Graphs without a $T_{2,2,3}$-minor

Daniel Slilaty*

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Abstract

Given three positive integers a, b, c, the fat triangle $T_{a,b,c}$ is the graph obtained from an ordinary triangle by replacing the three edges with parallel classes of a, b, and c edges. In this paper we give a structural characterization of all graphs without a $T_{2,2,3}$-minor.

1 Introduction

Given a fixed graph H, results characterizing the structure of graphs G without an H-minor have a well-established history going back as far as 1937 with Wagner's seminal result [11] for $H = K_5$. Results for other graphs H are numerous and are surveyed by Ding and Liu [4] and Diestel [3]. In all of the results listed in [3, 4], the graph H is simple.

Given three positive integers a, b, c, the *fat triangle* $T_{a,b,c}$ is the graph obtained from an ordinary triangle by replacing the three edges with parallel classes of a, b, and c edges. Fat triangles are, of course, not simple in general. Slilaty [9] gave a structural characterization of graphs without a $T_{2,2,2}$-minor which then also yields a characterization of bicircular matroids without a $U_{3,6}$-minor. A *doubled outerplanar embedding* is a planar embedding of a 2-connected outerplanar graph which is simple aside from some doubled chords with one embedded inside the Hamilton cycle and one outside.

Theorem 1.1 (Slilaty [9]). *If G is a 2-connected graph with minimum degree 3, then G has no $2C_3$-minor if and only if*

(1) $G \cong K_4$ or

(2) G is the topological dual graph of some doubled outerplanar embedding.

Theorem 1.1 is a very simple characterization; however, one might consider the dual outerplanar variety to be "degenerate" for the following reasons. If G is a 2-connected graph and e is an edge in G such that $G \backslash e$ has a cut vertex, then call e a 2-*essential* edge of G. Proposition 1.2 tells us that 2-essential edges can always be contracted in the search for a $T_{a,b,c}$-minor and Proposition 1.3 tells us that a topological dual graph of any doubled outerplanar embedding either has a 2-essential edge or has only two vertices (which can't, of course, contain a minor with three vertices). Furthermore, as cut vertices are easily identified in graphs, so are 2-essential edges. Thus it makes sense to try to characterize graphs with no $T_{2,2,2}$-minor and no 2-essential edges. Let us call a 2-connected graph with no 2-essential edges 2-*separation irreducible*. We thus obtain Corollary 1.4 which is an extremely succinct characterization of graphs with no $T_{2,2,2}$-minor.

*Department of Mathematics and Statistics, Wright State University, Dayton, OH 45435. Email daniel.slilaty@wright.edu.

Proposition 1.2. *Let $a, b, c \geq 2$ and let e be a 2-essential edge in a 2-connected graph G. Then G has a $T_{a,b,c}$-minor if and only if G/e has a $T_{a,b,c}$-minor.*

Proof. If G/e has a $T_{a,b,c}$-minor, then G has a $T_{a,b,c}$-minor by definition. Conversely suppose that G has a $T_{a,b,c}$-minor. Let K be a subgraph of G which has a contraction isomorphic to $T_{a,b,c}$. If $e \notin K$, then G/e has a $T_{a,b,c}$-minor. If $e \in K$, then because $T_{a,b,c}$ has no 2-essential edge, e must be contracted in obtaining $T_{a,b,c}$ from K. Thus G/e has a $T_{a,b,c}$-minor. □

Proposition 1.3. *If G has at least three vertices and is the topological dual graph of doubled outerplanar embedding, then G has a 2-essential edge.*

Proof. If $G = H^*$ where H is a doubled outerplanar embedding, then G is not 3-connected. This is because a 3-connected graph must contain a K_4-minor, the topological dual graph of K_4 is K_4, and K_4 is not outerplanar. Furthermore, if e is a chord of the Hamilton cycle of H, then let e also denote the dual edge in G. Because H/e has a cut vertex, $G \backslash e = (H/e)^*$ has a cut vertex. Thus e is 2-essential in G. □

Corollary 1.4. *If G is 2-connected and 2-separation irreducible, then G has no $T_{2,2,2}$-minor if and only if $G \cong K_4$.*

Proof. Follows from Theorem 1.1 and Propositions 1.2 and 1.3. □

The approach of characterizing the 2-separation irreducible members of a minor-closed class of graphs was also used by Sivaraman and Slilaty [7] to obtain similarly succinct characterizations of graphs having antivoltages over groups of order up to 6. In this paper, we push the characterizations of fat-triangle-free graphs one edge further and characterize the 2-separation irreducible graphs without a $T_{2,2,3}$-minor. The graph $2C_n$ is obtained from the cycle of length n by doubling every edge.

Theorem 1.5. *If G is a 2-connected and 2-separation irreducible, then G has no $T_{2,2,3}$-minor if and only if*

(1) $G \cong 2C_n$ for some $n \geq 3$ or

(2) G is a 3-connected minor of $K_{3,3}$, P_6, or K_4^{++}.

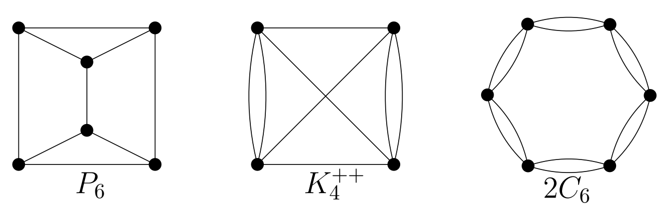

Figure 1: The graphs P_6, K_4^{++}, and $2C_6$.

2 Preliminaries

Our graph-theory terminology is mostly standard but we will go over some important and some less well-known terms and notations. A graph G is *separable* when there are two subgraphs G_1 and G_2 of G with at least one edge each such that $G_1 \cup G_2 = G$ and $G_1 \cap G_2$ is empty or a single vertex. A graph on at least $k+1$ vertices is k-connected when it is connected and there are no $t < k$ vertices in G whose removal leaves a disconnected subgraph. Thus a loopless graph with a least three vertices is 2-connected if and only if it is non-separable.

If G is a simple graph and $m \geq 2$ an integer, then by mG we mean the graph obtained from G by replacing each edge of G by m parallel edges on the same endpoints. The graph mK_2 is called an m-*multilink*. The cycle of length t is denoted by C_t and the wheel with t spokes is denoted by W_t.

Given two graphs G_1 and G_2 for which $G_1 \cap G_2$ is a single non-loop edge e along with its endpoints, the 2-*sum* along e is $G_1 \oplus_2 G_2 = (G_1 \cup G_2) - e$.

In this paper we will make use of the *canonical tree decomposition* of a non-separable graph G. For information on the canonical tree decomposition see one of [2], [6, pp.308–315], or [10] for a full description. In short, if G is non-separable, then there is a unique labeled tree T satisfying the following.

- Each vertex v in T is labeled with either a 3-connected simple graph, a cycle of length at least three, or mK_2 for some $m \geq 3$.
- No two cycle-labeled vertices are adjacent in T and no two multilink-labeled vertices are adjacent in T.
- If e is an edge of T whose endpoints are labeled with graphs G_1 and G_2, then e corresponds to an edge e_i in G_i.
- G is obtained by executing the 2-sums indicated by the vertex labels of T along the edges indicated by the edges of T.

An important consequence of this tree decomposition is that if T_0 is a subtree of T, then the graph G_0 obtained by executing the 2-sums indicated in T_0 is a minor of G. The canonical tree decomposition of a non-separable graph has been used to significant effect in analyzing minor-closed classes of graphs in which the number of 3-connected graphs in the class is finite [1, 5, 7, 8]. The effect of 2-separation irreducibility on the canonical tree decomposition is given by Proposition 2.1.

Proposition 2.1. *If T is the canonical tree decomposition of a non-separable graph G, then for any cycle-labeled vertex in T, every edge of that cycle is indicated in a 2-sum. In particular, no leaf of tree T is a cycle-labeled vertex.*

Proof. If e is an edge of a cycle in a cycle-labeled vertex c in T that is not indicated in a 2-sum, then because that cycle has length at least three, $G \backslash e$ would have a cut vertex, a contradiction. The second statement of the proposition now follows from the first. $\square$

3 Proofs

Proposition 3.1. *If G is 3-connected, loopless, and has no $T_{2,2,3}$-minor, then G is a 3-connected minor of $K_{3,3}$, P_6, or K_4^{++}.*

Proof. Let $\hat{G}$ be the simplification of G; that is, for each parallel class of edges, remove all of the edges but one. By Tutte's Wheel Theorem there is a sequence of 3-connected simple graphs $G_1, \ldots, G_t$ such that $G_1 = \hat{G}$, $G_{i+1} = G/e$ or $G \backslash e$ for some e, and $G_t = W_n$ for $n \geq 3$ where W_n is the n-spoked wheel. Consider cases for n

It cannot be that $n \geq 5$ because if G contains a W_n-minor for $n \geq 5$ then G contains a W_5-minor and W_5 contains $T_{2,2,3}$-minor. Thus G contains a $T_{2,2,3}$-minor, a contradiction.

Now suppose that $n = 4$. The only 3-connected simple graph having W_4 as a single-edge contraction or deletion is $K_{3,3}$ or P_6. Therefore $\hat{G} \cong W_4$, $K_{3,3}$, or P_6. We leave it to the reader to check that if any edge of $\hat{G} \cong W_4$, $K_{3,3}$, or P_6 is doubled then the resulting graph has a $T_{2,2,3}$-minor. Thus $G = \hat{G} \cong W_4$, $K_{3,3}$, or P_6, a desired outcome.

Finally, suppose that $n = 3$. There is no 3-connected simple graph having $W_3 \cong K_4$ as a single-edge contraction or deletion. Hence $\hat{G} \cong K_4$. The reader can check that the graph obtained from K_4

by tripling one edge has a $T_{2,2,3}$-minor. The same is true for the graph obtained from K_4 by doubling two incident edges. Thus $K_4 \subseteq G \subseteq K_4^{++}$, as required. □

Lemma 3.2. *If G is a 3-connected graph and e is an edge of G, then G has a K_4-minor which contains e.*

Proof. This is a well-known result which follows from Theorem 12.3.9 in [6]. □

Proof of Theorem 1.5. If G is 3-connected, then our result follows from Proposition 3.1. Thus we may assume that G is 2-connected but not 3-connected for the remainder of the proof. Let T be the canonical tree decomposition of G.

First, we claim that T does not have more than one vertex labeled by a 3-connected simple graph. By way of contradiction assume that T has two vertices, call them a and b, labeled by 3-connected simple graphs. Choose a and b so that the ab-path in T has shortest possible length. Thus the internal vertices of P (if any) alternate between multilink and cycle labels. Lemma 3.2 now implies that G contains $K_4 \oplus_2 K_4$ as a minor. This graph, however, has a $T_{2,2,3}$-minor, a contradiction.

Second, we claim that T does not have any vertex labeled with a 3-connected simple graph. Assume by way of contradiction that it does. Because G is not 3-connected and T has at most one vertex labeled with a 3-connected simple graph, T must contain a cycle-labeled vertex, call it c. Because the cycle for c has length at least three, Proposition 2.1 implies that the graph with the tree decomposition on the left of Figure 2 is a minor of G. This graph, however, has a $T_{2,2,3}$-minor, a contradiction.

Therefore each vertex of T is labeled with a cycle or multilink. Let c be a cycle-labeled vertex and consider this to be the root of T. Every edge of c is indicated in a 2-sum with a multilink of at least three links. This then implies that none of the multilinks have more than three edges because this would create a $T_{2,2,3}$-minor. So now, if the depth of T relative to root c is one, then $G \cong 2C_n$ for some $n \geq 3$, as required. If the depth of T relative to root c is more than one, then it must be at least three by Proposition 2.1. Thus G has a minor whose tree decomposition is as shown on the right of Figure 2. This graph, however, has a $T_{2,2,3}$-minor, a contradiction.

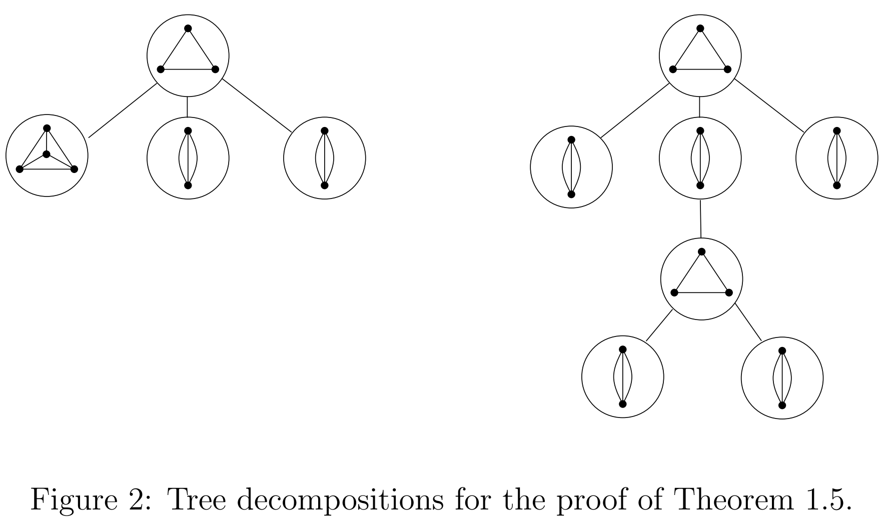

Figure 2: Tree decompositions for the proof of Theorem 1.5.

□

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