REPRESENTATIONS OF SIGNED GRAPHS

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Abstract. We extend the concept of graph representations modulo integers introduced by Erdös and Evans to graph representations over finite rings and generalize it to representations of signed graphs. We introduce several representation numbers and product dimensions of graphs and signed graphs and compute these quantities for a few special classes of signed graphs.

Key words and phrases. modular number representation; product dimension; pseudo-modular number representation; representation number; signed graph.

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1. Introduction

A graph Γ consists of a finite vertex set $V(\Gamma)$ together with an edge set $E(\Gamma)$, which is a family of two element subsets of $V(\Gamma)$. The complete graph on n vertices is denoted by K_n and edge complement graph of Γ is denoted by Γ^c. Given a group A, a gain graph $G\Gamma$ with gains in A consists of a finite vertex set $V(G\Gamma)$ together with an edge set

$$E(G\Gamma) \subseteq \{(v_1, a, v_2) \in V(G\Gamma) \times A \times V(G\Gamma) \mid v_1 \neq v_2\}/\sim,$$

where $\sim$ is the equivalence relation $(v_1, a, v_2) \sim (v_2, a^{-1}, v_1)$ for all $v_1 \neq v_2$ in $V(G\Gamma)$ and all $a \in A$. From the definition, we can see that in the edge set $E(G\Gamma)$, edges are elements $(v_1, a, v_2) = (v_2, a^{-1}, v_1)$. The middle coordinate a in edge (v_1, a, v_2) is called the gain of the edge from v_1 to v_2, and if the edge has a gain a from v_1 to v_2, then it has a gain a^{-1} from v_2 to v_1. Note that our definition of gain graphs can be considered as that of simple gain graphs since we did not allow loops or multi-edges. A gain graph $G\Gamma$ with gain group A is said to be complete if

$$E(G\Gamma) = \{(v_1, a, v_2) \in V(G\Gamma) \times A \times V(G\Gamma) \mid v_1 \neq v_2\}/\sim .$$

The complete gain graph on n vertices with gain group A will be denoted by GK_n^A, and it is closely related to Dowling geometry. The edge complement of a gain graph $G\Gamma$ is written $G\Gamma^c$. When the group satisfies $a^{-1} = a$ for all $a \in A$, i.e. the group A is an elementary abelian 2-group, the gain of an edge in $E(G\Gamma)$ does not depend on the orientation of the edge, and hence the gain graph with gains in A is simply a multi-graph with an A-edge labelling. A gain graph is called a signed graph if $|A| = 2$. We use $S\Gamma$ to denoted a signed graph and edges with trivial gain in $S\Gamma$ are called positive edges while edges with non-trivial gain in $S\Gamma$ are called negative edges. The edge complement of $S\Gamma$ is written $S\Gamma^c$. Given a signed graph $S\Gamma$, the positive graph of $S\Gamma$, denoted by $(S\Gamma)^+$, is the simple graph whose vertex set is that of $S\Gamma$ and whose edge set is the the set of positive edges of $S\Gamma$. The negative graph $(S\Gamma)^-$ of $S\Gamma$ is similarly defined. Given a simple graph Γ, the positive signed graph of Γ, denoted by $S(\Gamma^+)$, is the signed graph whose vertex set is that of Γ, whose positive edge set is the edge set of Γ and whose negative edge set is empty. Similarly, we denote by $S(\Gamma^-)$ the signed graph whose vertex set is that of Γ, whose negative edge set is the edge set of Γ and whose positive edge set is empty. We will use $S(\Gamma^\pm)$ for the signed graph whose vertex set is that of Γ and $(S(\Gamma^\pm))^+ = (S(\Gamma^\pm))^- = \Gamma$. Clearly $S(K_n^\pm)$ is the complete signed graph on n vertices.

In this paper we extend the concept of modular number representations of graphs introduced by Erdös and Evans in [6] to that of signed graphs and compute various representation numbers for some special graphs and signed graphs. Along the way we also define product dimension of a signed graph and determine the values of product dimension of a few signed graphs. Representation number and product dimension of signed graphs are defined in Section 2 and some computations are carried out in Section 5 and 6. The reader may find similarities between our results and those in [1, 2, 3, 4, 7, 8, 9, 10, 16, 17].

2. Preliminaries

Given graphs $\Gamma_1, \Gamma_2, \cdots, \Gamma_k$ with a common vertex set V, that is $V(\Gamma_i) = V$ for all $i = 1, 2, \cdots, k$, we define their edge-union graph, denoted by $\bigcup_{i=1}^k \Gamma_i$, to be the graph with

$$V\left(\bigcup_{i=1}^k \Gamma_i\right) = V \text{ and } E\left(\bigcup_{i=1}^k \Gamma_i\right) = \bigcup_{i=1}^k E(\Gamma_i)$$

and their edge-intersection graph, denoted by $\bigcap_{i=1}^k \Gamma_i$, to be the graph with

$$V\left(\bigcap_{i=1}^k \Gamma_i\right) = V \text{ and } E\left(\bigcap_{i=1}^k \Gamma_i\right) = \bigcap_{i=1}^k E(\Gamma_i).$$

Edge-union and edge-intersection of a family of gain graphs with a common vertex set can be similarly defined.

Let V be a finite set, R be a ring and $\theta \in R$. For each map $f : V \longrightarrow R$ we define a graph $\Gamma(f, R, \theta)$ with

$V(\Gamma(f, R, \theta)) = V$;

$E(\Gamma(f, R, \theta)) = \{\{v_1, v_2\} \subseteq V \mid v_1 \neq v_2, f(v_1) - \theta f(v_2) \text{ and } f(v_2) - \theta f(v_1) \text{ are units in } R\}$.

If $S \subseteq R$ is a subset of R and $f : V \longrightarrow R$ is a map, we define $\Gamma(f, R, S)$ to be the graph

$$\Gamma(f, R, S) = \bigcup_{\theta \in S} \Gamma(f, R, \theta).$$

When $V = R$, f is the identity map and $\theta = 1$, the graph $\Gamma(f, R, \theta)$ is the Cayley graph $\mathrm{Cay}(R, R^*)$ of the additive group of R with the generating set R^*, the set of all units of R, and these graphs are called unitary Cayley graphs in [5]. So for any bijection f, the graph $\Gamma(f, R, 1)$ will be denoted by $\mathrm{Cay}^+(R, R^*)$ and $\Gamma(f, R, -1)$ will be denoted by $\mathrm{Cay}^-(R, R^*)$. Note that this is merely a notation and $\mathrm{Cay}^-(R, R^*)$ is not a Cayley graph of any kind.

When θ is a unit, we define a gain graph $G\Gamma(f, R, \theta)$ with

$V(G\Gamma(f, R, \theta)) = V$;

$E(G\Gamma(f, R, \theta)) = \{(v_1, \theta, v_2) \in V \times \langle\theta\rangle \times V/\sim \ \mid v_1 \neq v_2, f(v_1) - \theta f(v_2) \text{ is a unit in } R\}$.

If A is a subgroup of R^* and $f : V \longrightarrow R$ is a map, we define a gain graph $G\Gamma(f, R, A)$ to be

$$G\Gamma(f, R, A) = \bigcup_{g \in A} G\Gamma(f, R, g).$$

When $V = R$, f is the identity map and $A = \{\pm 1\}$, the signed graph $S\Gamma(f, R, \{\pm 1\}) = G\Gamma(f, R, \{\pm 1\})$ will be denoted by $S\mathrm{Cay}(R, R^*)$. Clearly, $(S\mathrm{Cay}(R, R^*))^+ = \mathrm{Cay}^+(R, R^*)$ and $(S\mathrm{Cay}(R, R^*))^- = \mathrm{Cay}^-(R, R^*)$.

We now present the definition of graph representation given by Erdös and Evans [6] in a more general context and extend their definition to gain graphs.

Definition 2.1. Given a graph Γ, a ring R and an element $\theta \in R$, a map $f : V(\Gamma) \longrightarrow R$ is called an (R, θ)-pseudo-representation of Γ if $\Gamma = \Gamma(f, R, \theta)$. An (R, θ)-pseudo-representation is called an (R, θ)-representation if the map $f : V(\Gamma) \longrightarrow R$ is injective.

Definition 2.2. Given a gain graph $G\Gamma$ with a gain group A, a ring R and a faithful linear representation φ of the group A over R, i.e. $\varphi : A \longrightarrow R^*$ is an injective group homomorphism, a map $f : V(G\Gamma) \longrightarrow R$ is called an $(R, \varphi(A))$-pseudo-representation of $G\Gamma$ if $G\Gamma = G\Gamma(f, R, \varphi(A))$, that is $(v_1, a, v_2) \in E(G\Gamma)$ if and only if $(v_1, \varphi(a), v_2) \in E(G\Gamma(f, R, \varphi(A)))$ for all $v_1, v_2 \in V(G\Gamma)$ and all $a \in A$. An $(R, \varphi(A))$-pseudo-representation is called an $(R, \varphi(A))$-representation if the map $f : V(G\Gamma) \longrightarrow R$ is injective.

The following propositions are immediate consequences of Definition 2.1- 2.2 and simple algebra.

Proposition 2.3. *Let $\Gamma_1, \Gamma_2, \cdots, \Gamma_k$ be graphs with a common vertex set V, $R_1, R_2, \cdots, R_k$ be rings, $\theta_1 \in R_1, \theta_2 \in R_2, \cdots, \theta_k \in R_k$, and $f_1 : V \longrightarrow R_1$, $f_2 : V \longrightarrow R_2, \cdots, f_k : V \longrightarrow R_k$ be maps. Let $\Gamma = \Gamma_1 \cap \Gamma_2 \cap \cdots \cap \Gamma_k$, $R = R_1 \times R_2 \times \cdots \times R_k$, $\theta = (\theta_1, \theta_2, \cdots, \theta_k) \in R$ and $f : V \longrightarrow R$ be the map given by $f(v) = (f_1(v), f_2(v), \cdots, f_k(v))$ for all $v \in V$. Then*

$$\Gamma(f, R, \theta) = \bigcap_{i=1}^{k} \Gamma(f_i, R_i, \theta_i),$$

and therefore maps $f_i : V \longrightarrow R_i$ are (R_i, θ_i)-pseudo-representations of Γ_i for all $i = 1, 2, \cdots, k$ if and only if the map $f : V \longrightarrow R$ is an (R, θ)-pseudo-representation of Γ. When the maps

f_i are (R_i,θ_i)-pseudo-representation of Γ_i for all $i = 1,2,\cdots,k$, the map f is an (R,θ)-representation of Γ if one of f_i is an (R_i,θ_i)-representation of Γ_i for some $1 \leqslant i \leqslant k$.

Proposition 2.4. *Let $G\Gamma_1$, $G\Gamma_2$, $\cdots$, $G\Gamma_k$ be gain graphs with a common gain group A and a common vertex set V, R_1, R_2, $\cdots$, R_k be rings, $\varphi_1 : A \longrightarrow R_1^*$, $\varphi_2 : A \longrightarrow R_2^*$, $\cdots$, $\varphi_k : A \longrightarrow R_k^*$ be injective group homomorphisms, and $f_1 : V \longrightarrow R_1$, $f_2 : V \longrightarrow R_2$, $\cdots$, $f_k : V \longrightarrow R_k$ be maps. Let $G\Gamma = G\Gamma_1 \cap G\Gamma_2 \cap \cdots \cap G\Gamma_k$, $R = R_1 \times R_2 \times \cdots \times R_k$, $\varphi = (\varphi_1, \varphi_2, \cdots, \varphi_k) : A \longrightarrow R$ and $f : V \longrightarrow R$ be the map given by $f(v) = (f_1(v), f_2(v), \cdots, f_k(v))$ for all $v \in V$. Then*

$$G\Gamma(f,R,\varphi(A)) = \bigcap_{i=1}^{k} G\Gamma(f_i,R_i,\varphi_i(A)),$$

and therefore maps $f_i : V \longrightarrow R_i$ are $(R_i,\varphi_i(A))$-pseudo-representations of $G\Gamma_i$ for all $i = 1,2,\cdots,k$ if and only if the map $f : V \longrightarrow R$ is an $(R,\varphi(A))$-pseudo-representation of $G\Gamma$. When f_i are $(R_i,\varphi_i(A))$-pseudo-representation of $G\Gamma_i$ for all $i = 1,2,\cdots,k$, the map f is an $(R,\varphi(A))$-representation if one of f_i is an $(R_i,\varphi_i(A))$-representation of $G\Gamma_i$ for some $1 \leqslant i \leqslant k$.

In this paper, we are mainly interested in $(\mathbb{Z}_n, 1)$- and $(\mathbb{Z}_n, -1)$-representations of graphs and $(\mathbb{Z}_n, \{\pm 1\})$-representations of signed graphs, where $\mathbb{Z}_n$ is the modular number ring for various integers $n > 1$. The following Chinese remainder theorem will be frequently used.

Theorem 2.5 (Chinese remainder theorem). *Let $m_1 > 1$, $m_2 > 1$, $\cdots$, $m_k > 1$ be pairwise relatively prime integers and let $m = m_1 m_2 \cdots m_k$. Then*

$$\mathbb{Z}_m \cong \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_k}.$$

We now give some examples of representations and pseudo-representations of graphs and signed graphs.

Example 2.6. Let n be a positive integer and R be a division ring such that $|R| \geqslant n$. Then every injective map $f : V(K_n) \longrightarrow R$ is an $(R,1)$-representation of K_n. In particular, if $p \geqslant n$ is a prime, then K_n has $(\mathbb{Z}_p, 1)$-representations.

Example 2.7. Let n be a positive integer and R be a division ring such that $|R| \geqslant n-1$. Let v and v' be two distinct vertices of the complete graph K_n and K' be the graph obtained from K_n by deleting the edge $\{v,v'\}$. Then every map $f : V(K') \longrightarrow R$ satisfying the conditions that $f(v) = f(v')$ and the restriction map $f : V(K') \setminus \{v'\} \longrightarrow R$ is injective is an $(R,1)$-pseudo-representation of K'. In particular, if $p \geqslant n-1$ is a prime, then K' has $(\mathbb{Z}_p, 1)$-pseudo-representations.

Example 2.8. Let n be a positive integer and R be a division ring. Let $\widetilde{R}$ be a subset of R such that $|\widetilde{R} \cap \{x,-x\}| = 1$ for all $x \in R$. Then every injective map $f : V(K_n) \longrightarrow \widetilde{R}$ is an $(R,-1)$-representation of K_n. In particular, if $p \geqslant 2n-1$ is a prime, then K_n has $(\mathbb{Z}_p, -1)$-representations.

Example 2.9. Let n be a positive integer and R be a division ring. Let $\widetilde{R}$ be a subset of R such that $|\widetilde{R} \cap \{x,-x\}| = 1$ for all $x \in R$. Let v and v' be two vertices of the complete graph K_n and K' be the graph obtained from K_n by deleting the edge $\{v,v'\}$. Then every map $f : V(K') \longrightarrow R$ satisfying the conditions that $f(v) + f(v') = 0$ and the image of the restriction map $f : V(K') \setminus \{v,v'\} \longrightarrow R$ is contained in $\widetilde{R} \setminus \{f(v), f(v')\}$ is an $(R,-1)$-pseudo-representation of K'. In particular, if p is an odd prime, then K' has $(\mathbb{Z}_p, -1)$-pseudo-representations.

Example 2.10. The representations given in Example 2.8 are also $(R, \{\pm 1\})$-representations of $S(K_n^{\pm})$. In particular, if $p \geqslant 2n-1$ is a prime, then $S(K_n^{\pm})$ has $(\mathbb{Z}_p, \{\pm 1\})$-representations. If the map f in Example 2.9 is injective, then the representation of K' is also a $(R, \{\pm 1\})$-representation of $S(K_n^{\pm}) \setminus (v,-1,v')$. In particular, if $p \geqslant 2n-3$ is an odd prime, then the signed graph obtained from the complete signed graph $S(K_n^{\pm})$ by deleting a negative edge has $(\mathbb{Z}_p, \{\pm 1\})$-representations. In Example 2.7, let $\widetilde{R}$ be as defined in Example 2.8. Then the pseudo-$(R,1)$-representation f of K' is a pseudo-$(R,\{\pm 1\})$-representation of $S(K_n^{\pm}) \setminus (v,1,v')$ if $f(v) = f(v') \neq 0$ and the image of the injective restriction map $f : V(S(K_n^{\pm})) \setminus \{v, v'\} \longrightarrow R$ is contained in $\widetilde{R} \setminus \{f(v), -f(v)\}$. In particular, if $p \geqslant 2n-3$ is an odd prime, then the signed graph obtained from the complete signed graph $S(K_n^{\pm})$ by deleting a positive edge has pseudo-$(\mathbb{Z}_p, \{\pm 1\})$-representations.

The following proposition is a generalization of the main theorem in [6].

Proposition 2.11. *Every graph has $(\mathbb{Z}_l, 1)$-representations and $(\mathbb{Z}_m, -1)$-representations for some positive integers $l > 1$ and $m > 1$. Every signed graph has $(\mathbb{Z}_n, \{\pm 1\})$-representations for some positive integer $n > 1$. Furthermore, the integers l, m and n may be chosen to be square-free, i.e. they are not divisible by the square of any primes.*

Proof. We will only give a proof for signed graphs and other cases can be similarly proved. Let $S\Gamma$ be a signed graph with v vertices and $V = V(S\Gamma)$. Let $S\Gamma^c = S(K_v^{\pm}) \setminus S\Gamma$ be its complement signed graph. For each edge $e \in E(S\Gamma^c)$, select a prime $p_e \geqslant 2v-3$ such that $p_e \neq p_{e'}$ when $e \neq e'$. By Example 2.10, we can find a $(\mathbb{Z}_{p_e}, \{\pm 1\})$-pseudo representation $f_e : V \longrightarrow \mathbb{Z}_{p_e}$ of $S(K_v^{\pm}) \setminus \{e\}$ for each $e \in E(S\Gamma^c)$. Finally, select a prime $p_0 \geqslant 2v-1$ that is not in $\{p_e | e \in E(S\Gamma^c)\}$. Again by Example 2.10, we can find a $(\mathbb{Z}_{p_0}, \{\pm 1\})$-representation $f_0 : V \longrightarrow \mathbb{Z}_{p_0}$ of the complete signed graph $S(K_v^{\pm})$. Let

$$n = p_0 \prod_{e \in E(S\Gamma^c)} p_e$$

and $f : V \longrightarrow \mathbb{Z}_n$ be the map defined in Proposition 2.4. By Theorem 2.5 and Proposition 2.4, the signed graph $S\Gamma = G\Gamma(f, \mathbb{Z}_n, \{\pm 1\})$ and $f : V \longrightarrow \mathbb{Z}_n$ is a $(\mathbb{Z}_n, \{\pm 1\})$-representation of $S\Gamma$. It is clear that n is square-free. □

Proposition 2.11 guarantees the existence of $(\mathbb{Z}_l, 1)$- and $(\mathbb{Z}_m, -1)$-representations for each graph Γ and $(\mathbb{Z}_n, \{\pm 1\})$-representations for each signed graph $S\Gamma$.

Definition 2.12. Let Γ be a graph and $S\Gamma$ be a signed graph. The least positive integer l for which the graph Γ has a $(\mathbb{Z}_l, 1)$-representation will be called the *plus representation number* of Γ, written $\mathrm{Rep}^+(\Gamma)$, while the least positive integer m for which the graph Γ has a $(\mathbb{Z}_m, -1)$-representation will be called the *minus representation number* of Γ, written $\mathrm{Rep}^-(\Gamma)$. The least positive integer n for which the signed graph $S\Gamma$ has a $(\mathbb{Z}_n, \{\pm 1\})$-representation will be called the *representation number* of $S\Gamma$, written $\mathrm{Rep}(S\Gamma)$.

Proposition 2.11 further guarantees the existence of $(\mathbb{Z}_l, 1)$- and $(\mathbb{Z}_m, -1)$-representations for each graph Γ and $(\mathbb{Z}_m, \{\pm 1\})$-representations for each signed graph $S\Gamma$ with l, m and n being square-free integers. As is pointed out in [9], the least number of prime factors needed for a square-free $(\mathbb{Z}_l, 1)$-representation of a graph Γ is studied in [14] and [18], and is called product dimension of Γ, written $\mathrm{pdim}(\Gamma)$. We extend the term as follows.

Definition 2.13. Let Γ be a graph and $S\Gamma$ be a signed graph. The least number of prime factors needed of a square-free integer l for which the graph Γ has a $(\mathbb{Z}_l, 1)$-representation will be called *the plus product dimension of* Γ, written $\mathrm{p}^+\mathrm{dim}(\Gamma)$, while the least number of prime factors needed of a square-free integer m for which the graph Γ has a $(\mathbb{Z}_m, -1)$-representation will be called *the minus product dimension of* Γ, written $\mathrm{p}^-\mathrm{dim}(\Gamma)$. The least number of prime factors needed of a square-free integer n for which the signed graph $S\Gamma$ has a $(\mathbb{Z}_n, \{\pm 1\})$-representation will be called *the product dimension of* $S\Gamma$, written $\mathrm{pdim}(S\Gamma)$.

We will compute representation numbers and product dimensions of various graphs and signed graphs in subsequent sections. We end this section with definition of a sequence of prime number valued functions that we will use in subsequent sections.

Definition 2.14. For each real number x, we define

$$p_0(x) = \text{ the smallest prime number } p \geqslant x.$$

For each integer $i > 0$ and each real number x, we recursively define

$$p_i(x) = \text{ the smallest prime number } p > p_{i-1}(x).$$

3. Representations and pseudo representations as graph morphisms and homomorphisms

We refer the reader to [15] for basics of category theory. In this section we take a brief look at representations and pseudo representations introduced in Section 2 in categorical settings to facilitate a better understanding of these maps. Given two graphs Γ_1 and Γ_2, a graph morphism $f\colon \Gamma_1 \longrightarrow \Gamma_2$ from Γ_1 to Γ_2 is a map $f\colon V(\Gamma_1) \longrightarrow V(\Gamma_2)$ such that for every edge $\{u, v\} \in E(\Gamma_1)$, either $f(u) = f(v)$ or $\{f(u), f(v)\} \in E(\Gamma_2)$. Finite simple graphs together with graph morphisms form a category which we denote by **GM**, the category of graphs with graph morphisms. The object class Obj(**GM**) of **GM** consists of all finite simple graphs and the morphism class Mor(**GM**) of **GM** consists of all graph morphisms. The category **GM** has a categorical product $\boxtimes$, commonly known as strong product or normal product, given by

$$\begin{aligned}
&V(\Gamma_1 \boxtimes \Gamma_2) = V(\Gamma_1) \times V(\Gamma_2) \text{ and} \\
&E(\Gamma_1 \boxtimes \Gamma_2) = \{\{(u_1, u_2), (v_1, v_2)\} \subseteq V(\Gamma_1) \times V(\Gamma_2) \mid u_1 = v_1 \text{ and } \{u_2, v_2\} \in E(\Gamma_2), \\
&\qquad \text{or } \{u_1, v_1\} \in E(\Gamma_1) \text{ and } u_2 = v_2, \text{ or } \{u_1, v_1\} \in E(\Gamma_1) \text{ and } \{u_2, v_2\} \in E(\Gamma_2)\}
\end{aligned}$$

for any two graphs Γ_1 and Γ_2. A graph morphism $f\colon \Gamma_1 \longrightarrow \Gamma_2$ in Mor(**GM**) is called a graph homomorphism [19] if $\{f(u), f(v)\} \in E(\Gamma_2)$ for all edges $\{u, v\} \in E(\Gamma_1)$. Finite simple graphs together with graph homomorphisms form a category [13] which we denote by **GH**, the category of graphs with graph homomorphisms. The object class Obj(**GH**) of **GH** consists of all finite simple graphs and the morphism class Mor(**GH**) of **GH** consists of all graph homomorphisms. The category **GH** is a subcategory of **GM** and it has its own categorical product $\times$, known as weak product or tensor product, given by

$$\begin{aligned}
&V(\Gamma_1 \times \Gamma_2) = V(\Gamma_1) \times V(\Gamma_2) \text{ and} \\
&E(\Gamma_1 \times \Gamma_2) = \{\{(u_1, u_2), (v_1, v_2)\} \subseteq V(\Gamma) \mid \{u_1, v_1\} \in E(\Gamma_1), \{u_2, v_2\} \in E(\Gamma_2)\}
\end{aligned}$$

for any two graphs Γ_1 and Γ_2. The natural projection maps $\pi_i\colon V(\Gamma_1) \times V(\Gamma_2) \longrightarrow V(\Gamma_i)$ given by $\pi_i(v_1, v_2) = v_i$, $i = 1, 2$, are graph morphisms for both products and they are also graph homomorphisms for tensor product.

Let **FR** be the category of finite rings and ring homomorphisms. The object class Obj(**FR**) of **FR** consists of all finite rings with identity and the morphism class Mor(**FR**) of **FR** consists of all ring homomorphisms. The category **FR** has a categorical product, the usual Cartesian product of rings. For each ring $R \in$ Obj(**FR**), we introduced two simple graphs $\mathrm{Cay}^+(R, R^*)$ and $\mathrm{Cay}^-(R, R^*)$ in Section 2. We further introduce graph morphisms from ring homomorphisms. In fact, for every pair of finite rings R_1 and R_2 in Obj(**FR**) and every ring homomorphism $f\colon R_1 \longrightarrow R_2$, the homomorphism f induces maps

$$f^+\colon V(\mathrm{Cay}^+(R_1, R_1^*)) \longrightarrow V(\mathrm{Cay}^+(R_2, R_2^*)) \text{ and } f^-\colon V(\mathrm{Cay}^-(R_1, R_1^*)) \longrightarrow V(\mathrm{Cay}^-(R_2, R_2^*)).$$

The map f^+ is a graph homomorphism in $\mathrm{Mor}(\mathbf{GH})$ and f^- is a graph morphism in $\mathrm{Mor}(\mathbf{GM})$. Therefore the map

$$\begin{aligned}&F^+\colon \mathbf{FR} \longrightarrow \mathbf{GH}, \text{ given by}\\ &F^+(R) = \mathrm{Cay}^+(R, R^*) \text{ for each ring } R \in \mathrm{Obj}(\mathbf{FR})\\ &F^+(f) = f^+ \text{ for each ring homomorphism } f \in \mathrm{Mor}(\mathbf{FR})\end{aligned}$$

is a functor, and the map

$$\begin{aligned}&F^-\colon \mathbf{FR} \longrightarrow \mathbf{GM}, \text{ given by}\\ &F^-(R) = \mathrm{Cay}^-(R, R^*) \text{ for each ring } R \in \mathrm{Obj}(\mathbf{FR})\\ &F^-(f) = f^- \text{ for each ring homomorphism } f \in \mathrm{Mor}(\mathbf{FR})\end{aligned}$$

is also a functor. It is pointed out in [5] that the functor $F^+\colon \mathbf{FR} \longrightarrow \mathbf{GH}$ preserves categorical products, that is, for any two rings $R_1, R_2 \in \mathrm{Obj}(\mathbf{FR})$, we have

$$\begin{aligned}F^+(R_1 \times R_2) =&\mathrm{Cay}^+(R_1 \times R_2, (R \times R_2)^*) = \mathrm{Cay}^+(R_1 \times R_2, R_1^* \times R_2^*)\\ =&\mathrm{Cay}^+(R_1, R_1^*) \times \mathrm{Cay}^+(R_2, R_2^*) = F^+(R_1) \times F^+(R_2)\end{aligned}$$

and F^+ takes projection ring homomorphisms to projection graph homomorphisms. However the functor $F^-\colon \mathbf{FR} \longrightarrow \mathbf{GM}$ does not preserve categorical products.

It was noted in [8, 9] that for any graph Γ and finite ring R, a map $f\colon V(\Gamma) \longrightarrow R$ is an $(R, 1)$-representation if and only if f is an embedding of Γ into the graph $\mathrm{Cay}^+(R, R^*)$ as an induced subgraph. This is also true for $(R, -1)$-representations of Γ, that is, f is an $(R, -1)$-representation of Γ if and only if f is an embedding of Γ into the graph $\mathrm{Cay}^-(R, R^*)$ as an induced subgraph. The following proposition characterizes $(R, 1)$-representations and $(R, -1)$-representations of a graph Γ.

Proposition 3.1. *Let Γ be a graph and R be a finite ring. A map $f\colon V(\Gamma) \longrightarrow R$ is an $(R, 1)$-representation if and only if both $f\colon \Gamma \longrightarrow \mathrm{Cay}^+(R, R^*)$ and $f\colon \Gamma^c \longrightarrow \mathrm{Cay}^+(R, R^*)^c$ are graph homomorphisms, while $f\colon V(\Gamma) \longrightarrow R$ is an $(R, -1)$-representation if and only if both $f\colon \Gamma \longrightarrow \mathrm{Cay}^-(R, R^*)$ and $f\colon \Gamma^c \longrightarrow \mathrm{Cay}^-(R, R^*)^c$ are graph homomorphisms.*

The map $f\colon V(\Gamma) \longrightarrow R$ is an $(R, 1)$-pseudo representation if and only if $f\colon \Gamma \longrightarrow \mathrm{Cay}^+(R, R^*)$ is a graph homomorphism whose image subgraph $f(\Gamma)$ in $\mathrm{Cay}^+(R, R^*)$ is an induced subgraph, and if the map $f\colon V(\Gamma) \longrightarrow R$ is an $(R, -1)$-pseudo representation, then $f\colon \Gamma \longrightarrow \mathrm{Cay}^-(R, R^*)$ is a graph morphism whose image subgraph $f(\Gamma)$ in $\mathrm{Cay}^-(R, R^*)$ is an induced subgraph. The following proposition describes $(R, 1)$-pseudo representations and $(R, -1)$-pseudo representations of a graph Γ.

Proposition 3.2. *Let Γ be a graph and R be a finite ring. A map $f\colon V(\Gamma) \longrightarrow R$ is an $(R, 1)$-pseudo representation if and only if $f\colon \Gamma \longrightarrow \mathrm{Cay}^+(R, R^*)$ is a graph homomorphism and $f\colon \Gamma^c \longrightarrow \mathrm{Cay}^+(R, R^*)^c$ is a graph morphism. If $f\colon V(\Gamma) \longrightarrow R$ is an $(R, -1)$-pseudo representation then both $f\colon \Gamma \longrightarrow \mathrm{Cay}^-(R, R^*)$ and $f\colon \Gamma^c \longrightarrow \mathrm{Cay}^-(R, R^*)^c$ are graph morphisms.*

Given two signed graphs $S\Gamma_1$ and $S\Gamma_2$, a signed graph morphism $f\colon S\Gamma_1 \longrightarrow S\Gamma_2$ is a map $f\colon V(S\Gamma_1) \longrightarrow V(S\Gamma_2)$ such that $f\colon (S\Gamma_1)^+ \longrightarrow (S\Gamma_2)^+$ and $f\colon (S\Gamma_1)^- \longrightarrow (S\Gamma_2)^-$ are graph morphisms, and a signed graph morphism $f\colon S\Gamma_1 \longrightarrow S\Gamma_2$ is called a graph homomorphism if $f\colon (S\Gamma_1)^+ \longrightarrow (S\Gamma_2)^+$ and $f\colon (S\Gamma_1)^- \longrightarrow (S\Gamma_2)^-$ are graph homomorphisms.

Finite simple signed graphs together with signed graph morphisms form a category which we denote by $\mathbf{SGM}$, the category of signed graphs with signed graph morphisms. The object class $\mathrm{Obj}(\mathbf{SGM})$ of $\mathbf{SGM}$ consists of all finite simple signed graphs and the morphism class $\mathrm{Mor}(\mathbf{SGM})$ of $\mathbf{SGM}$ consists of all signed graph morphisms. The category $\mathbf{SGM}$ has a categorical product $\boxtimes$, given by $(S\Gamma_1 \boxtimes S\Gamma_2)^+ = (S\Gamma_1)^+ \boxtimes (S\Gamma_2)^+$ and $(S\Gamma_1 \boxtimes S\Gamma_2)^- = (S\Gamma_1)^- \boxtimes (S\Gamma_2)^-$. Finite simple signed graphs together with signed graph homomorphisms form a category which we denote by **SGH**, the category of signed graphs with signed graph homomorphisms. The object class Obj(**SGH**) of **SGH** consists of all finite simple signed graphs and the morphism class Mor(**SGH**) of **SGH** consists of all signed graph homomorphisms. The category **SGH** has a categorical product $\times$, given by $(S\Gamma_1 \times S\Gamma_2)^+ = (S\Gamma_1)^+ \times (S\Gamma_2)^+$ and $(S\Gamma_1 \times S\Gamma_2)^- = (S\Gamma_1)^- \times (S\Gamma_2)^-$.

For each finite ring R and each signed graph $S\Gamma$, an $(R, \{\pm 1\})$-representation of the signed graph $S\Gamma$ is an embedding of $S\Gamma$ into the signed graph $S\mathrm{Cay}(R, R^*)$ as an induced signed subgraph. If $f\colon V(S\Gamma) \longrightarrow R$ is an $(R, \{\pm 1\})$-pseudo representation of Γ, then $f\colon S\Gamma \longrightarrow R$ is an $S\mathrm{Cay}(R, R^*)$ is a signed graph morphism whose image is an induced signed subgraph in $S\mathrm{Cay}(R, R^*)$.

4. Basic results

In this section, we prove some basic results that are needed in next two sections. We first generalize Theorem 1.2 in [8] from $(\mathbb{Z}_l, 1)$-representations to $(\mathbb{Z}_l, 1)$-pseudo representations.

Lemma 4.1. *Let Γ be a graph and p be a prime. Then the following statements are equivalent.*

1. *The graph Γ has a $(\mathbb{Z}_l, 1)$-representation for some positive integer l divisible by p;*
2. *The graph Γ has a $(\mathbb{Z}_l, 1)$-pseudo representation for some positive integer l divisible by p;*
3. *The prime $p \geqslant \chi(\Gamma)$, where $\chi(\Gamma)$ is the chromatic number of Γ.*

Proof. (1) $\Rightarrow$ (2) is trivial. It is clear that every $(\mathbb{Z}_l, 1)$-pseudo representation $f\colon V(\Gamma) \longrightarrow \mathbb{Z}_l$ of Γ is also a $\mathbb{Z}_l$-vertex coloring of Γ because f is a graph homomorphism from Γ to $\mathrm{Cay}^+(\mathbb{Z}_l, \mathbb{Z}_l^*)$, and therefore $l \geqslant \chi(\Gamma)$. If p divides l, from discussions in Section 3, the natural projection ring homomorphism $\pi\colon \mathbb{Z}_l \longrightarrow \mathbb{Z}_p$ induces a graph homomorphism $\pi^+\colon V(\mathrm{Cay}^+(\mathbb{Z}_l, \mathbb{Z}_l^*)) \longrightarrow V(\mathrm{Cay}^+(\mathbb{Z}_p, \mathbb{Z}_p^*))$. Therefore $\pi^+ \circ f\colon V(\Gamma) \longrightarrow V(\mathrm{Cay}^+(\mathbb{Z}_p, \mathbb{Z}_p^*)) = \mathbb{Z}_p$ is a graph homomorphism and $p \geqslant \chi(\Gamma)$. Hence (2) $\Rightarrow$ (3). Finally we show that (3) $\Rightarrow$ (1). For every graph Γ, there is a graph homomorphism $c\colon \Gamma \longrightarrow K_{\chi(\Gamma)}$, a $\chi(\Gamma)$-coloring of Γ. Let $\widetilde{\Gamma}$ be the complete multipartite graph obtained from the graph Γ and the coloring c by adding edges to Γ between all vertices of different color classes of c. Clearly $V(\widetilde{\Gamma}) = V(\Gamma)$ and $c\colon \widetilde{\Gamma} \longrightarrow K_{\chi(\Gamma)}$ is also a graph homomorphism. If $p \geqslant \chi(\Gamma)$, then there is an embedding $i\colon K_{\chi(\Gamma)} \longrightarrow K_p = \mathrm{Cay}^+(\mathbb{Z}_p, \mathbb{Z}_p^*)$ and the composition map

$$f = i \circ c\colon \widetilde{\Gamma} \longrightarrow \mathrm{Cay}^+(\mathbb{Z}_p, \mathbb{Z}_p^*)$$

is a $(\mathbb{Z}_p, 1)$-pseudo representation of $\widetilde{\Gamma}$. Now we use Proposition 2.11 to construct a $(\mathbb{Z}_m, 1)$-representation g of Γ for some integer $m > 1$ with $\gcd(m, p) = 1$ and then use Proposition 2.3 to obtained a $(\mathbb{Z}_{pm}, 1)$-representation of Γ by combining the $(\mathbb{Z}_p, 1)$-pseudo representation f of $\widetilde{\Gamma}$ and $(\mathbb{Z}_m, 1)$-representation g of Γ. □

Lemma 4.2. *Let Γ be a graph. Then the following statements are equivalent:*

1. *the graph Γ has a $(\mathbb{Z}_l, -1)$-representation for some positive even integer l;*
2. *the graph Γ has a $(\mathbb{Z}_l, -1)$-pseudo representation for some positive even integer l;*
3. *the graph Γ is a bipartite graph.*

Proof. (1) $\Rightarrow$ (2) is again trivial. Let l be an even integer and $\pi\colon \mathbb{Z}_l \longrightarrow \mathbb{Z}_2$ be the natural projection ring homomorphism. Then for any $(\mathbb{Z}_l, -1)$-pseudo representation $f\colon V(\Gamma) \longrightarrow \mathbb{Z}_l$ of Γ and any cycle C in Γ of length k, say

$$V(C) = \{v_1, v_2, \cdots, v_k\} \subseteq V(\Gamma) \text{ and } \{v_1, v_2\}, \{v_2, v_3\}, \cdots, \{v_{k-1}, v_k\}, \{v_k, v_1\} \in E(\Gamma),$$

we must have

$$\begin{aligned}
&\pi(f(v_1)) + \pi(f(v_2)) = \pi(f(v_1) + f(v_2)) = 1 \\
&\pi(f(v_2)) + \pi(f(v_3)) = \pi(f(v_2) + f(v_3)) = 1 \\
&\cdots\cdots \\
&\pi(f(v_{k-1})) + \pi(f(v_k)) = \pi(f(v_{k-1}) + f(v_k)) = 1 \\
&\pi(f(v_k)) + \pi(f(v_1)) = \pi(f(v_k) + f(v_1)) = 1.
\end{aligned}$$

By adding all equations above together, we find that $k = 0$ in $\mathbb{Z}_2$, which means that k is even. Hence (2) $\Rightarrow$ (3). The proof for (3) $\Rightarrow$ (1) is similar to that of the corresponding part in Lemma 4.1 since every $(\mathbb{Z}_2, 1)$–pseudo representation is also a $(\mathbb{Z}_2, -1)$–pseudo representation. □

Lemma 4.3. *Let $S\Gamma$ be a signed graph. Then the following statements are equivalent:*

(1) *the signed graph $S\Gamma$ has a $(\mathbb{Z}_l, \{\pm 1\})$-representation for some positive even integer l;*
(2) *the signed graph $S\Gamma$ has a $(\mathbb{Z}_l, \{\pm 1\})$-pseudo representation for some positive even integer l;*
(3) *the graphs $(S\Gamma)^+$ and $(S\Gamma)^-$ are both bipartite graphs and they share at least one bipartition.*

The following lemma is a straightforward consequence of the definition of representations of graphs and signed graphs.

Lemma 4.4. *Let $S\Gamma$ be a signed graph and $n > 1$ be an integer. A map $f\colon V(S\Gamma) \longrightarrow \mathbb{Z}_n$ is a $(\mathbb{Z}_n, \{\pm 1\})$-representation of $S\Gamma$ if and only if it is a $(\mathbb{Z}_n, 1)$-representation of $(\mathrm{S}\Gamma)^+$ and a $(\mathbb{Z}_n, -1)$-representation of $(S\Gamma)^-$. Consequently*

$$\mathrm{Rep}(S\Gamma) \geqslant \max\{\mathrm{Rep}^+((S\Gamma)^+), \mathrm{Rep}^-((S\Gamma)^-)\}.$$

From the proof of Proposition 2.11, we have the following trivial bounds on various product dimensions of graphs and signed graphs.

Proposition 4.5. *Let Γ be a graph and $S\Gamma$ be a signed graph. Then $\mathrm{p}^+\mathrm{dim}(\Gamma) \leqslant |E(\Gamma^c)| + 1$, $\mathrm{p}^-\mathrm{dim}(\Gamma) \leqslant |E(\Gamma^c)| + 1$ and $\mathrm{pdim}(S\Gamma) \leqslant |E(S\Gamma^c)| + 1$. Moreover,*

(1) *the plus product dimension $\mathrm{p}^+\mathrm{dim}(\Gamma) = 1$ if and only if $\Gamma = K_n$ for some positive integer n;*
(2) *the minus product dimension $\mathrm{p}^-\mathrm{dim}(\Gamma) = 1$ if and only if $\Gamma = K_{r_1, r_2, \cdots, r_t}$, a complete multipartite graph with $r_i \leqslant 2$ for all $i = 1, 2, \cdots, t$, i.e. Γ is K_n minus a matching; and*
(3) *the product dimension $\mathrm{pdim}(S\Gamma) = 1$ if and only if $(S\Gamma)^+ = K_n$ and $(S\Gamma)^- = K_{r_1, r_2, \cdots, r_t}$ with $r_i \leqslant 2$ for all $i = 1, 2, \cdots, t$ and $r_1 + r_2 + \cdots + r_t = n$.*

Proof. The inequalities follow directly from the constructions of representation given in the proof of Proposition 2.11. Since for each prime p, the graph $\mathrm{Cay}^+(\mathbb{Z}_p, \mathbb{Z}_p^*) = K_p$, every induced subgraph of $\mathrm{Cay}^+(\mathbb{Z}_p, \mathbb{Z}_p^*)$ is a complete graph and every complete graph is an induced subgraph of $\mathrm{Cay}^+(\mathbb{Z}_p, \mathbb{Z}_p^*)$ for some prime p. This completes proof of (1). Also for each prime p, the graph $\mathrm{Cay}^-(\mathbb{Z}_p, \mathbb{Z}_p^*) = K_{1,2,2,\cdots,2}$ when p is odd and K_2 when $p = 2$, and every induced subgraph of $\mathrm{Cay}^-(\mathbb{Z}_p, \mathbb{Z}_p^*)$ is a complete multipartite graph $K_{r_1, r_2, \cdots, r_t}$ with $r_i \leqslant 2$ for all $i = 1, 2, \cdots, t$ and $r_1 + r_2 + \cdots + r_t \leqslant p$. Every such a complete multipartite graph $K_{r_1, r_2, \cdots, r_t}$ is an induced subgraph of $\mathrm{Cay}^-(\mathbb{Z}_p, \mathbb{Z}_p^*)$ for some prime p. Hence (2) follows. (3) follows from (1) and (2). □

Lemma 4.6. *Let $l > 1$ be an integer and $p > 1$ be the least prime dividing l. Let $n \geqslant 2$ be an integer.*

(1) *If K_n^c has a $(\mathbb{Z}_l, 1)$-representation, then $l \geqslant pn$.*

(2) *If K_n^c has a $(\mathbb{Z}_l, -1)$-representation, then either $l \geqslant 2n$ or $n = 2$ and $l = 3$.*

Proof. Suppose $l \leqslant pn - 1$ and $f\colon V(K_n^c) \longrightarrow \mathbb{Z}_l$ is a $(\mathbb{Z}_l, 1)$-representation of K_n^c. Let $V(K_n^c) = \{v_1, v_2, \cdots, v_n\}$. By pigeonhole principle, two of

$$f(v_1), f(v_2), \cdots, f(v_n), f(v_1)+1, f(v_2)+1, \cdots, f(v_n)+1, f(v_1)+2, f(v_2)+2, \cdots,$$
$$f(v_n)+2, \cdots, f(v_1)+(p-1), f(v_2)+(p-1), \cdots, f(v_n)+(p-1)$$

are the same, say $f(v_i)+u = f(v_j)+v$ with $v \neq u$. Since p is the least prime dividing l, the element $f(v_i)-f(v_j) = v-u$ is a unit in $\mathbb{Z}_l$. This contradicts the assumption that $f\colon V(K_n^c) \longrightarrow \mathbb{Z}_l$ is a $(\mathbb{Z}_l, 1)$-representation of K_n^c, and therefore $l \geqslant pn$ and (1) is proved.

Now suppose $l \leqslant 2n-1$ and $f\colon V(K_n^c) \longrightarrow \mathbb{Z}_l$ is a $(\mathbb{Z}_l, -1)$-representation of K_n^c. By pigeonhole principle, two of $f(v_1), f(v_2), \cdots, f(v_n), -f(v_1)+1, -f(v_2)+1, \cdots, -f(v_n)+1$ are the same, say $f(v_i) = -f(v_j)+1$. If $i \neq j$, then $f(v_i)+f(v_j) = 1$ is a unit in $\mathbb{Z}_l$, contradicting the assumption that $f\colon V(K_n^c) \longrightarrow \mathbb{Z}_l$ is a $(\mathbb{Z}_l, -1)$-representation of K_n^c. Hence $i = j$, and $2f(v_i) = 1$ and 2 is a unit in $\mathbb{Z}_l$, which implies that l is odd. Since $l \leqslant 2n-1$ and elements

$$f(v_1), f(v_2), \cdots, f(v_{i-1}), f(v_{i+1}), \cdots, f(v_n), -f(v_1)+1, -f(v_2)+1, \cdots, -f(v_n)+1$$

are distinct in $\mathbb{Z}_l$, we find that $l = 2n-1$ and for every $x \in \mathbb{Z}_l$ and $x \neq 2^{-1} = n = f(v_i)$, either $x = f(v_j)$ or $1-x = f(v_j)$ for some $j \neq i$. When $x = n-1$, there is a $v_s \in V(K_n^c)$ such that $f(v_s) = n-1$ or $n+1$. Since $f(v_i)+f(v_s)$ must not be a unit in $\mathbb{Z}_l$ and $2^{-1}+(n+1) = n+n+1 = 2$ is a unit, we get $f(v_s) = n-1$. When $x = 1$. there is a $v_t \in V(K_n^c)$ such that $f(v_t) = 1$ or 0. Since $f(v_i)+f(v_t)$ must not be a unit in $\mathbb{Z}_l$, we get $f(v_t) = 1$. But then $f(v_s)+f(v_t) = n-1+1 = n$ is a unit, and hence $v_s = v_t$ and $n-1 = 1$ in $\mathbb{Z}_l$, which implies $0 = l = 2n-1 = 2(n-1)+1 = 3$ in $\mathbb{Z}_l$. Therefore $n = 2$ and $l = 3$ and (2) is proved. □

Proposition 4.7. *Let $n \geqslant 3$. The representation numbers*

$$\mathrm{Rep}^+(K_n^c) = \mathrm{Rep}^-(K_n^c) = \mathrm{Rep}(S(K_n^\pm)^c) = 2n$$

and the product dimensions

$$\mathrm{p}^+\mathrm{dim}(K_n^c) = \mathrm{p}^-\mathrm{dim}(K_n^c) = \mathrm{pdim}(S(K_n^\pm)^c) = 2.$$

Also the representation numbers

$$\mathrm{Rep}^+(K_2^c) = \mathrm{Rep}(S(K_2^\pm)^c) = 4 \textit{ and } \mathrm{Rep}^-(K_2^c) = 3$$

and the product dimensions

$$\mathrm{p}^+\mathrm{dim}(K_2^c) = \mathrm{pdim}(S(K_2^\pm)^c) = 2 \textit{ and } \mathrm{p}^-\mathrm{dim}(K_2^c) = 1.$$

Proof. From Proposition 4.5, we have $\mathrm{p}^+\mathrm{dim}(K_n^c) \geqslant 2$, $\mathrm{pdim}(S(K_n^\pm)^c) \geqslant 2$, and $\mathrm{p}^-\mathrm{dim}(K_n^c) \geqslant 2$ when $n \geqslant 3$ and $\mathrm{p}^-\mathrm{dim}(K_2^c) = 1$. For every integer $m \geqslant n$, the map $f\colon V(K_n) \longrightarrow \mathbb{Z}_{2m}$ with

$$f(V(K_n)) = \{0, 2, 4, \ldots, 2(n-1)\}$$

is a $(\mathbb{Z}_{2m}, 1)$-representation of the graph K_n^c, a $(\mathbb{Z}_{2m}, -1)$-representation of the graph K_n^c, and a $(\mathbb{Z}_{2m}, \{\pm 1\})$-representation of the signed graph $S(K_n^\pm)^c$. When m is a prime, we find that $\mathrm{p}^+\mathrm{dim}(K_n^c) = \mathrm{p}^-\mathrm{dim}(K_n^c) = \mathrm{pdim}(S(K_n^\pm)^c) = 2$. The rest follows from Lemma 4.6. □

5. Signed graphs with a complete positive graph

In this section we determine $\mathrm{Rep}(S\Gamma)$ and $\mathrm{pdim}(S\Gamma)$ for signed graphs $S\Gamma$ when $(S\Gamma)^+$ are complete graphs. We first show that every modular representation of such a signed graph can be obtained from a square-free modular representation.

Lemma 5.1. *Let $S\Gamma$ be a signed graph with n vertices such that $(S\Gamma)^+ = K_n$. Let d and $m > 1$ be positive integers such that $d|m$. The signed graph $S\Gamma$ has a $(\mathbb{Z}_{dm}, \{\pm 1\})$-representation if and only if $S\Gamma$ has a $(\mathbb{Z}_m, \{\pm 1\})$-representation.*

Proof. Notice that for any integer $l > 1$, a $(\mathbb{Z}_l, \{\pm 1\})$-representation $f : V(S\Gamma) \longrightarrow \mathbb{Z}_l$ of the signed graph $S\Gamma$ is simply equivalent to a map $f : V(S\Gamma) \longrightarrow \mathbb{Z}$ satifying

(1) $\gcd(f(u) - f(v), l) = 1$ for all $u \neq v$ in $V(S\Gamma)$;
(2) $\gcd(f(u) + f(v), l) = 1$ for all $\{u, v\} \in E((S\Gamma)^-)$;
(3) $\gcd(f(u) + f(v), l) \neq 1$ for all $\{u, v\} \notin E((S\Gamma)^-)$.

Since $d|m$, for any integer $x \in \mathbb{Z}$, we have $\gcd(x, dm) = 1$ if and only if $\gcd(x, m) = 1$. Therefore f is a $(\mathbb{Z}_{dm}, \{\pm 1\})$-representation of $S\Gamma$ if and only if f is a $(\mathbb{Z}_m, \{\pm 1\})$-representation of $S\Gamma$. □

Next we give a necessary condition and a sufficient condition for a signed graph whose positive graph is complete to have a $(\mathbb{Z}_m, \{\pm 1\})$-representation.

Proposition 5.2. *Let $S\Gamma$ be a signed graph with n vertices such that $(S\Gamma)^+ = K_n$ and $m > 1$ be an integer. Let ν be the matching number of $((S\Gamma)^-)^c$, If $S\Gamma$ has a $(\mathbb{Z}_m, \{\pm 1\})$-representation, then every prime factor of m is no less than $\max(n, 2n - 2\nu - 1)$ and the number of distinct prime factors of m is at least $\chi'(((S\Gamma)^-)^c)$, where $\chi'(((S\Gamma)^-)^c)$ is the edge chromatic number of $((S\Gamma)^-)^c$.*

Proof. Let $f : V(S\Gamma) \longrightarrow \mathbb{Z}_m$ be a $(\mathbb{Z}_m, \{\pm 1\})$-representation of $S\Gamma$. Since f is also a $(\mathbb{Z}_m, 1)$-representation of $(S\Gamma)^+ = K_n$, by Lemma 4.1, each factor $p > 1$ of m is no less than $\chi(K_n) = n$. Since f is a $(\mathbb{Z}_m, -1)$-representation of $(S\Gamma)^-$, for each edge $\{u, v\} \in E(((S\Gamma)^-)^c)$, there is a prime $p|m$ such that $p|(f(u) + f(v))$. We now define a $\mathbb{Z}_m$-edge labeling $l : E(((S\Gamma)^-)^c) \longrightarrow \mathbb{Z}_m$ of the graph $((S\Gamma)^-)^c$ by $l(\{u, v\}) = f(u) + f(v)$ for all $\{u, v\} \in E(((S\Gamma)^-)^c)$. Then for each prime $p|m$, the natural projection ring homomorphism $\pi_p : \mathbb{Z}_m \longrightarrow \mathbb{Z}_p$ induces a $\mathbb{Z}_p$-edge labeling $\pi_p \circ l$ of $((S\Gamma)^-)^c$. Let $E_p = (\pi_p \circ l)^{-1}(0)$. Then E_p is an independent edge set, i.e. a matching, in $E(((S\Gamma)^-)^c)$ as $(S\Gamma)^+ = K_n$, and

$$\bigcup_{p|m} E_p = E(((S\Gamma)^-)^c).$$

From the independence of E_p for each $p|m$, there is an edge coloring of $((S\Gamma)^-)^c$ with colors the prime factors of m (if prime factors of m are $p_1 > p_2 > \cdots > p_k$, then we color edges in E_{p_1} with p_1, edges in $E_{p_2} \setminus E_{p_1}$ with p_2, edges in $E_{p_3} \setminus (E_{p_1} \cup E_{p_2})$ with p_3, $\cdots$, edges in $E_{p_k} \setminus (E_{p_1} \cup E_{p_2} \cup \cdots \cup E_{p_{k-1}})$ with p_k). Hence the number of prime factors of m is at least $\chi'(((S\Gamma)^-)^c)$.

Finally, for each $p|m$, the induced $\mathbb{Z}_p$-vertex labeling $\pi_p \circ f : V(S\Gamma) \longrightarrow \mathbb{Z}_p$ satisfies

(i) for each $\{u, v\} \in E_p$, $\{\pi_p \circ f(u), \pi_p \circ f(v)\} = \{x, -x\}$ for some $0 \neq x \in \mathbb{Z}_p$;
(ii) for each vertex w of $S\Gamma$ that is not incident with any edge in E_p, $-\pi_p \circ f(w)$ is not a $\mathbb{Z}_p$-label of any vertex of $S\Gamma$ except when $\pi_p \circ f(w) = 0$ because $E_p = (\pi_p \circ f)^{-1}(0)$.

Therefore

$$p \geqslant 2|E_p| + 1 + 2(n - (2|E_p| + 1)) = 2n - 2|E_p| - 1.$$

From $|E_p| \leqslant \nu$, we get $p \geqslant 2n - 2\nu - 1$, and hence $p \geqslant \max(n, 2n - 2\nu - 1)$. □

Proposition 5.3. *Let $S\Gamma$ be a signed graph with n vertices such that $(S\Gamma)^+ = K_n$. Let E_1, E_2, $\cdots$, E_k be a family of edge independent subsets (matchings) of $((S\Gamma)^-)^c$ such that*

$$\bigcup_{i=1}^{k} E_i = E(((S\Gamma)^-)^c).$$

If $p_1, p_2, \cdots, p_k$ are distinct primes satisfying $p_i \geqslant \max(n, 2n-2|E_i|-1)$ for all $i = 1, 2, \cdots, k$, then $S\Gamma$ has a $(\mathbb{Z}_m, \{\pm 1\})$-representation, where $m = p_1 p_2 \cdots p_k$.

Proof. For each $1 \leqslant i \leqslant k$, the condition $p_i \geqslant \max(n, 2n-2|E_i|-1)$ ensures that there is a $\mathbb{Z}_{p_i}$-vertex coloring $f_i \colon V((S\Gamma)^+) \longrightarrow \mathbb{Z}_{p_i}$ of $(S\Gamma)^+ = K_n$ such that for any two distinct vertices u and v in $V(\Gamma)$, the sum $f_i(u) + f_i(v) = 0$ if and only if $\{u, v\} \in E_i$. In other words, the signed graph $S\Gamma(f_i, \mathbb{Z}_{p_i}, \{\pm 1\})$ is given by

$$V(S\Gamma(f_i, \mathbb{Z}_{p_i}, \{\pm 1\})) = V(S\Gamma), (S\Gamma(f_i, \mathbb{Z}_{p_i}, \{\pm 1\}))^+ = K_n, \text{ and } (S\Gamma(f_i, \mathbb{Z}_{p_i}, \{\pm 1\}))^- = K_n \setminus E_i$$

and f_i is a $(\mathbb{Z}_{p_i}, \{\pm 1\})$-representation of $S\Gamma(f_i, \mathbb{Z}_{p_i}, \{\pm 1\})$ for all $i = 1, 2, \cdots, k$. By Proposition 2.4 and Proposition 2.5, the product map

$$f \colon \bigcap_{i=1}^{k} S\Gamma(f_i, \mathbb{Z}_{p_i}, \{\pm 1\}) \longrightarrow \mathbb{Z}_m$$

is a $(\mathbb{Z}_m, \{\pm 1\})$-representation. It is clear that

$$\begin{aligned}
V\left(\bigcap_{i=1}^{k} S\Gamma(f_i, \mathbb{Z}_{p_i}, \{\pm 1\})\right) &= V(S\Gamma), \\
\left(\bigcap_{i=1}^{k} S\Gamma(f_i, \mathbb{Z}_{p_i}, \{\pm 1\})\right)^+ &= \bigcap_{i=1}^{k} (S\Gamma(f_i, \mathbb{Z}_{p_i}, \{\pm 1\}))^+ = K_n = (S\Gamma)^+, \\
\left(\bigcap_{i=1}^{k} S\Gamma(f_i, \mathbb{Z}_{p_i}, \{\pm 1\})\right)^- &= \bigcap_{i=1}^{k} (S\Gamma(f_i, \mathbb{Z}_{p_i}, \{\pm 1\}))^- = \bigcap_{i=1}^{k} K_n \setminus E_i \\
&= K_n \setminus \left(\bigcup_{i=1}^{k} E_i\right) = (S\Gamma)^-.
\end{aligned}$$

Therefore

$$\bigcap_{i=1}^{k} S\Gamma(f_i, \mathbb{Z}_{p_i}, \{\pm 1\}) = S\Gamma$$

and f is a $(\mathbb{Z}_m, \{\pm 1\})$-representation of $S\Gamma$. □

Remark 5.4. Note that the matchings in Proposition 5.3 are not required to be disjoint, and hence we can choose them to be all maximal.

Proposition 5.2 and 5.3 determine the product dimension of $S\Gamma$ with $(S\Gamma)^+$ complete. It is similar to but different from Proposition 2.3 in [14].

Corollary 5.5. *Let $S\Gamma$ be a signed graph with n vertices such that $(S\Gamma)^+ = K_n$. Then*

$$\operatorname{pdim}(S\Gamma) = \chi'((S\Gamma)^-).$$

Combining Lemma 5.1 and Proposition 5.2 and 5.3, we have

Theorem 5.6. *Let $S\Gamma$ be a signed graph with n vertices such that $(S\Gamma)^+ = K_n$ and $m > 1$ be an integer.*

(a) *If $E(((S\Gamma)^-)^c)$ is a perfect matching, then $S\Gamma$ has a $(\mathbb{Z}_m, \{\pm 1\})$-representation if and only if for every prime factor p of m, $p \geqslant n$. In particular*

$$\operatorname{Rep}(S\Gamma) = p_0(n) \text{ and } \operatorname{pdim}(S\Gamma) = 1.$$

(b) *If* $E(((S\Gamma)^-)^c)$ *is a matching but not perfect or empty, then* $S\Gamma$ *has a* $(\mathbb{Z}_m,\{\pm 1\})$*-representation if and only if for every prime factor* p *of* m, $p \geqslant 2n-2|E(((S\Gamma)^-)^c)|-1$. *In particular*

$$\operatorname{Rep}(S\Gamma) = p_0(2n-2|E(((S\Gamma)^-)^c)|-1) \text{ and } \operatorname{pdim}(S\Gamma) = 1.$$

Corollary 5.7. $\operatorname{Rep}(S(K_n^{\pm})) = p_0(2n-1)$ *and* $\operatorname{pdim}(S(K_n^{\pm})) = 1$.

Theorem 5.8. *Let* $S\Gamma$ *be a signed graph with* n *vertices such that* $(S\Gamma)^+ = K_n$ *and* $m > 1$ *be an integer. Suppose* $E((S\Gamma)^-) = E(K_n) \setminus E(K_r)$ *for some* $r \leqslant n$.

(a) *If* r *is odd, then* $S\Gamma$ *has a* $(\mathbb{Z}_m,\{\pm 1\})$*-representation if and only if* m *has at least* r *distinct prime factors and for every prime factor* p *of* m, $p \geqslant 2n-r$. *In particular*

$$\operatorname{Rep}(S\Gamma) = \prod_{i=0}^{r-1} p_i(2n-r) \text{ and } \operatorname{pdim}(S\Gamma) = r.$$

(b) *If* r *is even and* $r < n$, *then* $S\Gamma$ *has a* $(\mathbb{Z}_m,\{\pm 1\})$*-representation if and only if* m *has at least* $r-1$ *distinct prime factors and for every prime factor* p *of* m, $p \geqslant 2n-r-1$. *In particular*

$$\operatorname{Rep}(S\Gamma) = \prod_{i=0}^{r-2} p_i(2n-r-1) \text{ and } \operatorname{pdim}(S\Gamma) = r-1.$$

(c) *If* r *is even and* $r = n$, *then* $S\Gamma$ *has a* $(\mathbb{Z}_m,\{\pm 1\})$*-representation if and only if* m *has at least* $n-1$ *distinct prime factors and for every prime factor* p *of* m, $p \geqslant n$. *In particular*

$$\operatorname{Rep}(S\Gamma) = \prod_{i=0}^{n-2} p_i(n) \text{ and } \operatorname{pdim}(S\Gamma) = n-1.$$

Corollary 5.9. *When* n *is even*

$$\operatorname{Rep}(S(K_n^{+})) = \prod_{i=0}^{n-2} p_i(n) \text{ and } \operatorname{pdim}(S(K_n^{+})) = n-1,$$

and when n *is odd*

$$\operatorname{Rep}(S(K_n^{+})) = \prod_{i=0}^{n-1} p_i(n) \text{ and } \operatorname{pdim}(S(K_n^{+})) = n.$$

Theorem 5.10. *Let* $S\Gamma$ *be a signed graph with* $n \geqslant 3$ *vertices such that* $(S\Gamma)^+ = K_n$ *and* $m > 1$ *be an integer. Suppose* $E((S\Gamma)^-) = E(K_n) \setminus E(C_r)$, *where* C_r *is a cycle of length* r *for some* $3 \leqslant r \leqslant n$.

(a) *If* r *is odd, then* $S\Gamma$ *has a* $(\mathbb{Z}_m,\{\pm 1\})$*-representation if and only if* m *has at least* 3 *distinct prime factors and for every prime factor* p *of* m, $p \geqslant 2n-r$. *In particular*

$$\operatorname{Rep}(S\Gamma) = p_0(2n-r)p_1(2n-r)p_2(2n-r) \text{ and } \operatorname{pdim}(S\Gamma) = 3.$$

(b) *If* r *is even and* $r < n$, *then* $S\Gamma$ *has a* $(\mathbb{Z}_m,\{\pm 1\})$*-representation if and only if* m *has at least* 2 *distinct prime factors and for every prime factor* p *of* m, $p \geqslant 2n-r-1$. *In particular*

$$\operatorname{Rep}(S\Gamma) = p_0(2n-r-1)p_1(2n-r-1) \text{ and } \operatorname{pdim}(S\Gamma) = 2.$$

(c) *If* r *is even and* $r = n$, *then* Γ *has a* $(\mathbb{Z}_m,\{\pm 1\})$*-representation if and only if* m *has at least* 2 *distinct prime factors and for every prime factor* p *of* m, $p \geqslant n$. *In particular*

$$\operatorname{Rep}(S\Gamma) = p_0(n)p_1(n) \text{ and } \operatorname{pdim}(S\Gamma) = 2.$$

Theorem 5.11. *Let* $S\Gamma$ *be a signed graph such that* $(S\Gamma)^+ = K_n$ *and* $m > 1$ *be an integer. Suppose* $E((S\Gamma)^-) = E(K_n) \setminus E(K_{r,lr})$ *for some positive integers* l *and* r *with* $(l+1)r \leqslant n$.

(a) *If $n = 2r$, then $S\Gamma$ has a $(\mathbb{Z}_m, \{\pm 1\})$-representation if and only if m has at least r distinct prime factors and for every prime factor p of m, $p \geqslant n$. In particular*

$$\mathrm{Rep}(S\Gamma) = \prod_{i=0}^{r-1} p_i(n) \text{ and } \mathrm{pdim}(S\Gamma) = r.$$

(b) *If $n \geqslant 2r + 1$, then $S\Gamma$ has a $(\mathbb{Z}_m, \{\pm 1\})$-representation if and only if m has at least lr distinct prime factors and for every prime factor p of m, $p \geqslant 2n - 2r - 1$. In particular*

$$\mathrm{Rep}(S\Gamma) = \prod_{i=0}^{lr-1} p_i(2n - 2r - 1) \text{ and } \mathrm{pdim}(S\Gamma) = lr.$$

6. Signed graph with a complete negative graph and no positive edges

In this section we study $S(K_n^-)$. We first determine its product dimension.

Theorem 6.1. $\mathrm{pdim}(S(K_n^-)) = 2.$

Proof. By Proposition 4.5, we know that $\mathrm{pdim}(S(K_n^-)) \geqslant 2$. Let $f_1 \colon V(K_n^-) \longrightarrow \mathbb{Z}_3$ be the constant map given by $f(v) = 1$ for all $v \in V(K_n^-)$. Then f is a $(\mathbb{Z}_3, \{\pm 1\})$-pseudo representation of $S(K_n^-)$. from Example 2.10, for any prime $p \geqslant 2n - 1$, there is a $(\mathbb{Z}_p, \{\pm 1\})$-representation of $S(K_n^{\pm})$. By Proposition 2.4, there is a $(\mathbb{Z}_{3p}, \{\pm 1\})$-representation of $S(K_n^{\pm})$ for any prime $p > \max\{3, 2n - 2\}$. Hence $\mathrm{pdim}(S(K_n^-)) = 2$. □

The proof of Theorem 6.1 also proves the following lemma.

Lemma 6.2. *If $p \geqslant 2n - 1$ is prime, then $S(K_n^-)$ has a $(\mathbb{Z}_{3p}, \{\pm 1\})$-representation and hence* $\mathrm{Rep}(S(K_n^-)) \geqslant 3p$.

In order to calculate $\mathrm{Rep}(S(K_n^-))$, we need a theorem of Frankl [11]. It is an Erdös-Ko-Rado theorem for direct products. For the convenience of the reader we include a proof of the modified theorem which determines the sets that attain the maximum size. The proof is virtually identical to that of Frankl [11].

Theorem 6.3. *Let X be an n-element set, and $X_1, \ldots, X_k$ is a partition of X with $|X_i| = n_i$ for $i = 1, 2, \ldots, k$. Define*

$$\mathcal{H} = \{F \in \binom{X}{k} : |F \cap X_i| = 1 \text{ for } i = 1, 2, \ldots, k\}.$$

Suppose that $\mathcal{F} \subseteq \mathcal{H}$ is intersecting and $1 \leq n_1 < n_2 \leq \cdots \leqslant n_k$. Then $|\mathcal{F}| \leqslant n_2 \ldots n_k$, and equality holds if and only if

$$\mathcal{F} = \mathcal{F}_j = \{F \in \mathcal{H} : F \cap X_1 = \{x_j\}\}, \quad j = 1, 2, \ldots, n_1$$

where $X_1 = \{x_1, x_2, \ldots, x_{n_1}\}$.

Proof. Note that if $n_1 = 1$, then the conclusion is trivial. Now we assume $2 \leq n_1 < n_2 \leq \cdots \leq n_k$. For each $1 \leq i \leq k$, let $M_i = J_{n_i} - I_{n_i}$, where J_{n_i} is the all-one matrix of order n_i and I_{n_i} is the identity matrix of order n_i. The eigenvalues of M_i are $\lambda_i = n_i - 1$ with multiplicity 1, and $\mu_i = -1$ with multiplicity $n_i - 1$. Let $M = M_1 \otimes M_2 \otimes \cdots \otimes M_k$ be the tensor product of M_i's. Index the rows and columns of M by the members in $\mathcal{H}$. Then M is a symmetric $\{1, 0\}$-matrix where an entry is 1 if and only if the intersection of the corresponding row and column indices are disjoint. The eigenvalues of M are the products of those M_i's. So the largest eigenvalue of M is $\lambda = (n_1 - 1) \ldots (n_k - 1)$ with multiplicity 1, and a corresponding eigenvector is $\mathbf{1}$, the all-one vector. The smallest eigenvalue is $\mu = -(n_2 - 1) \ldots (n_k - 1)$ with multiplicity $n_1 - 1$. Let

$$N = M - \mu I - \frac{\lambda - \mu}{|\mathcal{H}|} J,$$

where I is the identity matrix of order $|\mathcal{H}|$, and J is the all-one matrix of order $|\mathcal{H}|$. N is positive semidefinite. Let $\mathcal{F} \subset \mathcal{H}$ be an intersecting set and let $\mathbf{v}$ be its characteristic vector. Then

$$0 \leq \mathbf{v}N\mathbf{v}^t = \mathbf{v}M\mathbf{v}^t - \mu\mathbf{v}I\mathbf{v}^t - \frac{\lambda-\mu}{|\mathcal{H}|}\mathbf{v}J\mathbf{v}^t = -\mu|\mathcal{F}| - \frac{\lambda-\mu}{|\mathcal{H}|}|\mathcal{F}|^2.$$

Therefore, $|\mathcal{F}|/|\mathcal{H}| \leq -\mu/(\lambda-\mu) = 1/n_1$, and equality holds if and only if $\mathbf{v}$ is in the N-eigenspace of 0. Consider any N-eigenvector $\mathbf{v}$ of 0. Write $\mathbf{v} = \mathbf{x}+\mathbf{y}$, where $\mathbf{x} \in \langle \mathbf{1} \rangle$ and $\mathbf{y}$ is a J-eigenvector of 0. Then

$$\mathbf{0} = N\mathbf{v}^t = N\mathbf{x}^t + N\mathbf{y}^t = \lambda\mathbf{x}^t - \mu\mathbf{x}^t - (\lambda-\mu)\mathbf{x}^t + M\mathbf{y}^t - \mu\mathbf{y}^t = (M-\mu I)\mathbf{y}^t.$$

So $\mathbf{y}$ is an $(M-\mu I)$-eigenvector of 0. Note that the $(M-\mu I)$-eigenspace of 0 equals the M-eigenspace of μ, which has dimension (n_1-1). Hence the N-eigenspace of 0 has dimension n_1. Consider the characteristic vector $\mathbf{v}_j$ of each intersecting family $\mathcal{F}_j$. There are n_1 such linearly independent vectors, each is an N-eigenvector of 0. Therefore, $\mathbf{v} = \sum_{j=1}^{n_1} a_j\mathbf{v}_j$, and such a $\mathbf{v}$ is a characteristic vector for some $\mathcal{F} \subset \mathcal{H}$ if and only if $\mathbf{v} = \mathbf{v}_j$ for some $j = 1, 2, \ldots, n_1$. In other words, $|\mathcal{F}| = n_2 n_3 \ldots n_k$ if and only if $\mathcal{F} = \mathcal{F}_j$ for some $j = 1, 2, \ldots, n_1$. □

Remark 6.4. In Theorem 6.3, it is clear that $\mathcal{H} = X_1 \times X_2 \times \cdots \times X_k$ and $\mathcal{F}$ is a subset in $\mathcal{H}$ such that every two elements in $\mathcal{F}$ share at least one coordinate.

It is trivial to see that $\mathrm{Rep}(S(K_2^-)) = 6$. When $n \geqslant 3$, by Lemma 4.3, the signed graph only has $(\mathbb{Z}_l, \{\pm 1\})$-representations or $(\mathbb{Z}_l, \{\pm 1\})$-pseudo representations for odd integers l.

Theorem 6.5. *Let $m > 1$ be an odd integer having exactly k distinct prime divisors, among which p is the smallest. The signed graph $S(K_n^-), n \geqslant 3$ has a $(\mathbb{Z}_m, \{\pm 1\})$-representation if and only if*

$$n \leqslant \frac{\phi(m)}{2^{k-1}(p-1)} + (k-1),$$

where ϕ is the Euler's totient function. Hence

$$\mathrm{Rep}(S(K_n^-)) = \text{ the least positive odd integer } m \text{ such that } n \leqslant \frac{\phi(m)}{2^{k-1}(p-1)} + (k-1),$$

where k is the number of prime divisors of m and p is the smallest prime divisor of m.

Proof. By Definition 2.2, the signed graph $S(K_n^-)$ has a $(\mathbb{Z}_m, \{\pm 1\})$-representation if and only if there are modular numbers $x_1, x_2, \cdots, x_n$ in $\mathbb{Z}_m$ such that $x_i + x_j \in \mathbb{Z}_m^*$ and $x_i - x_j \notin \mathbb{Z}_m^*$ for all $1 \leqslant i < j \leqslant n$. Let Φ_m be the graph with

$$V(\Phi_m) = \mathbb{Z}_m \text{ and } E(\Phi_m) = \{\{a,b\} | a, b \in \mathbb{Z}_m, a+b \in \mathbb{Z}_m^* \text{ and } a-b \notin \mathbb{Z}_m^*\}$$

and $\omega(\Phi_m)$ be the clique number of Φ_m. It is clear that $S(K_n^-)$ has a $(\mathbb{Z}_m, \{\pm 1\})$-representation if and only if $n \leqslant \omega(\Phi_m)$. We now compute $\omega(\Phi_m)$. Let W be a clique in Φ_m, W_{un} be the set of all unit modular numbers in W and W_{nu} be the set of all non-unit modular numbers in W. We will first estimate $|W_{\mathrm{un}}|$ and $|W_{\mathrm{nu}}|$ separately and then estimate $|W|$. Let $m = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ be the prime factorization of m with $p = p_1 < p_2 < \cdots < p_k$. Let $m' = p_1 p_2 \cdots p_k$, $\pi\colon \mathbb{Z}_m \longrightarrow \mathbb{Z}_{m'}$ and $\pi_i\colon \mathbb{Z}_{m'} \longrightarrow \mathbb{Z}_{p_i}$ be the natural projection ring homomorphisms for all $i = 1, 2, \cdots, k$.

An Observation: A modular number $x \in \mathbb{Z}_m$ is a unit if and only if the modular number $\pi(x) \in \mathbb{Z}_{m'}$ is a unit.

This observation implies the following claims.

Claim 1: a set U of unit modular numbers in $\mathbb{Z}_m$ forms a clique in Φ_m if and only if the set of unit modular numbers $\pi(U)$ in $\mathbb{Z}_{m'}$ forms a clique in $\Phi_{m'}$.

This is because if U forms a clique in Φ_m, then for any two $x, y \in U$ with $\pi(x) \neq \pi(y)$, the modular number $\pi(x) + \pi(y) = \pi(x+y)$ is a unit in $\mathbb{Z}_{m'}$ and $\pi(x) - \pi(y) = \pi(x-y)$ is not a unit in $\mathbb{Z}_{m'}$, which means that $\pi(U)$ in $\mathbb{Z}_{m'}$ forms a clique in $\Phi_{m'}$. Conversely, if $\pi(U)$ forms a clique in $\Phi_{m'}$, then for any $x, y \in U$, the modular number $\pi(x-y) = \pi(x) - \pi(y)$ is not a unit in $\mathbb{Z}_{m'}$ as it is either 0 or a non-zero non-unit since $\pi(U)$ forms a clique in $\Phi_{m'}$. Similarly, $\pi(x+y) = \pi(x) + \pi(y)$ is a unit in $\mathbb{Z}_{m'}$ either due to the fact that $\pi(U)$ forms a clique in $\Phi_{m'}$ when $\pi(x) \neq \pi(y)$ or due to the fact that m is odd and x is a unit when $\pi(x) = \pi(y)$.

From $U \subseteq \pi^{-1}(\pi(U))$ for any subset U in $\mathbb{Z}_m$ and Claim 1, we know that any maximum clique M of Φ_m that consists of only unit modular numbers in $\mathbb{Z}_m$ must be of the form $M = \pi^{-1}(M')$, where M' is a maximum clique in $\Phi_{m'}$ that consists of only unit modular numbers in $\mathbb{Z}_{m'}$ and $|W_{\text{un}}| \leqslant |M| = |\text{Ker}(\pi)||M'| = \frac{m}{m'}|M'|$. We now use Theorem 2.5 and Theorem 6.3 to determine M'. Theorem 2.5 shows that $\mathbb{Z}_{m'} \cong \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_k}$ as well as $\mathbb{Z}^*_{m'} \cong \mathbb{Z}^*_{p_1} \times \mathbb{Z}^*_{p_2} \times \cdots \times \mathbb{Z}^*_{p_k}$. Since $M' \subseteq \mathbb{Z}^*_{m'}$ is a clique in $\Phi_{m'}$, for every $x, y \in M'$ their components x_i and y_i in $\mathbb{Z}_{p_i}$ satisfy $x_i + y_i \neq 0$ for all $i = 1, 2, \cdots, k$. This means that for all $i = 1, 2, \cdots, k$ there are subsets $X_i \subseteq \mathbb{Z}^*_{p_i}$ with $|X_i \cap \{x, -x\}| = 1$ for all $x \in \mathbb{Z}^*_{p_i}$ such that $M' \subseteq X_1 \times X_2 \times \cdots \times X_k$. Let X be the disjoint union of X_i for $i = 1, 2, \cdots, k$. By Remark 6.4, $\mathcal{H} = X_1 \times X_2 \times \cdots \times X_k$ and $M' \subseteq \mathcal{H}$ is a clique in $\Phi_{m'}$ if and only if M' is intersecting. By Theorem 6.3, $|M'| \leqslant \frac{p_2-1}{2}\frac{p_3-1}{2}\cdots\frac{p_k-1}{2}$ and $|M'| = \frac{p_2-1}{2}\frac{p_3-1}{2}\cdots\frac{p_k-1}{2}$ if and only if $M' = \{u_1\} \times X_2 \times X_3 \times \cdots \times X_k$ for some fixed $u_1 \in X_1$. Hence

$$|W_{\text{un}}| \leqslant p_1^{e_1-1}p_2^{e_2-1}\cdots p_k^{e_k-1}\frac{p_2-1}{2}\frac{p_3-1}{2}\cdots\frac{p_k-1}{2} = \frac{\phi(m)}{2^{k-1}(p-1)}$$

and equality holds if and only if $W_{\text{un}} = \pi^{-1}(\{u_1\} \times X_2 \times X_3 \times \cdots \times X_k)$.

Claim 2: a set V of non-unit modular numbers in $\mathbb{Z}_m$ forms a clique in Φ_m if and only if the set of non-unit modular numbers $\pi(V)$ in $\mathbb{Z}_{m'}$ forms a clique in $\Phi_{m'}$ and $\pi|_V : V \longrightarrow \pi(V)$ is a bijection.

This is because if $x, y \in V$ and $\pi(x) = \pi(y)$, then $\pi(x+y) = \pi(x) + \pi(y) = 2\pi(x)$ is not a unit as x is not a unit. Therefore $\pi|_V : V \longrightarrow \pi(V)$ is a bijection. The rest of the claim follows from the observation.

For any clique V in $\Phi_{m'}$ that consists of only non-unit modular numbers in $\mathbb{Z}_{m'} \cong \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_k}$ and for any two distinct $x = (x_1, x_2, \cdots, x_k)$ and $y = (y_1, y_2, \cdots, y_k)$ in N, there exist $1 \leqslant i, j \leqslant k$ such that $x_i = 0$ and $y_j = 0$ and $i \neq j$ as x and y are non-units and $x + y$ is a unit. This shows that $|V| \leqslant k$ and $|W_{\text{nu}}| \leqslant k$. Furthermore, if such a clique V has $|V| = k$, then for each $1 \leqslant i \leqslant k$ there is a unique $x \in V$ such that $x_i = 0$ and $x_j \neq 0$ for all $j \neq i$ and $1 \leqslant j \leqslant k$.

Combining the bounds for $|W_{\text{un}}|$ and $|W_{\text{nu}}|$, we have

$$|W| = |W_{\text{un}}| + |W_{\text{nu}}| \leqslant \frac{\phi(m)}{2^{k-1}(p-1)} + k.$$

We now show that the equality can not be reached. Suppose W is a clique in Φ_m such that $|W| = \frac{\phi(m)}{2^{k-1}(p-1)} + k$. Then $|W_{\text{un}}| = \frac{\phi(m)}{2^{k-1}(p-1)}$ and $W_{\text{un}} = \pi^{-1}(\{u_1\} \times X_2 \times X_3 \times \cdots \times X_k)$ for some $u_1 \in \mathbb{Z}^*_{p_1}$ and for all $i = 1, 2, \cdots, k$, some subsets $X_i \subseteq \mathbb{Z}^*_{p_i}$ with $|X_i \cap \{x, -x\}| = 1$ for all $x \in \mathbb{Z}^*_{p_i}$, while $|W_{\text{nu}}| = k$ and each element x in W_{nu} has exactly one coordinate x_i that is not a unit in $\mathbb{Z}_{p_i^{e_i}}$. Particularly, there is a unique element x in W such that its first coordinate x_1 is not a unit in $\mathbb{Z}_{p_1^{e_1}}$. This implies that for each $y \in W_{\text{un}}$, the first coordinate of $y - x$ is a unit as $\pi_1 \circ \pi(y-x) = u_1 \neq 0$. In order for x being adjacent to $y \in W_{\text{un}}$ in Φ_m, we must have a $2 \leqslant j(y) \leqslant k$ such that $\pi_{j(y)} \circ \pi(y-x) = 0$. Hence $|W_{\text{un}}| < \frac{\phi(m)}{2^{k-1}(p-1)}$, a contradiction. Therefore

$$|W| \leqslant \frac{\phi(m)}{2^{k-1}(p-1)} + (k-1).$$

Table 1. $\mathrm{Rep}(S(K_n^-))$ for small n.

n	$\mathrm{Rep}(S(K_n^-))$	n	$\mathrm{Rep}(S(K_n^-))$	n	$\mathrm{Rep}(S(K_n^-))$
$[3\dots 3]$	$9 = 3^2$	$[4\dots 4]$	$21 = 3\times 7$	$[5\dots 5]$	$25 = 5^2$
$[6\dots 9]$	$27 = 3^3$	$[10\dots 10]$	$57 = 3\times 19$	$[11\dots 12]$	$69 = 3\times 23$
$[13\dots 27]$	$81 = 3^4$	$[28\dots 28]$	$171 = 3^2\times 19$	$[29\dots 30]$	$177 = 3\times 59$
$[31\dots 31]$	$183 = 3\times 61$	$[32\dots 34]$	$201 = 3\times 67$	$[35\dots 36]$	$213 = 3\times 71$
$[37\dots 37]$	$219 = 3\times 73$	$[38\dots 40]$	$237 = 3\times 79$	$[41\dots 81]$	$243 = 3^5$
$[82\dots 82]$	$489 = 3\times 163$	$[83\dots 84]$	$501 = 3\times 167$	$[85\dots 87]$	$519 = 3\times 173$
$[88\dots 88]$	$531 = 3^2\times 59$	$[89\dots 90]$	$537 = 3\times 179$	$[91\dots 91]$	$543 = 3\times 181$
$[92\dots 96]$	$573 = 3\times 191$	$[97\dots 97]$	$579 = 3\times 193$	$[98\dots 99]$	$591 = 3\times 197$
$[100\dots 100]$	$597 = 3\times 199$	$[101\dots 125]$	$625 = 5^4$	$[126\dots 243]$	$729 = 3^6$
$[244\dots 244]$	$1461 = 3\times 487$	$[245\dots 246]$	$1473 = 3\times 491$	$[247\dots 250]$	$1497 = 3\times 499$

On the other hand, the set

$$W = \pi^{-1}(\{1\}\times X_2\times X_3\times\cdots\times X_k)$$
$$\bigcup\{(1,0,1,1,\cdots,1),(1,1,0,1,\cdots,1),(1,1,1,0,\cdots,1),\cdots,(1,1,1,1,\cdots,1)\}$$

is a clique of size $\frac{\phi(m)}{2^{k-1}(p-1)}+(k-1)$ in Φ_m, where for each $i=2,3,\cdots,k$, the set X_i is a subset of $\mathbb{Z}_{p_i}^*$ satisfying $1\in X_i$ and $|X_i\cap\{x,-x\}|=1$ for all $x\in\mathbb{Z}_{p_i}^*$. So

$$\omega(\Phi_m) = \frac{\phi(m)}{2^{k-1}(p-1)}+(k-1).$$

□

Proposition 6.6. *For each integer $n>1$, the representation number*

$$3n \leqslant \mathrm{Rep}(S(K_n^-)) < 6.3n.$$

Proof. The lower bound $\mathrm{Rep}(S(K_n^-))\geqslant 3n$ follows Lemma 4.6 when $n\geqslant 3$ and the fact that $\mathrm{Rep}(S(K_2^-))=6$. For the upper bound, the Table 1 in [12] shows that when $2n-1\geqslant 213$, there is a prime p satisfies $2n-1<p\leqslant(1.05)(2n-1)$. By Lemma 6.2, the representation number $\mathrm{Rep}(S(K_n^-))\leqslant 3p\leqslant 3(1.05)(2n-1)<6.3n$. When $2n-1<213$, our computations in Table 1 also shows that $\mathrm{Rep}(S(K_n^-))<6.3n$. Therefore $3n\leqslant\mathrm{Rep}(S(K_n^-))<6.3n$ for all $n>1$. □

Lemma 6.7. *Suppose* $\mathrm{Rep}(S(K_n^-))=p_1^{e_1}p_2^{e_2}\cdots p_k^{e_k}$, *where* $k\geqslant 1$, $e_1,e_2,\dots,e_k\geqslant 1$, $3\leqslant p_1<p_2<\cdots<p_k$ *are distinct primes. Then* $k=1$ *or* 2.

Proof. Write $m=p_1^{e_1}p_2^{e_2}\cdots p_k^{e_k}$. We may assume

$$n=\omega(\Phi_m)=\frac{\phi(m)}{2^{k-1}(p_1-1)}+(k-1).$$

If $k\geqslant 3$, then

$$\begin{aligned}\frac{n}{k-1} &= \frac{p_1^{e_1-1}}{k-1}\cdot\frac{p_2^{e_2-1}(p_2-1)}{2}\cdot\frac{p_3^{e_3-1}(p_3-1)}{2}\cdot\frac{p_4^{e_4-1}(p_4-1)}{2}\cdots\frac{p_k^{e_k-1}(p_k-1)}{2}+1\\ &= \frac{p_2^{e_2-1}(p_2-1)}{2}\cdot\frac{p_3^{e_3-1}(p_3-1)}{2}\cdot\left(\frac{p_4^{e_4-1}(p_4-1)}{2}\cdots\frac{p_k^{e_k-1}(p_k-1)}{2}\cdot\frac{p_1^{e_1-1}}{k-1}\right)+1\\ &\geqslant \frac{5-1}{2}\cdot\frac{7-1}{2}\cdot\frac{1}{2}+1=4\end{aligned}$$

and therefore $k-1 \leqslant \frac{n}{4}$. Note that

$$n = \frac{\phi(m)}{2^{k-1}(p_1-1)} + (k-1) \leqslant \frac{m}{2^{k-1}p_1} + (k-1).$$

Hence $m \geqslant 2^{k-1}p_1[n-(k-1)] \geqslant 2^2 \cdot 3(n-\frac{n}{4}) \geqslant 6.3n$, which contradicts Proposition 6.6. Therefore, $k=1$ or 2. □

Proposition 6.8. *Let $n \geqslant 3$ be an integer.*

(1) *If* $\mathrm{Rep}(S(K_n^-)) = p^k$ *for some prime p, then $p = 3$ or 5.*
(2) *If* $\mathrm{Rep}(S(K_n^-)) = p_1^{e_1}p_2^{e_2}$ *for some primes $p_1 < p_2$, then $p_1 = 3$.*

Proof. (1) If $\mathrm{Rep}(S(K_n^-)) = p^k$, we may assume $n = \omega(\Phi_{p^k}) = p^{k-1}$. By Proposition 6.6,

$$\mathrm{Rep}(S(K_n^-)) = p^k < 6.3n = 6.3p^{k-1}$$

and $p < 6.3$. Hence $p = 3$ or 5.

(2) Write $m = p_1^{e_1}p_2^{e_2}$ and we may assume

$$n = \omega(\Phi_m) = \frac{p_1^{e_1-1}p_2^{e_2-1}(p_2-1)}{2} + 1 = \frac{m}{2p_1}(1-\frac{1}{p_2}) + 1.$$

If $p_1 \geqslant 5$, then $m = \mathrm{Rep}((S(K_n^-)) = 2p_1(n-1)\frac{p_2}{p_2-1} \geqslant 10(n-1) \geqslant 6.3n$ when $n \geqslant 4$, which contradicts Proposition 6.6. It is easy to see that if $n=3$, $p_1=3$. Therefore, $p_1 = 3$ for all $n \geqslant 3$. □

To summarize, we have the following conclusion.

Theorem 6.9. *The representation number* $\mathrm{Rep}(S(K_n^-))$ *is the smallest integer in the following forms:*

(1) 3^s *with* $3^{s-1} \geqslant n$;
(2) 5^s *with* $5^{s-1} \geqslant n$;
(3) 3^sp^t *with* $3^{s-1}p^{t-1}(\frac{p-1}{2}) + 1 \geqslant n$, *where $s, t \geqslant 1$, and $p > 3$ is a prime.*

Remark 6.10. (1) If $\mathrm{Rep}(S(K_n^-)) = 3^sp^t$ and p is large enough, then there is a prime q such that $p^t - p^{t-1} < q < p^t$ for any $t \geqslant 3$. Write $m = 3^sp^t$ and we may assume $n = \omega(\Phi_m) = 3^{s-1}p^{t-1} \cdot \frac{p-1}{2} + 1$. Consider $m' = 3^sq$. Then $\omega(\Phi_{m'}) = 3^{s-1} \cdot \frac{q-1}{2} + 1 \geqslant n$ and $m' < m$. Therefore, $\mathrm{Rep}(S(K_n^-)) \leqslant m'$. Hence $t \leqslant 2$.

(2) Moreover, assume Oppermann's Conjecture is true: there is a prime between $n^2 - n$ and n^2 when n is large enough. Then with the same argument, we have that if $\mathrm{Rep}(S(K_n^-)) = 3^sp^t$ and p is large enough, then $t = 1$.

(3) We ran a computer program to compute $\mathrm{Rep}(S(K_n^-))$ up to $n = 10,000,000$, and $\mathrm{Rep}(S(K_n^-)) = 3^s, 5^s$, or 3^sp ($p > 3$ a prime).

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