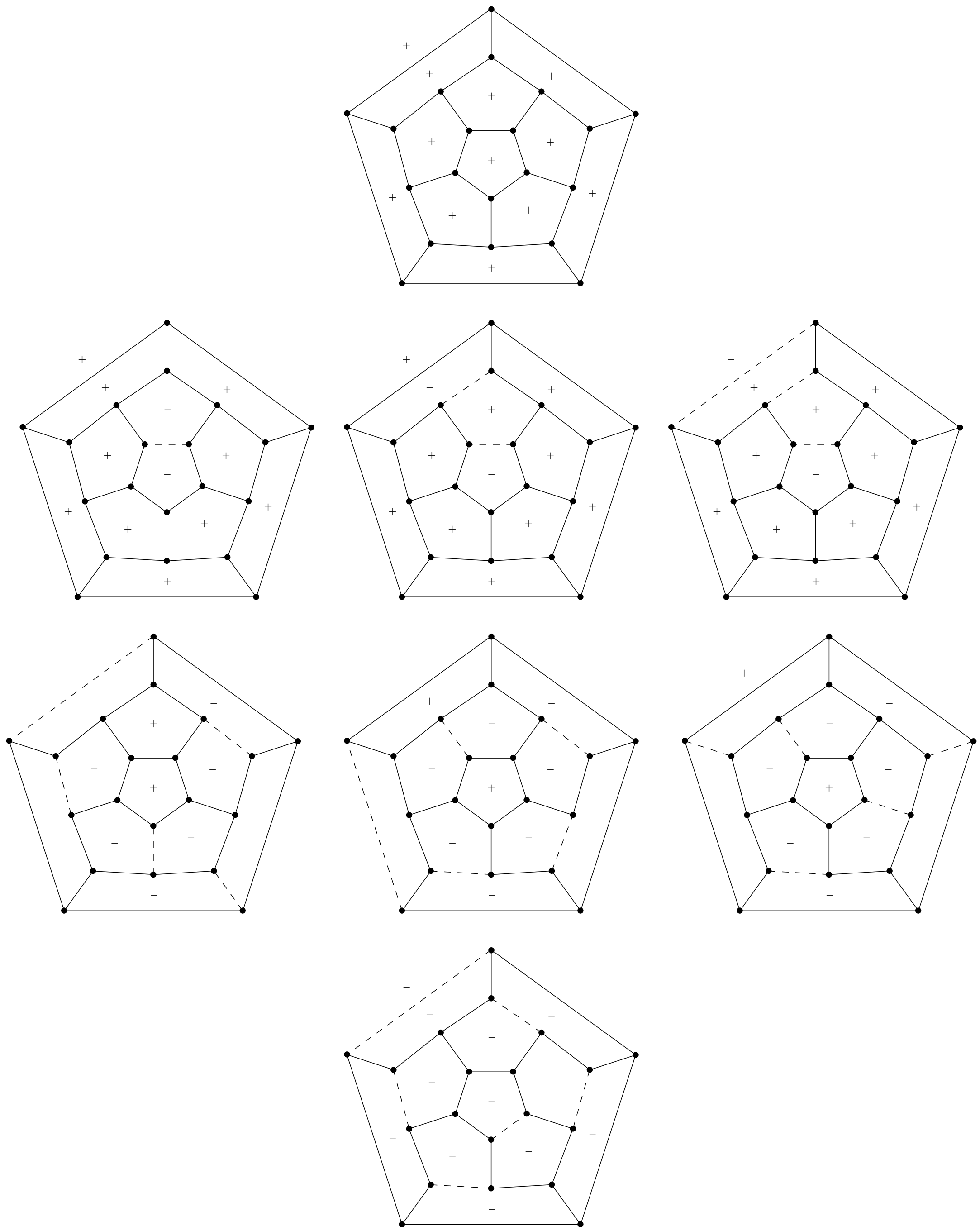

Figure 10: Signings of D_{12} with 0, 2, 10, and 12 negative faces.

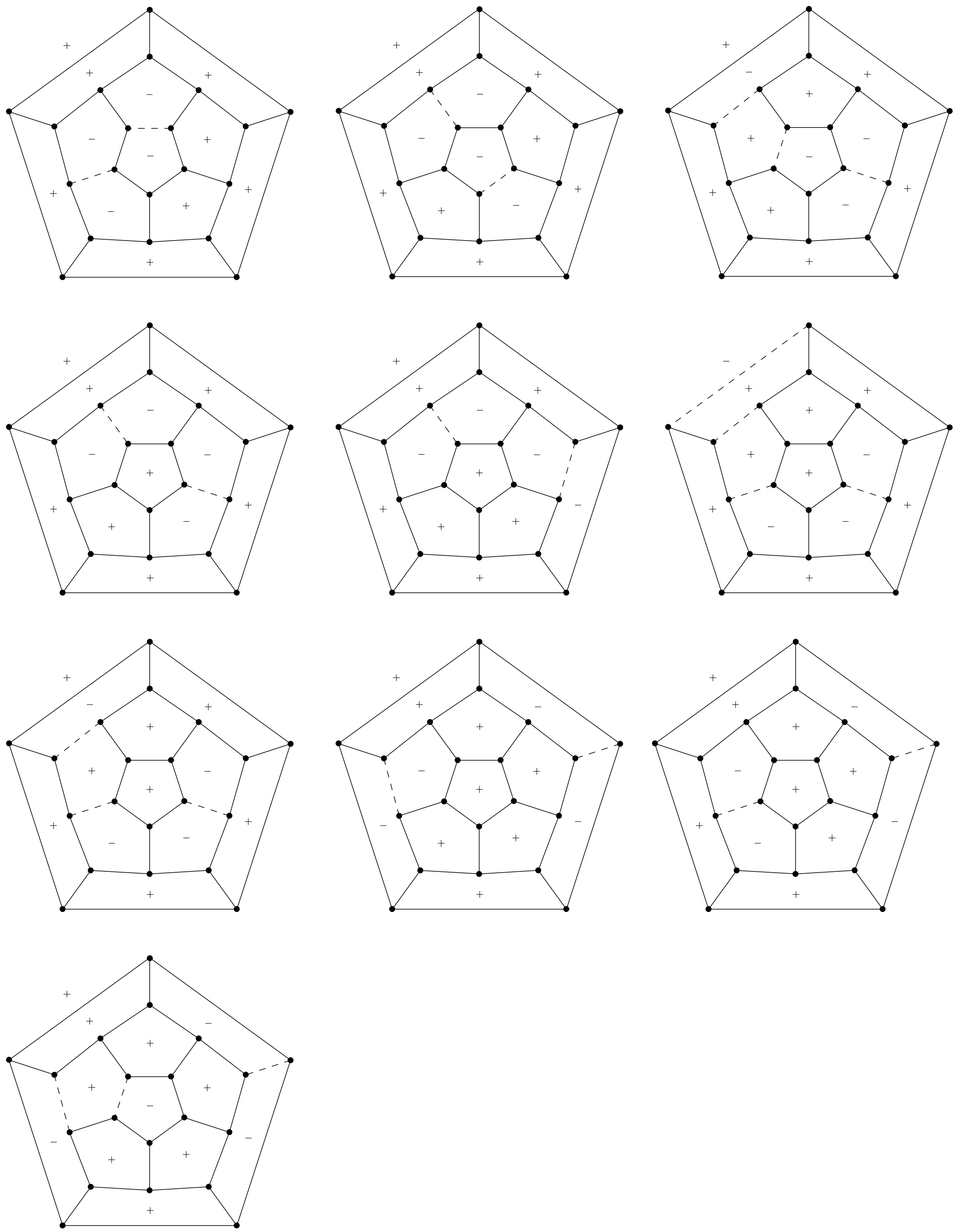

Figure 11: Signings of D_{12} with 4 negative faces.

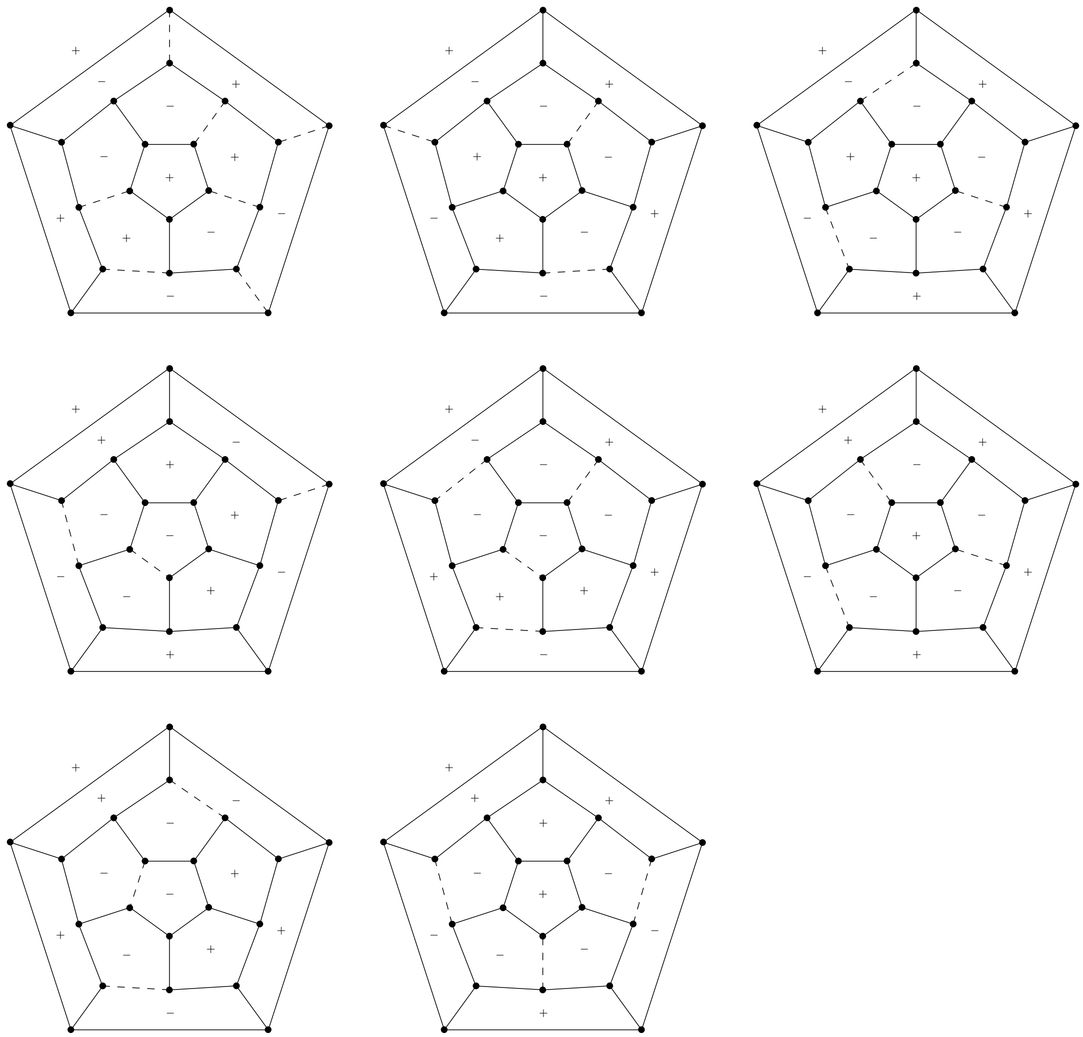

Figure 12: Signings of D_{12} with 6 negative faces that are not sign symmetric.

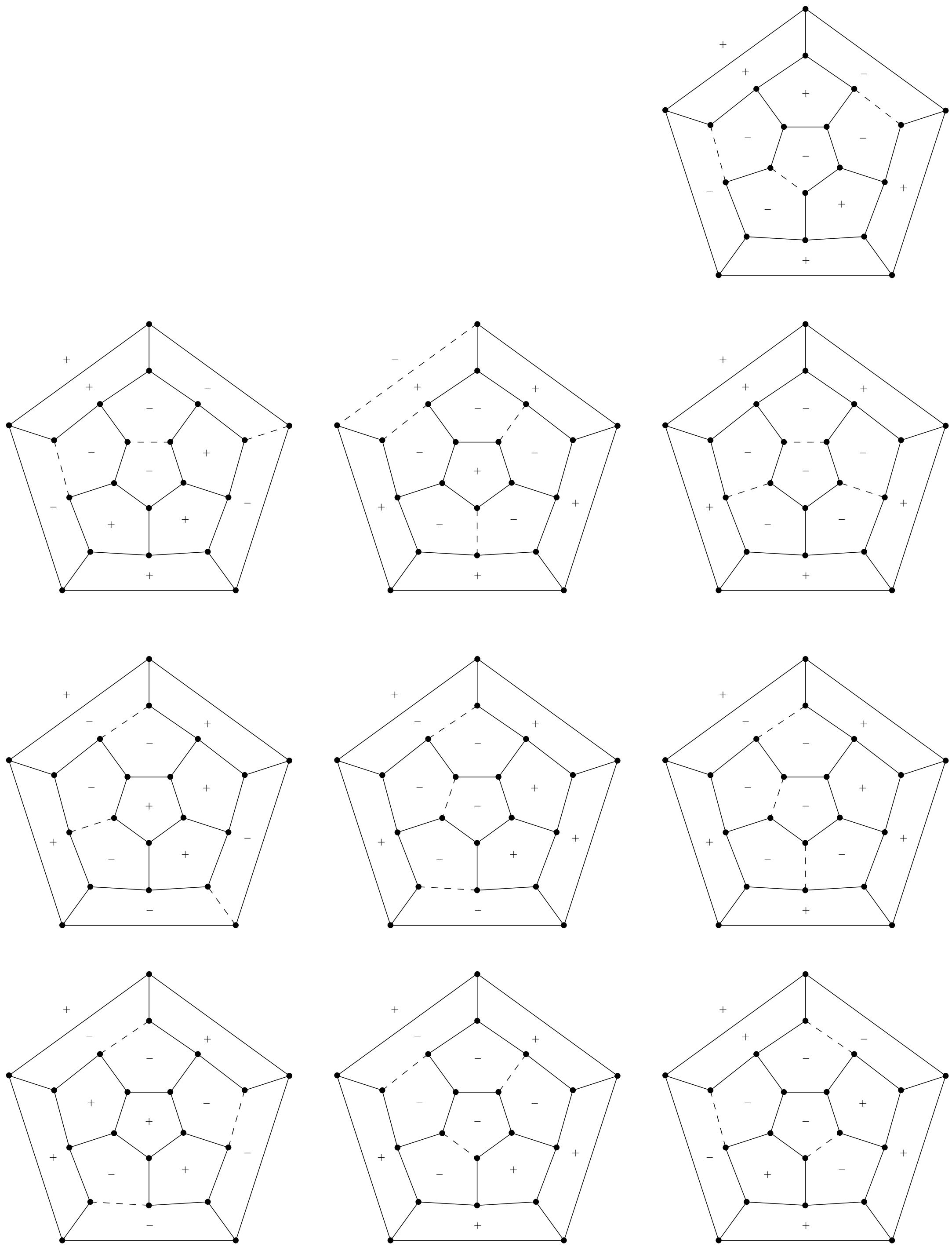

Figure 13: Signings of D_{12} with 6 negative faces that are sign symmetric.

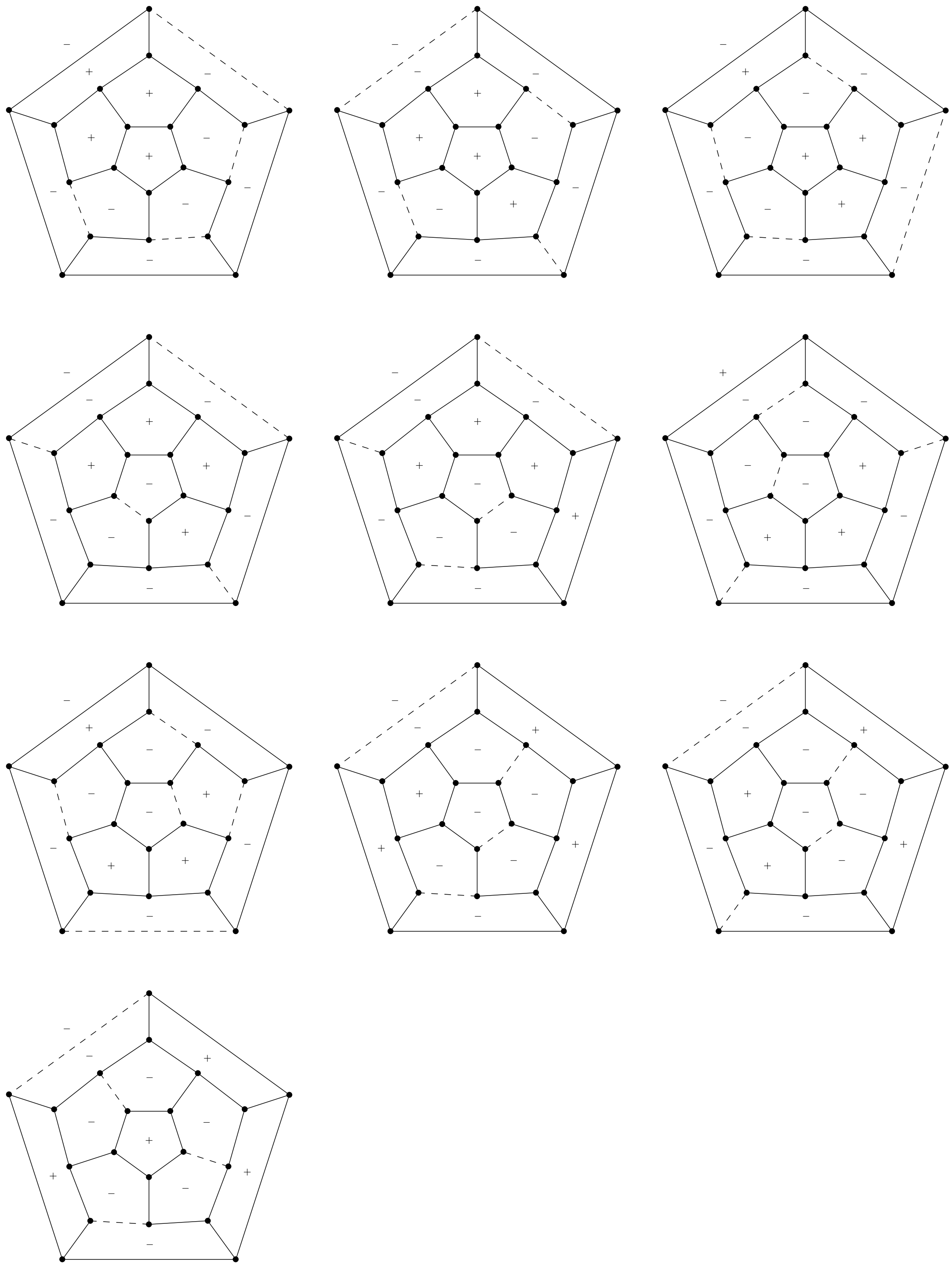

Figure 14: Signings of D_{12} with 8 negative faces.