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# GRAPHS WITHOUT A $2C_3$-MINOR AND BICIRCULAR MATROIDS WITHOUT A $U_{3,6}$-MINOR

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ABSTRACT. In this note we characterize all graphs without a $2C_3$-minor. A consequence of this result is a characterization of the bicircular matroids with no $U_{3,6}$-minor.

## 1. Introduction

We assume the reader has a basic familiarity with matroid theory as in [5]; however, it isn't completely necessary to read this note. Given a fixed graph $H$, results characterizing the structure of graphs $G$ without an $H$-minor have a well-established history going back as far as 1937 with Wagner's seminal result [7] for $H = K_5$. Recently Ding and Liu [3] surveyed the known results for 3-connected graphs $H$ and an older survey by Diestel [2] lists results for some other small graphs. In all of the results listed in [2, 3], the graph $H$ is simple. The graph $2C_3$ is obtained from the cycle of length 3 by doubling each edge. The graph $2C_3$ is of interest in matroid theory in that a bicircular matroid $B(G)$ is isomorphic to $U_{3,6}$ if and only if $G \cong 2C_3$ up to removal of isolated vertices (see [8, Lemma 2.12] or [1, Theorem 4.11]).

The main result of this note is Theorem 1.1 which describes the very limited structure that a graph with no $2C_3$-minor can have. We remark that Theorem 1.1 is enough to characterize all graphs without a $2C_3$-minor because: $G$ has a $2C_3$-minor if and only if some block of $G$ has a $2C_3$-minor and if $G$ has a vertex $v$ of degree 2, then $G$ has a $2C_3$-minor if and only if the graph obtained from $G$ by smoothing out $v$ has a $2C_3$-minor. We also prove Theorem 1.2.

Let $G$ be an outerplanar simple graph. Thus $G$ consists of a Hamilton cycle $H$ along with a set of chords $C$. Let $C'$ be a disjoint copy of $C$. Embed $G \cup C'$ with chords $C$ inside $H$ and chords $C'$

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outside $H$. A *doubled outerplanar embedding* is any graph $K$ contained between $H$ and $G \cup C'$ with embedding inherited from $G \cup C'$.

**Theorem 1.1.** *If $G$ is a connected and nonseparable graph with minimum degree 3, then $G$ has no $2C_3$-minor if and only if*

(1) *$G \cong K_4$ or*

(2) *$G$ is the topological dual graph of some doubled outerplanar embedding.*

**Theorem 1.2.** *If $G$ is 3-connected and loopless, then $G \cong K_4$ or $G$ contains a $2C_3$-minor.*

## 2. Proofs

Given a graph $G$, a *$k$-separation* is an expression $G = G_1 \cup G_2$ in which each $G_i$ has at least $k$ edges and $G_1 \cap G_2$ is a set of $k$ vertices. A connected graph is *separable* when it has a 1-separation. A graph $G$ is *nonseparable* when it is connected and has no 1-separation. A *link* is an edge in a graph that is not a loop. Note that every edge in a nonseparable graph is a link. A connected graph $G$ is *$k$-connected* when it has at least $k+1$ vertices and it has no set of $t < k$ vertices whose removal leaves a disconnected subgraph.

*Proof of Theorem 1.1.* Assume that $G \cong K_4$ or $G = H^*$ where $H$ is a doubled outerplanar embedding. It is important to note that the graph of a doubled outerplanar embedding is still an outerplanar graph. If $G \cong K_4$, then $G$ has no $2C_3$-minor. If $G = H^*$, then $H$ has no $K_{2,3}$-minor. (It is well known that a graph $G$ is outerplanar if and only if it has no $K_{2,3}$- or $K_4$-minor.) Since any embedding of $2C_3$ in the plane has topological dual graph isomorphic to $K_{2,3}$, we get that $G$ has no $2C_3$-minor.

Conversely, suppose that $G$ has no $2C_3$-minor. The reader can check that $K_{3,3}$ and $K_5$ both contain $2C_3$-minors and hence $G$ is planar. Let $H$ be the topological dual graph of some embedding of $G$ in the plane. Note that $H$ has no faces of length two because $G$ has minimum degree 3. Furthermore, since $G$ is nonseparable, so must be $H$. Now $|V(H)| > 2$ because $G$ has minimum degree 3. Since $|V(H)| \geq 3$, $H$ is 2-connected. Let $H_v$ be the graph obtained from $H$ by adjoining an apex vertex to all other vertices of $H$. Thus $H_v$ is 3-connected. If $H_v$ is planar, then $H$ is outerplanar and has an embedding in the plane without faces of length 2. Thus $H$ is a doubled outerplanar embedding, a desired result. If $H_v$ is non-planar, then by a theorem of D.W. Hall ([4] or see [5, 12.2.11]) either $H_v \cong K_5$ along with maybe some doubled edges or $H_v$ contains a $K_{3,3}$-subdivision. In the former case, $H \cong K_4$ along with maybe some doubled edges. If an edge of $K_4$ is doubled, however, the resulting graph has a $2C_3$-minor, a contradiction. Thus $G \cong K_4$, a desired outcome. In the latter case $H$ contains a $K_{2,3}$-subdivision and so $G$ contains a $2C_3$-minor, a contradiction. □

*Proof of Theorem 1.2.* Let $\hat{G}$ be the simplification of $G$; that is, for each class of parallel links, delete all but one of them. Thus $\hat{G}$ is 3-connected and simple. By Tutte's Wheel Theorem ([6] or see [5, Theorem 8.8.4]) there is a sequence of 3-connected simple graphs $G_1, \ldots, G_t$ such that $G_1 = \hat{G}$, $G_{i+1} = G_i/e$ or $G_i \backslash e$, and $G_t \cong W_n$ for $n \geq 3$ where $W_n$ is the $n$-spoked wheel. If $n \geq 4$, then $G_t$

has a $2C_3$-minor and therefore so does $G$. So suppose that $G_t \cong W_3 \cong K_4$. If $G = G_t$, then we are done. So suppose that $G_t$ is a proper minor of $G$. Since there is no 3-connected simple graph $H$ for which $H/e$ or $H\backslash e$ is $K_4$, we must have that $\hat{G} = G_1 = G_t \cong K_4$. Since $G_t$ is a proper minor of $G$, $G$ contains $K_4$ along with one doubled edge. This contains a $2C_3$-minor, as required. □

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