Signed Ramsey Numbers

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Abstract

Let $r(s,t)$ be the classical 2-color Ramsey number; that is, the smallest integer n such that any edge 2-colored K_n contains either a monochromatic K_s of color 1 or K_t of color 2. Define the *signed Ramsey number* $r_\pm(s,t)$ to be the smallest integer n for which any signing of K_n has a subgraph which switches to $-K_s$ or $+K_t$. We prove the following results.

(1) $r_\pm(s,t) = r_\pm(t,s)$

(2) $r_\pm(s,t) \geq \left\lfloor \frac{s-1}{2} \right\rfloor (t-1)$

(3) $r_\pm(s,t) \leq r(s-1,t-1)+1$

(4) $r_\pm(3,t) = t$

(5) $r_\pm(4,4) = 7$

(6) $r_\pm(4,5) = 8$

(7) $r_\pm(4,6) = 10$

(8) $3\left\lfloor \frac{t}{2} \right\rfloor \leq r_\pm(4,t+1) \leq 3t-1$

1 Introduction and Preliminaries

Let $r(s,t)$ be the classical 2-color Ramsey number; that is, the smallest integer n such that any edge 2-colored K_n contains either a monochromatic K_s of color 1 or K_t of color 2. Define the *signed Ramsey number* $r_\pm(s,t)$ to be the smallest integer n for which any signing of K_n has a subgraph which switches to $-K_s$ or $+K_t$; that is, differs from $-K_s$ or $+K_t$ by sign reversal on an edge cut. Our results are those listed in the abstract which are all proven in Section 2. Some of these results are contained in the M.S. thesis of Mutar [4]. One feature that makes the signed Ramsey number $r_\pm(s,t)$ interesting when contrasted with the classical number $r(s,t)$ is that the number of colored subgraphs sought in determining $r(s,t)$ is exactly two, despite the values of s and t, while the number of signed subgraphs sought for determining $r_\pm(s,t)$ grows linearly with s and t (see Proposition 1.1).

The rest of this section is a short overview of the pertinent ideas concerning signed graphs. A *signed graph* is a pair (G,σ) in which G is a graph and $\sigma\colon E(G) \to \{+,-\}$. Two signed graphs (G,σ)

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and (G, ψ) are *switching equivalent* if the set of edges along which σ and ψ differ form an edge cut of G; that is, there is $X \subseteq V(G)$ (possibly empty) such that for each edge e in G, $\sigma(e) \neq \psi(e)$ if and only if e has exactly one endpoint in X. A cycle in (G, σ) is called *positive* (respectively, *negative*) when the product of signs on its edges is positive (respectively, negative). Zaslavsky [6] showed that (G, σ) and (G, ψ) have the same positive and negative cycles if and only if they are switching equivalent. This tells us that switching equivalence is an equivalence relation on signed graphs with underlying graph G. Let $[(G, \sigma)]$ denote the equivalence class for (G, σ).

Two signed graphs are *isomorphic* when there is an isomorphism between their underlying graphs which preserves signs of edges. Two signed graphs are *switching isomorphic* when there is an isomorphism between their underlying graphs which preserves signs of cycles. We will say that (G, σ) has an (H, ψ)-subgraph when there is a subgraph of (G, σ) which is isomorphic to (H, ψ). We will say that (G, σ) has a $[(H, \psi)]$-subgraph when there is a subgraph of (G, σ) which is switching isomorphic to (H, ψ).

By $+G$ we mean the signed graph whose underlying graph is G and whose every edge is positive. The signed graph $-G$ is similarly defined in which each edge is negative. Every cycle in each of the signed graphs of $[+G]$ is positive. Such signed graphs are called *balanced*. A cycle in a signed graph of $[-G]$ is positive if and only if it has even length. Such signed graphs are sometimes called *parity signed* or *antibalanced*. Note that $(K_n, \sigma) \in [+K_n]$ (respectively, $(K_n, \sigma) \in [-K_n]$) if and only if the set of negative edges (respectively, positive edges) is either empty or induces a subgraph isomorphic to $K_{a,b}$ for some $a + b = n$.

Proposition 1.1. *If $s, t \geq 3$, then $[+K_s]$ and $[-K_t]$ are disjoint sets containing, respectively, $\lfloor s/2 \rfloor + 1$ and $\lfloor t/2 \rfloor + 1$ signed graphs up to isomorphism.*

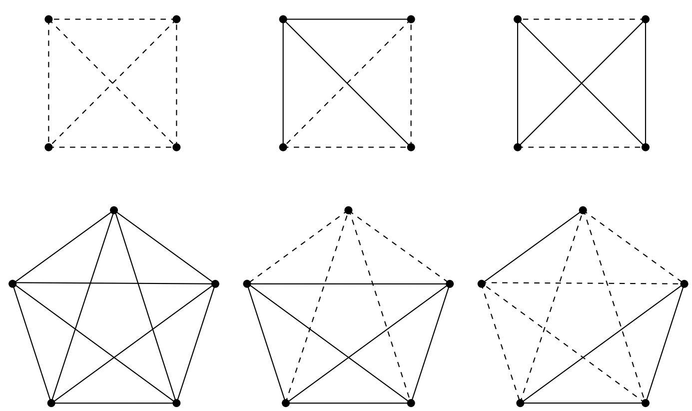

Figure 1: The signed graphs in $[-K_4]$ and $[+K_5]$ up to isomorphism. Dashed edges are negative and solid edges are positive.

When (G, σ) is a signed graph, we will use $N(G, \sigma)$ to denote the *negative subgraph*. This is the subgraph of (G, σ) formed by the negative edges along with the vertices incident to such edges. Define the *positive subgraph* $P(G, \sigma)$ similarly. We consider $N(G, \sigma)$ and $P(G, \sigma)$ to be ordinary (unsigned) graphs. Thus (K_n, σ) has a $[+K_t]$-subgraph (respectively a $[-K_s]$-subgraph) when $N(G, \sigma)$ (respectively $P(G, \sigma)$) has an induced $K_{a,b}$-subgraph where $a + b = t$ (respectively $a + b = s$).

2 Theorems and Proofs

Theorem 2.1. *$r_{\pm}(s, t) = r_{\pm}(t, s)$.*

Proof. The theorem follows from the fact that (K_n, σ) has a $[+K_t]$-subgraph if and only if $(K_n, -\sigma)$ has a $[-K_t]$-subgraph. □

Theorem 2.2. $r_\pm(s,t) \geq \left\lfloor \frac{s-1}{2} \right\rfloor (t-1)$.

Proof. Let $n = \left\lfloor \frac{s-1}{2} \right\rfloor (t-1)$. Partition $\{1, \dots, n\}$ into $t-1$ blocks of $\left\lfloor \frac{s-1}{2} \right\rfloor$ vertices each. Make the edges within blocks negative and edges between blocks positive. Thus a triangle in this signed K_n is positive if and only if it contains vertices in three distinct blocks. So now any t vertices chosen must contain two in the same block and so the induced signed K_t is not in $[+K_t]$ because not all triangles are positive. Furthermore, any s vertices chosen must contain vertices in at least three distinct blocks and so induces a signed K_s with at least one positive triangle and so is not in $[-K_s]$. □

Given a vertex v in a graph G the edge cut consisting of all edges having v as an endpoint is called a *vertex edge cut*. In a signed complete graph (K_n, σ) we may switch the vertex edge cuts of the negative neighbors of v to make every edge incident to v positive. In this case, we say that the signing σ in (K_n, σ) is *homogenized* at v. We do not expect that the upper bound in Theorem 2.3 is ever sharp outside of the cases for $r_\pm(4,4)$ and $r_\pm(3,n)$.

Theorem 2.3. $r_\pm(s,t) \leq r(s-1, t-1) + 1$.

Proof. If (K_n, σ) is homogenized at v, then a $-K_{s-1}$- or $+K_{t-1}$-subgraph in $(K_n - v, \sigma)$ will yield a $[-K_s]$- or $+K_t$-subgraph when v is attached. Thus $r_\pm(s,t) \leq r(s-1,t-1) + 1$. □

Theorem 2.4. $r_\pm(3,t) = t$.

Proof. The signed graph $+K_{t-1}$ proves that $r_\pm(3,t) > t-1$ and Theorem 2.3 along with the fact that $r(2,t) = t$ completes the proof. □

Theorem 2.5. $r_\pm(4,4) = 7$.

Proof. Theorem 2.3 yields $r_\pm(4,4) \leq 7$. The signed graph (K_6, σ) for which $N(K_6, \sigma)$ is a pentagon has no $[-K_4]$- nor $[+K_4]$-subgraph. □

It is known [1] that $r(3,4) = 9$. Combining this with Theorem 2.3 yields $r_\pm(4,5) \leq 10$. Our next result, however, is that $r_\pm(4,5) = 8$ and hence we have our first example of the upper bound in Theorem 2.3 not being sharp. In the proofs of Theorems 2.6 and 2.7, $K_2 + K_2$ denotes a matching of size two.

Theorem 2.6. $r_\pm(4,5) = 8$.

Proof. For the lower bound consider (K_7, σ) in which $N(K_7, \sigma)$ is a pentagon. Any four vertices chosen induce a subgraph which is not in $[-K_4]$ and any five vertices chosen induce a subgraph which is not in $[+K_5]$ (see Figure 1).

To prove that $r_\pm(4,5) \leq 8$ consider (K_8, σ) which is homogenized at v. Assume by way of contradiction that (K_8, σ) has no $[+K_4]$- nor $[-K_5]$-subgraph. Four immediate facts are as follows. One, the vertex-deleted subgraph $(K_8 - v, \sigma)$ contains no $-K_3$- nor $+K_4$-subgraph because otherwise we immediately get a $[-K_4]$- or $+K_5$-subgraph when we attach v. Two, the negative subgraph $N = N(K_8 - v, \sigma)$ has no induced $K_2 + K_2$ because otherwise $(K_8 - v, \sigma)$ has a $[-K_4]$-subgraph (see Figure 1). Three, the degree of a vertex in N is at most 3 because otherwise $(K_8 - v, \sigma)$ would have a $-K_3$- or $+K_4$-subgraph. Four, the independence number $\alpha(N)$ of N is at most 3 because otherwise $(K_8 - v, \sigma)$ would have a $+K_4$-subgraph.

Note that if a vertex w of N has degree 0, then the $(K_8 - \{v, w\}, \sigma)$ is a signed K_6 which is guaranteed to have a $-K_3$- or $+K_3$-subgraph. The former does not exist and the latter would form a $+K_5$ when v and w are attached. So for the remainder of the proof we may now assume that the degree of a vertex in N is in $\{1,2,3\}$.

Now assume that $d_N(x) = 1$. Let y be the neighbor of x. The degree of y is at most 3 and so there are three vertices adjacent to neither x nor y in N. Because N has no induced $K_2 + K_2$, these three vertices must be independent; however, then the three vertices along with x form an independent set of size 4, a contradiction. So for the remainder of the proof we may assume that the degree of a vertex in N is 2 or 3.

Assume that N has two adjacent degree-2 vertices x and y. Let $x' \neq y$ be the other neighbor of x and $y' \neq x$ the other neighbor of y. Thus there are three vertices in N adjacent to neither x nor y. Again, the assumption of no induced $K_2 + K_2$ yields the contradiction $\alpha(N) \geq 4$. So for the remainder of the proof, we assume that degree-2 vertices have both neighbors of degree 3.

Because N has 7 vertices, there is at least one vertex, call it x, with $d_N(x) = 2$. Let y be one neighbor of x, $x_1 \neq y$ the other neighbor of x, and y_1 and y_2 the other two neighbors of y. Let $A = \{x_1, y_1, y_2\}$ and let $B = \{b_1, b_2\}$ be the set containing the remaining two vertices of N. Because N has no induced $K_2 + K_2$, $(b_1, b_2) \notin E(N)$. Because N has no triangle, the set of edges with both endpoints in A is a subset of $\{(x_1, y_1), (x_1, y_2)\}$. If A supports both of these edges, then the induced subgraph on $\{x, x_1, y, y_1, y_2\}$ is isomorphic to $K_{2,3}$. Thus $(K_8 - v, \sigma)$ contains a $[+K_5]$-subgraph (see Figure 1).

If A supports no edges, then because $d_N(x_1) = 3$, $(x_1, b_1), (x_1, b_2) \in E(N)$. Since N has no induced $K_2 + K_2$ and $(x_1, b_1), (x_1, b_2), (y, y_1), (y, y_2) \in E(N)$ we must now have that $(y_i, b_j) \in E(N)$ for all $i, j \in \{1, 2\}$. Thus $A \cup B$ induces a $K_{2,3}$-subgraph and so $(K_8 - v, \sigma)$ contains a $[+K_5]$-subgraph.

If A supports one edge, then without loss of generality $(x_1, y_1) \in E(N)$ and $(x_1, y_2) \notin E(N)$. Because $d_N(y_2) = 3$, $(y_2, b_1), (y_2, b_2) \in E(N)$. Because N has no induced $K_2 + K_2$ and $(x, x_1), (y_2, b_1), (y_2, b_2) \in E(N)$ we must have that $(x_1, b_1), (x_1, b_2) \in E(N)$ which makes $d_N(x_1) \geq 4$, a contradiction. □

Theorem 2.7. $r_\pm(4, 6) = 10$

Proof. For the lower bound consider (K_9, σ) whose negative subgraph $N(K_9, \sigma)$ is as shown in Figure 2. Because $N(K_9 - v, \sigma)$ is bipartite, it contains no triangle. Also, because vertices a and 4 have degree 4, they cannot be used in any potential induced $K_2 + K_2$-subgraph. However the edges of $N(K_9 - \{v, a, 4\}, \sigma)$ form a a path of length three. Thus $N(K_9 - v, \sigma)$ has no induced $K_2 + K_2$. These imply that (K_9, σ) has no $[-K_4]$-subgraph.

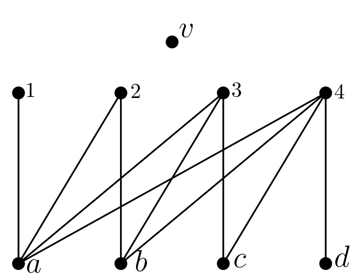

Figure 2: A graph $N(K_9, \sigma)$ showing that $r_\pm(4, 6) > 9$.

The independence number of $N(K_9 - v, \sigma)$ is 4 so (K_9, σ) has no $+K_6$-subgraph. By considering degrees, we immediately get that $N(K_9 - v, \sigma)$ has no induced $K_{1,5}$-, $K_{2,4}$-, or $K_{3,3}$-subgraph. Thus (K_9, σ) has no $[+K_6]$-subgraph.

To show that $r_\pm(4, 6) \leq 10$, consider any signed complete graph (K_{10}, σ) and assume by switching that it is homogenized at v. Throughout the remainder of the proof, let $N = N(K_{10} - v, \sigma)$. Furthermore, assume that (K_{10}, σ) has no $[-K_4]$-subgraph. Thus N is triangle free and has no induced

K_2+K_2. Also, assume that (K_{10},σ) has no $+K_6$-subgraph. This implies that $\alpha(N)\le 4$ which, along with N being triangle free, implies that the max degree of N is at most 4. In Case 1 assume that N has an isolated vertex. In Case 2, assume that $\delta(N)=1$. In Case 3, assume that $\delta(N)=2$. In Case 4, assume that $\delta(N)\ge 3$.

Case 1 First, assume that N has two isolated vertices, w_1 and w_2. Now $N-\{w_1,w_2\}$ has 7 vertices. Since $r(3,3)=6$ and N is triangle free, there is an independent set A of size three in $N-\{w_1,w_2\}$. Thus $A\cup\{w_1,w_2\}$ is independent of size 5, a contradiction. Thus N has a unique isolated vertex, call it w.

Assume that $N-w$ has a vertex x of degree 4. Since N is triangle free, the four neighbors of x, call them $\{x_1,x_2,x_3,x_4\}$ are independent. Thus $\{x_1,x_2,x_3,x_4,w\}$ is independent in N, a contradiction. So now assume that the max degree of $N-w$ is 3. Now consider the minimum degree $\delta(N-w)$. If $\delta(N-w)\le 2$, then let x be a vertex of degree at most 2 and let y be one of the neighbors of x in $N-w$. Because $d_N(y)\le 3$, there are at least three vertices y_1,y_2,y_3 in $N-w$ which are adjacent to neither x nor y. Because $(x,y)\in E(N)$ and N has no induced K_2+K_2, $\{y_1,y_2,y_3\}$ is independent in N. Thus $\{y_1,y_2,y_3,x,w\}$ are independent in N, a contradiction. So now $N-w$ must be 3-regular and triangle free. Thus $N-w$ is either the 3-dimensional cube or the Wagner graph (i.e., the 4-rung Möbius ladder). If $N-w$ is the cube, then $\alpha(N-w)=4$ so $\alpha(N)=5$, a contradiction. If $N-w$ is the Wagner graph, then N has an induced K_2+K_2, a contradiction.

Case 2 If $d_N(x)=1$, then let y be the neighbor of x. Now N has at least four vertices, y_1,y_2,y_3,y_4 adjacent to neither x nor y. Because $(x,y)\in E(N)$ and N has no induced K_2+K_2, $\{y_1,y_2,y_3,y_4\}$ is independent in N. Thus $\{x,y_1,y_2,y_3,y_4\}$ is independent in N, a contradiction.

Case 3 Let x be a vertex with $d_N(x)=2$ and let y be a neighbor of x. If $d_N(y)\le 3$, then there are four vertices y_1,y_2,y_3,y_4 adjacent to neither x nor y. As in Case 2 we get that $\{x,y_1,y_2,y_3,y_4\}$ is independent in N, a contradiction. Thus both neighbors of x have degree four. Let $x_1\ne y$ be the other neighbor of x and y_1,y_2,y_3 be the other three neighbors of y. Because N is triangle free, $x_1\notin\{y_1,y_2,y_3\}$ and $\{y_1,y_2,y_3\}$ is independent. Let $A=\{x_1,y_1,y_2,y_3\}$ and $B=\{b_1,b_2,b_3\}=V(N)-A$. Because N has no induced K_2+K_2 and $(x,y)\in E(N)$, B must be independent.

Now, since $d_N(x_1)=4$, the number of edges in N with both endpoints in A is at most three. If A supports no edges, then $(x_1,b_i)\in E(N)$ for all i. In order to prevent (x_1,b_i) from inducing a K_2+K_2 with any (y,y_j), we must have that $(b_i,y_j)\in E(N)$ for all i and j. Thus N has an induced $K_{3,3}$ which yields a $[+K_6]$-subgraph in $(K_{10}-v,\sigma)$, a desired outcome. If A supports exactly one edge, then without loss of generality we assume that $(x_1,y_1)\in E(N)$. Since $d_N(x_1)=4$, we may assume that $(x_1,b_1),(x_1,b_2)\in E(N)$ and $(x_1,b_3)\notin E(N)$. Since $d_N(b_3)\ge 2$, we may assume that $(b_3,y_2)\in E(N)$; however, this makes $(x,x_1)\cup(b_3,y_2)$ an induced K_2+K_2, a contradiction. If A supports exactly two edges, then without loss of generality we assume that $(x_1,y_1),(x_1,y_2)\in E(N)$. Because $d_N(x_1)=4$, we can assume that $(x_1,b_1)\in E(N)$. Because $(x_1,b_1)\cup(y,y_3)$ is not an induced K_2+K_2, $(b_1,y_3)\in E(N)$. Now since there is no induced K_2+K_2 and $(x,x_1)\in E(N)$, we must have that $(b_2,y_3),(b_3,y_3)\notin E(N)$. Thus $d_N(b_1)=d_N(y_3)=2$; however, at the beginning of the case, we showed that an arbitrary vertex of degree 2 in N (called x at the beginning of Case 3) must have both of its neighbors of degree 4, a contradiction. If A supports exactly three edges, then $\{x,x_1,y,y_1,y_2,y_3\}$ induce a $K_{2,4}$-subgraph of N and so $(K_{10}-v,\sigma)$ has a $[+K_6]$-subgraph, a desired outcome.

Case 4 It cannot be that N is 4-regular because every 4-regular graph on 9 vertices contains a triangle. (This is an easy exercise which is left to the reader or the reader may use the algorithm by Meringer [3] to calculate all 4-regular graphs on 9 vertices and display that all 16 of them have girth 3.) Thus N must have a vertex of degree 3. In Case 4.1 say that N has two adjacent vertices of degree 3. In Case 4.2, the degree-3 vertices of N are independent.

Case 4.1 Suppose that there is $(x,y)\in E(N)$ such that $d_N(x)=d_N(x)=3$. Let x_1,x_2,y_1,y_2 be the other neighbors of x and y. Because N is triangle free, $(x_1, x_2), (y_1, y_2) \notin E(N)$ and $\{x_1, x_2\} \cap \{y_1, y_2\} = \emptyset$. Let $A = \{x_1, x_2, y_1, y_2\}$ and $B = \{b_1, b_2, b_3\} = V(N) - A$. Again, B must be independent. Now A can only support up to four edges of the form (x_i, y_j) for $i, j \in \{1, 2\}$.

If A supports no edges, then each vertex of B is adjacent to at least three vertices of A. Without loss of generality $(b_1, x_1), (b_1, x_2), (b_1, y_1) \in E(N)$; however, since (y, y_2) does not induce a $K_2 + K_2$ with either (b_1, x_1) or (b_1, x_2), we must also have $(b_1, y_2) \in E(N)$. Thus $A \cup B$ induces $K_{3,4}$ and so $(K_{10} - v, \sigma)$ has a $[+K_6]$-subgraph, a desired result.

If A supports just one edge, then we may assume that it is (x_1, y_1). Since each $d_N(b_i) \geq 3$ and N is triangle free, for each i we must have $(b_i, x_2), (b_i, y_2) \in E(N)$. Again because $d_N(b_i) \geq 3$, we get without loss of generality that $(b_1, x_1), (b_2, x_1) \in E(N)$. We now have four edges attached to x_1 so we must have that $(b_3, y_1) \in E(N)$; however, now edges (b_3, y_2) and (x, x_1) are in an induced $K_2 + K_2$, a contradiction.

If A supports two edges, then without loss of generality either $(x_1, y_1), (x_2, y_2) \in E(N)$ or $(x_1, y_1), (x_1, y_2) \in E(N)$. In the former case, $d_N(b_i) \geq 3$ is not possible without creating a triangle in N using edge (x_1, y_1) or (x_2, y_2), a contradiction. In the latter case, because each $d_N(b_i) \geq 3$ and N is triangle free, we must have that b_1 is adjacent to each vertex in $\{x_2, y_1, y_2\}$. The same is true for b_2 and b_3; however, this yields the contradiction that $d_N(y_1) \geq 5$.

If A supports three or four edges, then without loss of generality $(x_1, y_1), (x_2, y_2) \in E(N)$. Again, $d_N(b_i) \geq 3$ is not possible without creating a triangle in N using edge (x_1, y_1) or (x_2, y_2), a contradiction.

Case 4.2 Take $(x, y) \in E(N)$ such that $d_N(x) = 3$. Thus $d_N(y) = 4$. Let x_1, x_2, y_1, y_2, y_3 be the other neighbors of x and y. Because N is triangle free, $\{x_1, x_2\}$ is independent, $\{y_1, y_2, y_3\}$ is independent, and $\{x_1, x_2\} \cap \{y_1, y_2, y_3\} = \emptyset$. Let $A = \{x_1, x_2, y_1, y_2, y_3\}$ and $B = \{b_1, b_2\} = V(N) - A$. Again, B must be independent. In this case $d_N(x_i) = 4$ and $d_N(y_j) \geq 3$. Because $|B| = 2$ this will require each x_i to be adjacent to at least one y_j. Furthermore, we may assume that x_i is not adjacent to each vertex in $\{y_1, y_2, y_3\}$ because then $\{x, x_i, y, y_1, y_2, y_3\}$ induces $K_{2,4}$ which yields a $[+K_6]$-subgraph in $(K_{10} - v, \sigma)$ and we are done. In Case 4.2.1 assume that x_1 and x_2 are both adjacent to two vertices in $\{y_1, y_2, y_3\}$. In Case 4.2.2 assume that x_1 is adjacent to two vertices and x_2 is adjacent to one vertex in $\{y_1, y_2, y_3\}$. In Case 4.2.3 assume that x_1 and x_2 are both adjacent to one vertex each in $\{y_1, y_2, y_3\}$.

Case 4.2.1 Without loss of generality, the edges from $\{x_1, x_2\}$ to $\{y_1, y_2, y_3\}$ are either

$$(x_1, y_1), (x_1, y_2), (x_2, y_1), (x_2, y_2) \text{ or } (x_1, y_1), (x_1, y_3), (x_2, y_2), (x_2, y_3).$$

In the former case, $d_N(y_3) \geq 3$ implies that $(y_3, b_1), (y_3, b_2) \in E(N)$. These two edges and the fact that N has no induced $K_2 + K_2$ force $(x_i, b_j) \in E(N)$ for all $i, j \in \{1, 2\}$; however, this implies $d_N(x_i) \geq 5$, a contradiction. In the latter case, b_1 cannot be adjacent to any three vertices in A without creating a triangle save for y_1, y_2, y_3. The same is true for b_2, but this would force $d_N(y_3) \geq 5$, a contradiction.

Case 4.2.2 Without loss of generality, the edges from $\{x_1, x_2\}$ to $\{y_1, y_2, y_3\}$ are either

$$(x_1, y_1), (x_1, y_2), (x_2, y_2) \text{ or } (x_1, y_1), (x_1, y_2), (x_2, y_3).$$

In the former case, $d_N(y_3) \geq 3$ implies that $(y_3, b_1), (y_3, b_2)$. Since neither of these two edges can be in an induced $K_2 + K_2$ with (x, x_i), we get that $(x_i, b_j) \in E(N)$ for each $i, j \in \{1, 2\}$; however, now $d_N(x_1) \geq 5$, a contradiction. In the latter case, $d_N(x_1) = 4$ implies without loss of generality that $(x_1, b_1) \in E(N)$. Since $d_N(b_1) \geq 3$, b_1 must be adjacent to two other vertices in A; however, there is no way to choose two vertices of A to be adjacent to b_1 without creating a triangle, a contradiction.

Case 4.2.3 Without loss of generality, the edges from $\{x_1, x_2\}$ to $\{y_1, y_2, y_3\}$ are $(x_1, y_1), (x_2, y_1)$ or $(x_1, y_1), (x_2, y_2)$. In the former case, the degree $d_N(x_i) = 4$ and $d_N(y_j) \geq 3$ imply that $\{x_1, x_2, y_2, y_3\} \cup$

$\{b_1, b_2\}$ induce $K_{2,4}$ in N which yields a $[+K_6]$-subgraph in $(K_{10} - v, \sigma)$, a desired result. In the latter case $d_N(y_3) \geq 3$ implies that $(y_3, b_1), (y_3, b_2) \in E(N)$. These edges along with the fact that N has no induced $K_2 + K_2$ imply that $(x_i, b_j) \in E(N)$ for each $i, j \in \{1, 2\}$. Now $d_N(y_j) \geq 3$ and $d_N(b_j) \leq 4$ imply without loss of generality that $(b_1, y_1) \in E(N)$; however, this creates a triangle, a contradiction. □

Kim [2] proved that $r(3, t)$ has asymptotic order of magnitude $\frac{t^2}{\log(t)}$. In contrast to this Theorem 2.8 proves that $r_{\pm}(4, t)$ has linear asymptotic order of magnitude. This is expected because the number of signed graphs in $[+K_t]$ is linear in t rather than being a constant.

Theorem 2.8. $3\left\lfloor \frac{t}{2} \right\rfloor \leq r_{\pm}(4, t+1) \leq 3t - 1$

Proof. To prove the lower bound, consider the signed complete graph on $3\left\lfloor \frac{t}{2} \right\rfloor$ vertices which are partitioned into three blocks of order $\left\lfloor \frac{t}{2} \right\rfloor$ vertices in each. Make each edge within a block positive and edges between blocks negative. Now any 4 vertices chosen must contain some two vertices from the same block and so form a positive triangle with some third vertex of the 4. Thus there is no $[-K_4]$-subgraph. Also any $t + 1$ vertices must contain a vertex from each block. Any three such vertices form a negative triangle. Thus there is no $[+K_{t+1}]$-subgraph.

For the upper bound, consider any signed complete graph (K_{n+1}, σ) with $n = 3t - 2$. Normalize at v and consider the signed complete graph $(K_{n+1} - v, \sigma)$ on n vertices. If we assume that (K_{n+1}, σ) has no $[-K_4]$-subgraph, then the negative subgraph $N(K_{n+1} - v, \sigma)$ has no triangle and no induced $K_2 + K_2$. Wagon [5] proved that a graph without an induced $K_2 + K_2$ has chromatic number at most $\binom{\omega+1}{2}$ in which ω is the clique number. This makes the chromatic number of $N(K_{n+1} - v, \sigma)$ at most 3. The pigeonhole principle now guarantees an independent set of size at least t in $N(K_{n+1} - v, \sigma)$ which guarantees an independent set in $N(K_{n+1}, \sigma)$ of size $t + 1$ which implies the existence of a $+K_{t+1}$-subgraph. □

Statements and Declarations

The authors state that there are no financial or non-financial interests that are directly or indirectly related to this work. There is no external data which is directly relevant to this work.

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