Matrix representations of frame and lifted-graphic matroids correspond to gain functions

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Abstract

Let M be a 3-connected matroid and let $\mathbb{F}$ be a field. Let A be a matrix over $\mathbb{F}$ representing M and let $(G, \mathcal{B})$ be a biased graph representing M. We characterize the relationship between A and $(G, \mathcal{B})$, settling four conjectures of Zaslavsky. We show that for each matrix representation A and each biased graph representation $(G, \mathcal{B})$ of M, A is projectively equivalent to a canonical matrix representation arising from G as a gain graph over $\mathbb{F}^+$ or $\mathbb{F}^\times$ realizing $\mathcal{B}$. Further, we show that the projective equivalence classes of matrix representations of M are in one-to-one correspondence with the switching equivalence classes of gain graphs arising from $(G, \mathcal{B})$, except in one degenerate case.

1 Introduction

An amazing theorem of Kahn & Kung [10] shows that there are only two matroid varieties containing 3-connected matroids: those given by (1) projective geometries over finite fields, and (2) Dowling geometries over finite groups. The matroids belonging to both of these varieties are precisely the simple

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frame matroids linearly representable over finite fields. Thus it is perhaps not surprising that these matroids have a central role in matroid structure theory [6, 8]. In this paper we characterize the relationship between the matrix and biased-graphic representations of the 3-connected matroids that are members of both of these important classes. While on the surface there is no obvious reason that there should be any such relationship, these representations are in fact inherently linked.

For a vertex v in a graph G, denote by $\mathrm{star}(v)$ the set of edges incident to v whose other end is in $V(G)-\{v\}$ (thus $\mathrm{star}(v)$ contains no loops). Let M be a matroid. Suppose there is a graph G with $E(G) = E(M)$ such that (i) the edge set of each component of G has rank in M no larger than the number of its vertices, (ii) for each vertex $v \in V(G)$, $\mathrm{cl}_M(E(G-v)) \subseteq E(G)-\mathrm{star}(v)$, and (iii) no circuit of M induces a subgraph of G with more than two components. Such a graph G is a *framework* for M; a matroid that has a framework is *quasi-graphic* ([9] is the introductory paper). Setting $\mathcal{B} = \{C : C \text{ is a cycle of } G \text{ and a circuit of } M\}$ yields a *biased graph* $(G, \mathcal{B})$: that is, a graph together with a distinguished collection $\mathcal{B}$ of its cycles obeying a certain condition (precise definitions are given in Section 2). Cycles in $\mathcal{B}$ are said to be *balanced*, and are otherwise *unbalanced*.

Suppose M also has a representation as a matrix A over a field $\mathbb{F}$: the columns of A are indexed by $E(M)$ such that a subset of elements of M is independent precisely when its columns are linearly independent. That is, M is *linearly representable over* $\mathbb{F}$; we write $M = M(A)$. It is shown in [1] (and in [9] in the case that M is 3-connected) that when M is both linearly representable over a field and quasi-graphic, then M is either a *frame matroid* or a *lifted-graphic matroid*. When M is frame or lifted-graphic, the biased graph $(G, \mathcal{B})$ completely determines M. We write $M = F(G, \mathcal{B})$ when M is a frame matroid, and $M = L(G, \mathcal{B})$ when M is lifted-graphic, and say the biased graph $(G, \mathcal{B})$ *represents* M.

We ask the question: given a 3-connected matroid M, a matrix A over a field, and a biased graph $(G, \mathcal{B})$, such that $M = M(A) = F(G, \mathcal{B})$ (respectively, $M = M(A) = L(G, \mathcal{B})$), what relationship is there between A and $(G, \mathcal{B})$? In the remainder of this introduction we provide a brief high-level overview of our answer; precise definitions are given in Section 2.

1.1 Projective equivalence, gain graphs, and canonical representations

A pair of matrices representing a matroid over a field $\mathbb{F}$ are *projectively equivalent* if one may be obtained from the other via a sequence of the following elementary operations: interchange two rows, multiply a row by a nonzero element of $\mathbb{F}$, replace a row by its sum with another row, add or remove a zero row, interchange two columns (along with their labels), or multiply a column by a nonzero element of $\mathbb{F}$. (A pair of matrices are generally said to be *equivalent* if one may be obtained from the other via such a sequence along with possibly applying an automorphism of $\mathbb{F}$ to the entries of one of the matrices; our results do not require field automorphisms. Projective equivalence has sometimes been called *strong equivalence* in the literature.) Our main tool for showing projective equivalence of a pair of matrix representations is the following well-known result of Brylawski and Lucas.

Proposition 1.1 ([11], Proposition 6.3.12)**.** *Let A and B be $r \times n$ matrices over a field with the columns of each labelled, in order, by e_1, e_2, $\dots$, e_n, with $r \geq 1$. Then A and B are projectively equivalent representations of a matroid on $\{e_1, e_2, \dots, e_n\}$ if and only if there is a non-singular matrix T and a non-singular diagonal matrix S such that $TAS = B$.*

The connection between matrix and biased graph representations is provided by *gain graphs* (also known as *group-labelled graphs*). We show that if a matroid M has a representation as a matrix over a field $\mathbb{F}$ and has a representation as a biased graph $(G, \mathcal{B})$, then an element of $\mathbb{F}$ may be assigned to each edge of the graph G such that G becomes a gain graph, over either the additive or multiplicative group of $\mathbb{F}$, realizing $\mathcal{B}$. Conversely, Zaslavsky [19] showed that every such gain graph yields a *canonical matrix representation* A for M over $\mathbb{F}$, as follows.

When the gain graph is over the additive group of $\mathbb{F}$, A is the matrix with rows indexed by $V(G) \cup v_0$, where $v_0 \notin V(G)$, and columns indexed by $E(G)$ in which, if e is assigned element $g \in \mathbb{F}$, has distinct endpoints u, v, and is directed from u to v, then entry $a_{v_0 e} = g$, $a_{ue} = 1$, $a_{ve} = -1$, and all remaining entries in column e are zero; if e is a loop then $a_{v_0 e} = g$ and all remaining entries in column e are zero. Thus A consists of the oriented incidence matrix of G with one additional row containing the elements of $\mathbb{F}$ that are assigned to each edge $e \in E(G)$; we say A is a *canonical lift matrix*. When the gain graph is over the multiplicative group of $\mathbb{F}$, A is the matrix with rows indexed by $V(G)$ and columns indexed by $E(G)$ in which, if e is assigned element $g \in \mathbb{F}$, has distinct endpoints u, v, and is directed from u to v, then $a_{ue} = 1$, $a_{ve} = -g$, and the remaining entries in column e are zero; if e is a loop incident to vertex v then $a_{ve} = 1 - g$ and all other entries in column e are zero. We say A is a *canonical frame matrix*.

An assignment of elements of a group Γ to the oriented edges of a graph is a Γ-*gain function*. A graph equipped with a Γ-gain function is a Γ-*gain graph*. *Switching*, and when Γ is the additive group of a field, *scaling*, are two operations that may be applied to a Γ-gain function to obtain a new Γ-gain function on the same graph. These operations re-assign elements of Γ to $E(G)$ in such a way that the collection of cycles in $\mathcal{B}$ determined by the new gain function remains unchanged. Thus the biased graph determined by a pair of gain graphs, one resulting from a switching or scaling operation applied to the other, remains the same. So it makes sense to consider switching and scaling as biased graphic analogs to the elementary matrix operations for projective equivalence. In fact, as we shall see, the connection between switching and scaling on gain graphs and projectively equivalent matrix representations is deeper than merely analogous.

As will be clear by the citations in Section 2, the notions of frame matroids, lifted-graphic matroids, their representations by biased graphs, gain graphs, their associated canonical matrix representations, and the operations of switching and scaling, are all due to Thomas Zaslavsky, presented in his seminal series of papers [15, 17, 18, 19].

1.2 Main results

Switching and scaling naturally partition the set of Γ-gain functions on a graph into equivalence classes in which two gain functions are equivalent if one may be obtained from the other by switching, or when Γ is the additive group of a field, by switching and scaling. When $(G, \mathcal{B})$ is a biased graph realized by a Γ-gain graph, these are the *switching classes* of Γ-realizations of $(G, \mathcal{B})$. When the group is the additive group $\mathbb{F}^+$ or multiplicative group $\mathbb{F}^\times$ of a field $\mathbb{F}$, it is straightforward to show that gain functions belonging to the same switching class yield projectively equivalent canonical matrix representations [19]. In [19] Zaslavsky conjectured that the converse also holds. Leaving aside some technicalities, Conjectures 2.8 and 4.8 of [19] are essentially as follows.

Conjecture 1 (Zaslavsky [19])**.** *Let G be a graph and let $\mathbb{F}$ be a field. The canonical frame matrices given by two $\mathbb{F}^\times$-gain functions φ and ψ on G are projectively equivalent if and only if φ and ψ are switching equivalent.*

Conjecture 2 (Zaslavsky [19])**.** *Let G be a graph and let $\mathbb{F}$ be a field. The* *canonical lift matrices given by two $\mathbb{F}^+$-gain functions φ and ψ on G are projectively equivalent if and only if φ and ψ are switching-and-scaling equivalent.*

A *joint* is an unbalanced loop in a biased graph. It is a property of the switching operation that the gain on a joint remains unchanged by every switching operation (provided Γ is abelian; more precisely, the gain on a joint is conjugated by switching). Thus one of the technicalities to be dealt with in Conjectures 1 and 2 is the following. Let φ and ψ be $\mathbb{F}^\times$-gain functions on a graph G. Suppose $\varphi(e) = \psi(e)$ for every edge e that is not a joint, but that there is a joint e' for which $\varphi(e') \neq \psi(e')$ and neither $\varphi(e')$ nor $\psi(e')$ is zero. Since a switching operation can alter neither $\varphi(e')$ nor $\psi(e')$, φ and ψ are not switching equivalent. Yet clearly the canonical matrices defined by φ and ψ are projectively equivalent, since each of their columns representing the element e' may be scaled so that the single nonzero entry in each is equal to any element of $\mathbb{F}^\times$. Our first main result is that for gain graphs representing 3-connected matroids, this issue with gains on loops provides the only counterexamples to Conjectures 1 and 2.

Theorem 1. *Let M be a 3-connected matroid of rank greater than two. Let $(G, \mathcal{B})$ be a loopless biased graph representing M and let $\mathbb{F}$ be a field.*

(i) *The canonical frame matrices given by two $\mathbb{F}^\times$-gain functions φ and ψ realizing $(G, \mathcal{B})$ are projectively equivalent if and only if φ and ψ are switching equivalent.*

(ii) *The canonical lift matrices given by two $\mathbb{F}^+$-gain functions φ and ψ realizing $(G, \mathcal{B})$ are projectively equivalent if and only if φ and ψ are switching-and-scaling equivalent.*

(iii) *Provided $(G, \mathcal{B})$ has no vertex whose deletion leaves a biased graph with no unbalanced cycles, no canonical frame matrix representation is projectively equivalent to any canonical lift matrix representation of M.*

In fact, we can say more. Theorem 1 follows from the stronger statements of our Theorems 5.1 and 5.4. Together these also give necessary and sufficient conditions for projective equivalence of a canonical lift matrix and a canonical frame matrix representation. Examples that are not 3-connected for which the conclusions of Theorem 1 do not hold are not difficult to construct, so the hypothesis that M be 3-connected is necessary.

Let M be a linearly representable frame or lifted-graphic matroid. Not only may M have projectively inequivalent matrix representations arising as canonical matrix representations from the same biased graph, but there may also be different biased graphs representing M. We say a matrix representation is *particular to* the biased graph $(G, \mathcal{B})$ when it is a canonical matrix arising from a gain function realizing $(G, \mathcal{B})$. Zaslavsky has conjectured the following.

Conjecture 3 ([17])**.** *Let $(G,\mathcal{B})$ be a biased graph, where G is sufficiently connected, and let $\mathbb{F}$ be a field. If $F(G,\mathcal{B})$ (resp., $L(G,\mathcal{B})$) is linearly representable over $\mathbb{F}$ then $F(G,\mathcal{B})$ (resp., $L(G,\mathcal{B})$) has a canonical representation particular to $(G,\mathcal{B})$.*

Zaslavsky subsequently further conjectured the following.

Conjecture 4 (Zaslavsky, personal communication)**.** *Let $(G,\mathcal{B})$ be a biased graph, where G is sufficiently connected, and let $\mathbb{F}$ be a field. Every $\mathbb{F}$-representation of $F(G,\mathcal{B})$ (respectively $L(G,\mathcal{B})$) is projectively equivalent to a canonical representation.*

Geelen, Gerards, and Whittle prove Conjecture 4 for 3-connected matroids in [7], though the result is not explicitly stated (it appears in the proof of their Theorem 1.4). Surprisingly, a result even stronger than Conjectures 3 and 4 holds:

Theorem 2. *Let M be a 3-connected matroid, and let $\mathbb{F}$ be a field. Let A be a matrix over $\mathbb{F}$ representing M and let $(G,\mathcal{B})$ be a biased graph representing M. Then A is projectively equivalent to a canonical representation particular to $(G,\mathcal{B})$.*

Theorem 2 follows from the stronger statement of Theorem 5.5. Together Theorems 5.1, 5.4, and 5.5 imply our next main result. We need just a few more definitions before we can state it precisely. A biased graph is *balanced* when all of its cycles are balanced, *almost-balanced* when it is not balanced but there is a vertex that is contained in every unbalanced cycle of length at least two, and *properly unbalanced* otherwise. A properly unbalanced biased graph with no pair of vertex-disjoint unbalanced cycles is *tangled*. If $(G,\mathcal{B})$ is balanced, then both $F(G,\mathcal{B})$ and $L(G,\mathcal{B})$ are equal to the cycle matroid $M(G)$ of G. Since $M(G)$ has a projectively unique matrix representation over every field, and every gain function realizing a balanced biased graph is switching equivalent to the gain function assigning the group identity to every element (this follows from Proposition 2.1 below), Conjectures 1-4 hold in this case. Thus we just need consider almost-balanced and properly unbalanced biased graphs. The collection of unbalanced cycles in almost-balanced biased graphs is highly structured and well understood (see [5]). For each almost-balanced biased graph $(G,\mathcal{B})$ there is a family of almost-balanced biased graphs $\mathcal{R}_{(G,\mathcal{B})}$, each of which represents the frame matroid $F(G,\mathcal{B})$. We denote the unique biased graph in $\mathcal{R}_{(G,\mathcal{B})}$ with the least number of loops by $(\widehat{G},\widehat{\mathcal{B}})$; this is also the unique biased graph in the collection for which $F(\widehat{G},\widehat{\mathcal{B}}) = L(\widehat{G},\widehat{\mathcal{B}})$.

Let M be a 3-connected matroid with rank greater than two. Let $\mathbb{F}$ be a field and let $(G, \mathcal{B})$ be a biased graph representing M. Let $\mathcal{S}_{\mathbb{F}^\times}(G, \mathcal{B})$ denote the collection of switching classes of $\mathbb{F}^\times$-gain functions realizing $(G, \mathcal{B})$, and let $\mathcal{S}_{\mathbb{F}^+}(G, \mathcal{B})$ denote the collection of switching-and-scaling classes of $\mathbb{F}^+$-gain functions realizing $(G, \mathcal{B})$. For each biased graph representing M and each field $\mathbb{F}$, define

$$\mathcal{S}_{\mathbb{F}}(G, \mathcal{B}) = \begin{cases} \mathcal{S}_{\mathbb{F}^\times}(G, \mathcal{B}) & \text{if } M = F(G, \mathcal{B}) \neq L(G, \mathcal{B}), \\ \mathcal{S}_{\mathbb{F}^+}(G, \mathcal{B}) & \text{if } M = L(G, \mathcal{B}) \neq F(G, \mathcal{B}), \\ \mathcal{S}_{\mathbb{F}^\times}(G, \mathcal{B}) \cup \mathcal{S}_{\mathbb{F}^+}(G, \mathcal{B}) & \text{if } (G, \mathcal{B}) \text{ is tangled}, \\ \mathcal{S}_{\mathbb{F}^+}(\widehat{G}, \widehat{\mathcal{B}}) & \text{if } (G, \mathcal{B}) \text{ is almost-balanced}. \end{cases}$$

Theorem 3. *Let M be a 3-connected matroid of rank greater than two. Let $\mathbb{F}$ be a field and let $(G, \mathcal{B})$ be a loopless biased graph representing M. The projective equivalence classes of matrices over $\mathbb{F}$ representing M are in one-to-one correspondence with the switching classes of gain functions in $\mathcal{S}_{\mathbb{F}}(G, \mathcal{B})$.*

The proofs of our main results use an inductive argument. For this purpose we determine a small collection of biased graphs, at least one of which must occur as a biased topological subgraph in every 2-connected biased graph. This investigation yields three results on unavoidable minors and topological minors of biased graphs that are of independent interest.

The graph obtained from a 3-cycle by replacing each edge with a pair of parallel edges is denoted by $2C_3$. The graph obtained from a 4-cycle by replacing each edge in a pair of non-adjacent edges with a pair of parallel edges is the *tube graph*, denoted by $2C_4''$. We show that the collection $\mathcal{G}_0$ consisting of the six biased $2C_3$'s with no balanced 2-cycle, the three biased tubes with no balanced 2-cycle, and the four biased K_4's with no balanced triangle, is the set of biased graphs that are minor-minimal amongst all 2-connected, properly unbalanced biased graphs. Contraction of a loop is never required to obtain one of these minors; such a minor is a *link minor*.

Theorem 4. *Every 2-connected properly unbalanced biased graph contains a biased graph in $\mathcal{G}_0$ as a link minor.*

We prove an analogue of Theorem 4 for biased topological subgraphs. Let P denote the biased graph consisting of the triangular prism with just its two triangles balanced (Figure 5 on page 25). Let $\mathcal{T}_0$ be the set of biased graphs consisting of those in $\mathcal{G}_0$ together with P and the two biased graphs obtained from P by contracting 1 and 2 edges, respectively, of the matching in P linking its two triangles.

Theorem 5. *Every 2-connected properly unbalanced biased graph contains as a biased subgraph a subdivision of a biased graph in $\mathcal{T}_0$.*

Theorems 4 and 5 are useful because switching inequivalence of gain functions over abelian groups can always be found (in the interesting case that there are no joints) on a small minor. This is the content of our final main result. Two biased graphs representing the 4-point line $U_{2,4}$ as a frame or lifted-graphic matroid are shown in Figure 7 on page 27; we denote these biased graphs by U_2 and U_3.

Theorem 6. *Let $(G, \mathcal{B})$ be a 2-connected, loopless, and properly unbalanced biased graph. Let Γ be an abelian group and let φ and ψ be Γ-realizations of $(G, \mathcal{B})$. Assume that φ and ψ are not switching equivalent. In the case that Γ is the additive group of a field, assume that φ and ψ are not switching-and-scaling equivalent. Then either*

(i) *$(G, \mathcal{B})$ has a link minor $(H, \mathcal{S}) \in \mathcal{G}_0$ such that the gain functions induced by φ and ψ on $E(H)$ are not switching equivalent (resp., not switching-and-scaling equivalent), or*

(ii) *$(G, \mathcal{B})$ has U_2 as a minor and U_3 as a link minor such that the gain functions induced by φ and ψ are not switching equivalent (resp., not switching-and-scaling equivalent) on the 2-cycle of U_2 nor on the theta subgraph of U_3.*

The remainder of the paper is structured as follows. In Section 2 we provide necessary preliminary notions. In Section 3 we prove Theorems 4, 5, and 6 on unavoidable minors and unavoidable biased topological subgraphs. In Section 4 we show that Theorems 1 and 2 hold for the set of unavoidable minors, and finally in Section 5 we prove Theorems 1, 2, and 3.

2 Preliminaries

We assume that the reader is familiar with matroid theory as in Oxley's standard text [11]. In this section we summarize those notions that are central to the results of this paper or are not standard in the literature, and introduce some required notation.

2.1 Graphs and biased graphs

Let G be a graph. We denote the subgraph of G induced by a subset $X \subseteq E(G)$ by $G[X]$. The set of vertices of $G[X]$ is denoted by $V(X)$. A *k-separation* of G is a partition (A, B) of $E(G)$ with $|A| \geq k$, $|B| \geq k$, and $|V(A) \cap V(B)| = k$. A *vertical k-separation* of G is a k-separation (A, B) of G with both $V(A) - V(B)$ and $V(B) - V(A)$ non-empty. A graph on at least $k+2$ vertices is *k-connected* if it has no vertical l-separation for any $l < k$. A graph on $k+1$ vertices is said to be k-connected if it has a spanning complete subgraph. Thus a highly connected graph may contain loops or parallel edges. We often need to distinguish between edges with distinct endpoints and loops; an edge with distinct endpoints is a *link*.

A *biased graph* is a pair $(G, \mathcal{B})$ where G is a graph and $\mathcal{B}$ is a collection of cycles of G with the property that no theta subgraph of G contains exactly two cycles in $\mathcal{B}$; a *theta* graph is the union of three internally disjoint paths linking a pair of vertices. Such a collection $\mathcal{B}$ is said to satisfy the *theta property*. Cycles in $\mathcal{B}$ are *balanced*; cycles not in $\mathcal{B}$ are *unbalanced*. A biased graph is *balanced* if all cycles are balanced, *unbalanced* if it contains an unbalanced cycle and *contrabalanced* if no cycle is balanced. Similarly, a subset of edges or a subgraph is *balanced*, *unbalanced*, or *contrabalanced*, according to whether all, not all, or none, respectively, of the cycles it induces or contains are balanced. A vertex v is a *balancing vertex* if every unbalanced cycle contains v. A biased graph $(G, \mathcal{B})$ is *k-connected* if G is k-connected. For two biased graphs $(G, \mathcal{B})$ and $(H, \mathcal{S})$ an *isomorphism* $\iota \colon (G, \mathcal{B}) \to (H, \mathcal{S})$ consists of an underlying graph isomorphism $\iota \colon G \to H$ that takes $\mathcal{B}$ to $\mathcal{S}$. We sometimes write $\Omega = (G, \mathcal{B})$ and speak of the biased graph Ω when there is no need to be explicit about the underlying graph G and its collection of balanced cycles $\mathcal{B}$.

We denote the set of all cycles in G by $\mathcal{C}(G)$. Let $(G, \mathcal{B})$ be a biased graph and e an edge in G. Define $(G, \mathcal{B}) \backslash e = (G \backslash e, \mathcal{B}|_{G \backslash e})$ where $\mathcal{B}|_{G \backslash e} = \mathcal{B} \cap \mathcal{C}(G \backslash e)$. If e is a link, then define $(G, \mathcal{B})/e = (G/e, \mathcal{B}|_{G/e})$ where $\mathcal{B}|_{G/e} = \{C \in \mathcal{C}(G/e) : C \in \mathcal{B} \text{ or } C \cup e \in \mathcal{B}\}$. If e is a balanced loop, then $(G, \mathcal{B})/e = (G, \mathcal{B}) \backslash e$. In order that contraction of a joint e of $(G, \mathcal{B})$ remain consistent with the operation of contraction of e in the lifted-graphic or frame matroid represented by $(G, \mathcal{B})$, two different contraction operations in $(G, \mathcal{B})$ are required, depending upon which matroid $(G, \mathcal{B})/e$ is to represent. Let e be a joint of $(G, \mathcal{B})$, incident to vertex v. To obtain a biased graph representing $L(G, \mathcal{B})/e$, define $(G, \mathcal{B})/e = (G \backslash e, \mathcal{C}(G \backslash e))$. To obtain a biased graph representing $F(G, \mathcal{B})/e$, define $(G, \mathcal{B})/e = (G', \mathcal{B}')$ where G' is obtained from G by adding each loop $e' \neq e$ incident to v to $\mathcal{B}$ and replacing each link f incident to v with a joint incident to its other endpoint. The collection $\mathcal{B}'$ is $\mathcal{B}$ restricted to the subgraph

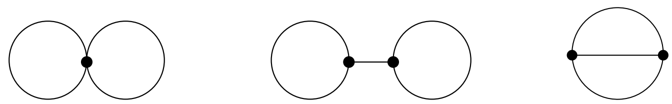


Figure 1: Circuits of the frame matroid.

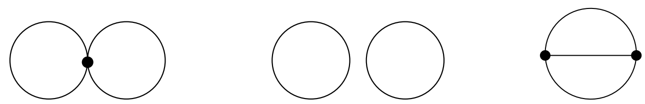


Figure 2: Circuits of the lift matroid.

$G-v$ along with any new balanced loops incident to v. Whenever contracting a joint, we will be explicit about which contraction operation is used, *lift-type* or *frame-type*, respectively. In fact, we will only require contraction of a joint once; this will be a frame-type contraction. All other minors we consider may be obtained without contracting joints. Such a minor is a *link minor*. We permit deletion of isolated vertices and do so without comment.

2.2 Matroids arising from biased graphs

A *frame matroid* is a matroid M to which a basis B may be added such that each $e \in E(M)$ is contained in the closure of some 2-element subset of B. A subset $C \subseteq E(G)$ is a circuit of the frame matroid $F(G,\mathcal{B})$ precisely when $C \in \mathcal{B}$ or C induces a subdivision of one of the graphs shown in Figure 1 containing no balanced cycle [17]. A matroid M is *lifted-graphic* if there is a single-element extension M_0 of M by an element e_0 such that M_0/e_0 is graphic. Dualizing Crapo's characterization [4] of single-element extensions of matroids, Zaslavsky observed [17] that if G is a graph such that $M_0/e_0 = M(G)$ then M has a natural description in terms of a biased graph $(G,\mathcal{B})$. A subset $C \subseteq E(G)$ is a circuit of the lifted-graphic matroid $L(G,\mathcal{B})$ precisely when $C \in \mathcal{B}$ or C induces a subdivision of one of the subgraphs shown in Figure 2 containing no balanced cycle [17]. Minors of biased graphs and their matroids agree: for any edge e of a biased graph $(G,\mathcal{B})$, $F(G,\mathcal{B})\backslash e = F((G,\mathcal{B})\backslash e)$, $F(G,\mathcal{B})/e = F((G,\mathcal{B})/e)$, $L(G,\mathcal{B})\backslash e = L((G,\mathcal{B})\backslash e)$, for any link e, $L(G,\mathcal{B})/e = L((G,\mathcal{B})/e)$, and if e is a joint then $L(G,\mathcal{B})/e = M(G\backslash e)$ [17].

Evidently whenever $(G,\mathcal{B})$ is a biased graph with no two vertex-disjoint unbalanced cycles, $F(G,\mathcal{B}) = L(G,\mathcal{B})$. For notational convenience and to avoid potential confusion, in the case that $(G,\mathcal{B})$ does not have a pair of vertex-disjoint unbalanced cycles we denote the matroid on ground set $E(G)$ that is equal to both $F(G, \mathcal{B})$ and $L(G, \mathcal{B})$ by $M(G, \mathcal{B})$.

2.3 Gain graphs

A *gain graph* is obtained from a graph by assigning a direction and an element of a group (a *gain*) to each edge of G [15]. Let G be a graph and let Γ be a group. Orient the edges of G by arbitrarily assigning a direction to each edge: for each edge $e \in E(G)$ choose one of its ends for its *tail* $\mathbf{t}(e)$. The other end of e is its *head* $\mathbf{h}(e)$; if e is a loop then $\mathbf{h}(e) = \mathbf{t}(e)$. We specify an oriented edge e by the ordered triple $(e; u, v)$ where $u = \mathbf{t}(e)$ and $v = \mathbf{h}(e)$; then e^{-1} is the ordered triple $(e; v, u)$. We think of e^{-1} as the oriented edge e traversed in the reverse direction. The collection of ordered triples $\{(e; u, v) : e \in E(G)$ has ends $u, v\}$, consisting of all oriented edges of G and their inverse orientations is denoted by $\vec{E}(G)$. As long as no confusion may arise, we write e for both the edge $e \in E(G)$ and for an ordered triple $(e; u, v) \in \vec{E}(G)$ specifying a direction of traversal of e. Let Γ be a group, and let $\gamma : \vec{E}(G) \to \Gamma$ be a function satisfying the following condition: for each link e, if Γ is multiplicative then $\gamma(e^{-1}) = \gamma(e)^{-1}$ while if Γ is additive then $\gamma(e^{-1}) = -\gamma(e)$. Such a function is a Γ-*gain function on* G; the pair (G, γ) is a Γ-*gain graph*.

A gain graph (G, γ) naturally gives rise to a biased graph $(G, \mathcal{B}_\gamma)$, where the membership of each cycle in the collection of balanced cycles $\mathcal{B}_\gamma$ is determined by the gains assigned to its edges, as follows. Let $W = (v_1, e_1, v_2, \ldots, v_n, e_n, v_{n+1})$, where for each i, v_i, v_{i+1} are the ends of edge e_i, be a walk in G. Define $\gamma(W) = \gamma(e_1; v_1, v_2)\gamma(e_2; v_2, v_3) \cdots \gamma(e_n; v_n, v_{n+1})$ when Γ is multiplicative and $\gamma(W) = \gamma(e_1; v_1, v_2) + \gamma(e_2; v_2, v_3) + \cdots + \gamma(e_n; v_n, v_{n+1})$ when Γ is additive. For each cycle C of G choose a closed Eulerian walk W_C in C, and define C to be *balanced* with respect to γ if $\gamma(W_C)$ is the identity element of Γ. Let $\mathcal{B}_\gamma$ be the collection of cycles in G that are balanced with respect to γ. Observe that if W'_C is another closed Eulerian walk in C then $\gamma(W_C)$ and $\gamma(W'_C)$ are conjugate. Thus $\mathcal{B}_\gamma$ is well-defined. If $(G, \mathcal{B})$ is a biased graph and γ is a Γ-gain function such that $\mathcal{B}_\gamma = \mathcal{B}$, then we say γ *realizes* $(G, \mathcal{B})$ and that γ is a Γ-*realization* of $(G, \mathcal{B})$.

Switching and scaling. Given a Γ-gain function γ on G and a function $\eta \colon V(G) \to \Gamma$, define the gain function γ^η by $\gamma^\eta(e) = \eta(\mathbf{t}(e))^{-1} \cdot \gamma(e) \cdot \eta(\mathbf{h}(e))$ if Γ is multiplicative and by $\gamma^\eta(e) = -\eta(\mathbf{t}(e)) + \gamma(e) + \eta(\mathbf{h}(e))$ if Γ is additive. The function η is a *switching function*. It is straightforward to check that a cycle C is balanced with respect to γ if and only if C is balanced with respect to γ^η, so

$\mathcal{B}_\gamma = \mathcal{B}_{\gamma^\eta}$. Observe also that for switching functions η_1 and η_2, $(\gamma^{\eta_1})^{\eta_2} = \gamma^{\eta_1\eta_2}$ or $\gamma^{\eta_1+\eta_2}$. Two Γ-gain functions φ and ψ are *switching equivalent* if there is a switching function η such that $\varphi^\eta = \psi$. In the case that Γ is the additive group of a field $\mathbb{F}$, we may choose any nonzero element $a \in \mathbb{F}$ and obtain a new gain function $a\varphi$ defined by $a \cdot \varphi(e)$ for each edge e. Clearly $\mathcal{B}_{a\varphi} = \mathcal{B}_\varphi$. We say $a\varphi$ is obtained from φ by *scaling*. When Γ is the additive group of a field, we say two gain functions φ and ψ are *switching-and-scaling equivalent* if there is a switching function η and scalar $a \in \mathbb{F}^\times$ such that $a\varphi^\eta = \psi$. Evidently, for a multiplicative group Γ the relation of being switching equivalent partitions the collection of Γ-gain functions on a graph into equivalence classes, its *switching classes*. Similarly, when Γ is the additive group of a field the relation the being switching-and-scaling equivalent partitions the collection of Γ-gain functions on a graph into its *switching-and-scaling classes*. Propositions 2.1 and 2.2 are immediate.

Proposition 2.1. *Let F be a forest in a graph G and let γ be a Γ-gain function on G. There is a switching function η such that $\gamma^\eta(e)$ is the identity for all edges e in F.*

Let F be a forest of a graph. A gain function γ is said to be *F-normalized* when $\gamma(e)$ is the identity for all edges e in F.

Proposition 2.2. *Let Γ be an abelian group, G be a graph, and F a maximal forest in G. If φ and ψ are two F-normalized Γ-gain functions on G, then φ and ψ are switching equivalent if and only if $\varphi = \psi$. In the case that Γ is the additive group of a field $\mathbb{F}$, φ and ψ are switching-and-scaling equivalent if and only if $\varphi = a\psi$ for some scalar $a \in \mathbb{F}^\times$.*

Minors and induced gain functions. Minors of gain graphs are defined so that they are consistent with those of their corresponding biased graphs. Let Γ be a group, and let (G, γ) be a Γ-gain graph. Every minor H of G has an *induced* Γ-gain function $\gamma|_H$ inherited from γ. Moreover, whenever $(H, \mathcal{S})$ is a biased graph that is a minor of $(G, \mathcal{B}_\gamma)$ then $(H, \mathcal{S})$ is realized by the induced gain function $\gamma|_H$ on $E(H)$ inherited from γ. We now define these notions and justify this claim.

Let e be an edge of G. We denote by $(G, \gamma)\backslash e$ and $(G, \gamma)/e$ the gain graphs obtained by deletion and contraction of e with their induced gain functions defined as follows. The induced gain function $\gamma|_{G\backslash e}$ on $G\backslash e$ is the restriction of γ to $E(G) - e$. If e is a loop assigned the identity element of Γ by γ, then $G/e = G\backslash e$ so again the induced gain function is just the restriction of γ to $E(G) - e$. If e is a link, then there is switching function η such that $\gamma^\eta(e)$ is the identity element of Γ. Define the induced gain function $\gamma|_{G/e}$ on G/e to be the restriction of γ^η to $E(G)-e$. Finally, suppose e is a loop with $\gamma(e)$ not equal to the identify element of Γ. Suppose e is incident to vertex $v \in V(G)$. For a lift-type contraction of e, $(G,\gamma)/e$ is the gain graph $(G\backslash e, \iota)$ where ι is the gain function assigning the identity element of Γ to every edge; declare ι to be the induced gain function on $(G,\gamma)/e$. For a frame-type contraction of e, $(G,\gamma)/e$ is the gain graph $(G', \gamma|_{G'})$ in which G' is the underlying graph of the biased graph $(G,\mathcal{B}_\gamma)/e$ obtained by the frame-type contraction of the joint e defined in Section 2.1 above, and $\gamma|_{G'}$ is the gain function whose restriction to $E(G-v)$ is equal to the restriction of γ to $E(G-v)$, that assigns the identity element of Γ to each balanced loop of $(G,\mathcal{B}_\gamma)/e$, and assigns $\gamma(e)$ to each new joint of $(G,\mathcal{B}_\gamma)/e$.

For link minors, induced gain functions can be defined globally (up to switching) as follows. Consider a biased graph $(G,\mathcal{B})$ and two disjoint subsets $K, D \subseteq E(G)$ where K does not contain any loops. Let $K' \subseteq K$ be a set of edges that induce a maximal forest in $G[K]$, and let $D' = D \cup (K\backslash K')$. Then $(G,\mathcal{B})/K\backslash D = (G,\mathcal{B})/K'\backslash D'$. Thus a link minor may always be obtained by contraction of an acyclic set. Consider a gain graph (G,γ) with disjoint subsets $K, D \subseteq E(G)$ where K does not contain a loop. We obtain an induced gain function $\gamma|_{G/K\backslash D}$ for $G/K\backslash D$ as follows. Choose a subset $K' \subseteq K$ such that $G[K']$ is a maximal forest contained in $G[K]$, and choose a maximal forest F of G containing K'. Let γ^η be the F-normalization of γ. Define $\gamma|_{G/K\backslash D}$ to be the restriction of γ^η to $E(G)-(K\cup D)$.

Proposition 2.3. *Let G be a graph and let F be the edge set of a forest in G. Let Γ be an abelian group (resp., the additive group of a field). Then φ and ψ are switching equivalent (resp., switching-and-scaling equivalent) Γ-gain functions on G if and only if $\varphi|_{G/F}$ and $\psi|_{G/F}$ are switching equivalent (resp., switching-and-scaling equivalent).*

Proof. Extend F to a maximal forest F_m in G and assume that φ and ψ are normalized on F_m. If φ and ψ are switching inequivalent (resp., switching-and-scaling inequivalent), then certainly $\varphi \neq \psi$ (resp., $\varphi \neq a\psi$ for any scalar a), and since both are normalized on F_m, their restrictions to $E(G)\backslash F_m$ are not equal (resp., neither is obtained by scaling the other). Now in G/F, the induced gain functions $\varphi|_{G/F}$ and $\psi|_{G/F}$ are normalized on the maximal forest $F_m\backslash F$ of G/F and $\varphi|_{G/F} \neq \psi|_{G/F}$. Thus by Proposition 2.2 $\varphi|_{G/F}$ and $\psi|_{G/F}$ are switching inequivalent (resp., switching-and-scaling inequivalent) on G/F.

Conversely, if φ and ψ are switching equivalent (resp., switching-and-scaling equivalent), then by Proposition 2.2, $\varphi = \psi$ (resp., $\varphi = a\psi$ for some scalar a), so $\varphi|_{G/F} = \psi|_{G/F}$ (resp., $\varphi|_{G/F} = a\psi|_{G/F}$). □

We say that biased graph $(G, \mathcal{B})$ has an $(H, \mathcal{S})$-minor (respectively, $(H, \mathcal{S})$-link minor) when there is a minor (resp., link minor) $(G', \mathcal{B}')$ of $(G, \mathcal{B})$ that is isomorphic to $(H, \mathcal{S})$. The following proposition now follows immediately from the definitions.

Proposition 2.4. *If γ is a Γ-realization of $(G, \mathcal{B})$ and $(G', \mathcal{B}')$ is a minor of $(G, \mathcal{B})$, then the induced gain function $\gamma|_{G'}$ is a Γ-realization of $(G', \mathcal{B}')$.*

When γ is a Γ-realization of $(G, \mathcal{B})$ and $(G', \mathcal{B}')$ is a minor of $(G, \mathcal{B})$, we say the Γ-realization $\gamma|_{G'}$ of $(G', \mathcal{B}|_{G'})$ is the *induced* Γ*-realization* of $(G', \mathcal{B}|_{G'})$.

Minors and canonical representations. Given a matroid M represented over a field by a matrix A, the operation of removing column e from A yields a matrix representation of $M \backslash e$. The operation of applying row operations so that column e contains a unique nonzero element equal to 1 and then removing column e along with the row in which column e is nonzero yields a matrix representation of M/e. With just a little more care, we may apply these usual operations to a canonical matrix representation to obtain a canonical matrix representation given by the corresponding induced gain function on the gain graph minor. For a canonical matrix A, and column e of A, denote by $A \backslash e$ the matrix obtained by removing column e from A. Denote by A/e a matrix obtained by applying row operations so that column e contains a unique nonzero element equal to 1, say in row i, and then removing column e and row i from A, subject to the following. If A is a canonical lift matrix and e is a link, then row i is not the "gains row" v_0. If A is a canonical frame matrix and if e is a joint, then also scale columns so that for each column $e' \neq e$ with a nonzero entry in row i and with a second nonzero entry in a row $j \neq i$, entry $A_{je'}$ is equal to $1 - \gamma(e)$.

Given an $\mathbb{F}^{\times}$-gain function φ on a graph G, denote by $A_F(G, \varphi)$ the canonical frame matrix defined by (G, φ) as described in Section 1.1. Similarly, for an $\mathbb{F}^{+}$-gain function ψ on G, denote by $A_L(G, \psi)$ the canonical lift matrix defined by (G, ψ), as described in Section 1.1. The following lemma is a straightforward consequence of the definitions.

Lemma 2.5. *Let G be a graph, let $\mathbb{F}$ be a field, and let $\varphi\colon \vec{E}(G) \to \mathbb{F}^{\times}$ and $\psi\colon \vec{E}(G) \to \mathbb{F}^{+}$ be gain functions. Let $e \in E(G)$.*

(i) *$A_F((G, \varphi)\backslash e) = A_F(G, \varphi)\backslash e$ and $A_L((G, \varphi)\backslash e) = A_L(G, \varphi)\backslash e$.*

(ii) *$A_F((G, \varphi)/e) = A_F(G, \varphi)/e$, where if e is a joint the contraction operation is of frame type.*

(iii) *$A_L((G, \psi)/e) = A_L(G, \psi)/e$, where if e is a joint the contraction operation is of lift type.*

Evidently, if A and B are projectively equivalent matrices over $\mathbb{F}$, then so too are $A\backslash e$ and $B\backslash e$ projectively equivalent, as are A/e and B/e.

2.4 Δ-Y and Y-Δ exchanges

The operations of Δ-Y and Y-Δ exchange in graphs and in matroids are well understood. Here we generalize these operations from graphs to biased graphs, and show that an exchange in a biased graph representation agrees with that in the matroid. We apply Δ-Y and Y-Δ exchanges in a biased graph $(G, \mathcal{B})$ by performing the usual operation in G, then appropriately defining a collection of balanced cycles to define the resulting biased graph. We use these tools in Sections 4 and 5.

Let $X = \{a, b, c\}$ be the edge set of a triangle T in a graph G, with $V(T) = \{x, y, z\}$, where $a = yz$, $b = xz$, and $c = xy$. Delete edges a, b, c from G, add a new vertex v to G, along with three new edges vx, vy, and vz, and label the new edges a, b, c so that $a = vx$, $b = vy$, and $c = vz$. The resulting graph, in which the triangle X is replaced with a $K_{1,3}$-subgraph with the same set of edges, is said to have been obtained from G by a Δ-Y *exchange*, and is denoted by $\Delta_X G$.

The reverse operation is a Y-Δ *exchange*: Let v be vertex of degree three in a graph G, whose incident edges $Y = \{a, b, c\}$ induce a $K_{1,3}$-subgraph of G. Let x, y, z be the neighbours of v in G, where $a = vx$, $b = vy$, and $c = vz$. Delete v along with its incident edges a, b, c, and add three new edges xy, yz, and xz, and label the new edges a, b, c so that $a = yz$, $b = xz$, and $c = xy$. The resulting graph, in which the $K_{1,3}$-subgraph Y is replaced with a triangle with the same set of edges, is said to have been obtained from G by a Y-Δ *exchange*, and is denoted by $\nabla_Y G$.

We extend these operations to biased graphs as follows. Let $(G, \mathcal{B})$ be a biased graph and let X be a balanced triangle of G. Define $\Delta_X \mathcal{B}$ to be the collection of cycles of $\Delta_X G$ given by

$$\{C \in \mathcal{B} : |C \cap X| = 0 \text{ or } 2\} \cup \{C \triangle X : C \in \mathcal{B} \text{ and } |C \cap X| = 1\}$$

where $\triangle$ denotes symmetric difference. Let Y be a $K_{1,3}$-subgraph of $(G, \mathcal{B})$. Define $\nabla_Y \mathcal{B}$ to be the collection of cycles of $\nabla_Y G$ consisting of Y together with the minimal nonempty members of the set

$$\{C : C \in \mathcal{B} \text{ and } |C \cap Y| = 0 \text{ or } 2\} \cup \{C \triangle Y : C \in \mathcal{B} \text{ and } |C \cap Y| = 2\}.$$

The proofs of the following two propositions are straightforward checks.

Proposition 2.6. *Let X be a balanced 3-cycle in a biased graph $(G, \mathcal{B})$. Then $(\Delta_X G, \Delta_X \mathcal{B})$ is a biased graph.*

Proposition 2.7. *Let Y be a $K_{1,3}$-subgraph of a biased graph $(G, \mathcal{B})$. Then $(\nabla_Y G, \nabla_Y \mathcal{B})$ is a biased graph.*

We denote the biased graph $(\Delta_X G, \Delta_X \mathcal{B})$ by $\Delta_X(G, \mathcal{B})$ and the biased graph $(\nabla_Y G, \nabla_Y \mathcal{B})$ by $\nabla_Y(G, \mathcal{B})$.

Observe that a Δ-Y exchange in a graph G is given by a 3-sum of a labelled K_4 and G. To perform a Δ-Y exchange in a matroid M, take a copy of $M(K_4)$ on ground set $\{a, b, c, a', b', c'\}$ labelled so that $\{a, b, c\}$ is a triangle, $\{a', b', c'\}$ is a triad, and each of $\{a, b', c'\}$, $\{a', b, c'\}$, and $\{a', b', c\}$ are triangles. Let $X = \{a, b, c\}$ be a triangle of M, and take the generalized parallel connection of M and $M(K_4)$ across X (see [11], Sections 11.4 & 11.5). Finally, delete X and relabel each of a', b', c' by a, b, c, respectively. The resulting matroid is said to have been obtained via a Δ-Y *exchange on* X, and is denoted $\Delta_X M$.

A Y-Δ exchange in a matroid M is defined via duality. Let Y be a triad of M. Then Y is a triangle of the dual M^*. Define $\nabla_Y M = (\Delta_Y M^*)^*$. We say $\nabla_Y M$ is obtained from M via a Y-Δ *exchange on* Y.

The following proposition is a straightforward consequence of the definitions. It can be proved by comparing flats.

Proposition 2.8. *Let $(G, \mathcal{B})$ be a biased graph.*

(i) *If X is a balanced triangle of $(G, \mathcal{B})$, then $F(\Delta_X(G, \mathcal{B})) = \Delta_X F(G, \mathcal{B})$ and $L(\Delta_X(G, \mathcal{B})) = \Delta_X L(G, \mathcal{B})$.*

(ii) *Let Y be a $K_{1,3}$-subgraph of G for which, if $(G, \mathcal{B})$ is unbalanced, $(G, \mathcal{B}) - E(Y)$ remains unbalanced. Then $F(\nabla_Y(G, \mathcal{B})) = \nabla_Y F(G, \mathcal{B})$ and $L(\nabla_Y(G, \mathcal{B})) = \nabla_Y L(G, \mathcal{B})$.*

The projective equivalence classes of matrix representations of a matroid are well-behaved under Δ-Y exchanges:

Proposition 2.9 (Whittle [13, Lemma 5.7])**.** *Let M' be a matroid obtained from the matroid M by a single Δ-Y exchange, and let $\mathbb{F}$ be a field.*

(i) *M is $\mathbb{F}$-representable if and only if M' is $\mathbb{F}$-representable.*

(ii) *The projective equivalence classes of $\mathbb{F}$-representations of M are in one-to-one correspondence with the projective equivalence classes of $\mathbb{F}$-representations of M'.*

Gain functions realizing a biased graph are similarly well-behaved under Δ-Y exchanges. Proposition 2.10 is an analogue of Proposition 2.9.

Proposition 2.10. *Let X be a balanced triangle of a biased graph $(G, \mathcal{B})$. Let Γ be the multiplicative (resp., additive) group of a field.*

(i) *φ is a Γ-realization of $(G, \mathcal{B})$ if and only if φ is a Γ-realization of $\Delta_X(G, \mathcal{B})$.*

(ii) *The switching (resp., switching-and-scaling) classes of Γ-realizations of $(G, \mathcal{B})$ are in one-to-one correspondence with the switching (resp., switching-and-scaling) classes of Γ-realizations of $\Delta_X(G, \mathcal{B})$.*

Proof. (i) Let F be a maximal forest of $\Delta_X G$ containing X. For each edge $e \in X$, $F - e$ is a maximal forest of G that contains two edges of X. By Proposition 2.2, every Γ-realization of $(G, \mathcal{B})$ is switching equivalent (resp., switching-and-scaling equivalent) to a unique (resp., unique up to scaling) $(F-e)$-normalized Γ-realization and every Γ-realization of $\Delta_X(G, \mathcal{B})$ is switching (resp., switching-and-scaling) equivalent to a unique (resp., unique up to scaling) F-normalized Γ-realization. Since X is a balanced triangle of $(G, \mathcal{B})$, every $(F - e)$-normalized Γ-realization of $(G, \mathcal{B})$ also has identity gain value on e. Thus a Γ-realization assigning identity gains to each edge in X is a Γ-realization of $(G, \mathcal{B})$ if and only if it is a Γ-realization of $\Delta_X(G, \mathcal{B})$. Thus we obtain a canonical bijection between between Γ-gain realizations of $(G, \mathcal{B})$ and $\Delta_X(G, \mathcal{B})$, so (ii) holds. $\square$

We now show that a Δ-Y exchange applied to a canonical representation agrees with the exchange applied to its gain graph. Over any field, a matrix representing the matroid $M(K_4)$ is projectively equivalent to the matrix $I(K_4)$ shown below.

$$I(K_4) = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & -1 \\ -1 & -1 & -1 & 0 & 0 & 0 \end{pmatrix}$$

The first three columns of this matrix represent a triad and the last three columns a triangle of K_4. Given a matrix A' such that $M(A')$ contains a triangle X, A' is projectively equivalent to a matrix A in which the columns corresponding to X are as the last three columns of $I(K_4)$, perhaps with one row omitted or with zero rows added. Let $\Delta_X A$ denote the matrix obtained from A by replacing the columns for X with first three columns of $I(K_4)$, where possibly the fourth row is omitted or additional zero rows are added. Then $M(\Delta_X A) = \Delta_X M(A)$ [2]. Similarly if $M(A')$ contains a triad Y, then A' is projectively equivalent to a matrix A in which the columns corresponding to Y are as the first three columns of $I(K_4)$, possibly with a row omitted or additional zero rows. Let $\nabla_Y A$ denote the matrix A obtained by replacing the columns of Y with those of the last three columns of $I(K_4)$. Then $M(\nabla_Y A) = \nabla_Y M(A)$ [2]. Thus the next fact follows immediately from Proposition 2.10.

Proposition 2.11. *Let $\mathbb{F}$ be a field and let (G,φ) be a Γ-gain graph, where $\Gamma \in \{\mathbb{F}^\times, \mathbb{F}^+\}$. Let X be a balanced triangle of $(G, \mathcal{B}_\varphi)$ and let Y be a $K_{1,3}$-subgraph of G.*

(i) $\Delta_X A_F(G,\varphi)$ *is a canonical frame matrix particular to* $(\Delta_X G, \varphi)$*,*

(ii) $\nabla_Y A_F(G,\varphi)$ *is a canonical frame matrix particular to* $(\nabla_Y G, \varphi)$*,*

(iii) $\Delta_X A_L(G,\varphi)$ *is a canonical lift matrix particular to* $(\Delta_X G, \varphi)$*,*

(iv) $\nabla_Y A_L(G,\varphi)$ *is a canonical lift matrix particular to* $(\nabla_Y G, \varphi)$*.*

2.5 Almost-balanced biased graphs: roll-ups

Let $(G,\mathcal{B})$ be an almost-balanced biased graph with the property that after deleting its joints it has a unique balancing vertex u. Let J be the set of joints of $(G,\mathcal{B})$ that are not incident to u, and denote by $\delta(u)$ the set of links incident to u. It is not difficult to check that the theta property implies that for each pair $e, e' \in \delta(u)$, either all cycles containing both e and e' are balanced or all cycles containing both e and e' are unbalanced (for details, see [5, Section 1]). Observe that, just as for a pair $e, e' \in \delta(u)$ for which every cycle containing both e and e' is balanced, for every pair $f, f' \in J$, every path linking the endpoint of f with the endpoint of f' together with f and f' is a circuit of $F(G,\mathcal{B})$. Define $\Sigma_G(u) = \Sigma(u) = \delta(u) \cup J$. Then for each pair of edges $e_1, e_2 \in \Sigma_G(u)$, either every minimal path linking the endpoints of e_1 and e_2 in $G-u$ together with $\{e_1, e_2\}$ forms a circuit in $F(G,\mathcal{B})$, or each such path together with $\{e_1, e_2\}$ is independent in $F(G,\mathcal{B})$. Define a relation $\sim$ on $\Sigma_G(u)$ in which $e_i \sim e_j$ if and only if $i = j$, or there is a balanced cycle containing e_i and e_j, or e_i and e_j are both in J. It is straightforward to check that $\sim$ is an equivalence relation [5, Lemma 1.4]. We call the $\sim$-equivalence classes partitioning $\Sigma(u)$ the *unbalancing classes* of $\Sigma(u)$, and denote the set of unbalancing classes of $\Sigma(u)$ by $\Sigma(u)/\!\sim$.

Now consider the biased graph $(\widehat{G}, \widehat{\mathcal{B}})$ obtained from $(G,\mathcal{B})$ by replacing each joint $e \in J$ incident to a vertex $v \neq u$ with a uv-link. Define $\widehat{\mathcal{B}}$ to be the set of those cycles having intersection of size 0 or 2 with each unbalancing class of $\Sigma_G(u)$. It is straightforward to check by comparing circuits that $F(\widehat{G}, \widehat{\mathcal{B}}) = F(G,\mathcal{B})$. Call $(\widehat{G}, \widehat{\mathcal{B}})$ the *unrolling of* $(G,\mathcal{B})$ *to* u. If J is empty, then set

$(\widehat{G}, \widehat{\mathcal{B}}) = (G, \mathcal{B})$. Observe that u is a balancing vertex of $(\widehat{G}, \widehat{\mathcal{B}})$, that $\Sigma_{\widehat{G}}(u) = \Sigma_G(u)$, and that $\Sigma_{\widehat{G}}(u)/\sim \, = \Sigma_G(u)/\sim$; that is, the unbalancing classes of $\Sigma_G(u)$ and of $\Sigma_{\widehat{G}}(u)$ are the same.

For each unbalancing class U of $\Sigma_{\widehat{G}}(u)$ there is a biased graph $(G_U, \mathcal{B}_U)$ for which $F(G_U, \mathcal{B}_U) = F(G, \mathcal{B}) = F(\widehat{G}, \widehat{\mathcal{B}})$, obtained from $(\widehat{G}, \widehat{\mathcal{B}})$ by replacing each link $e = uv \in U$ with a joint incident to v (this fact is straightforward to check by comparing circuits; it appears in [5, Proposition 2.2]). Call each such biased graph a *roll-up of* $(\widehat{G}, \widehat{\mathcal{B}})$ *from* u. It is a straightforward check that for each unbalancing class $U \in \Sigma_{\widehat{G}}(u)/\sim$, $\Sigma_{G_U}(u)/\sim \, = \Sigma_{\widehat{G}}(u)/\sim$. Note that J is an unbalancing class of $\Sigma_{\widehat{G}}(u)$, and that $(G_J, \mathcal{B}_J) = (G, \mathcal{B})$. Define $\mathcal{R}_{(G,\mathcal{B})}$ to be the set of biased graphs consisting of $(\widehat{G}, \widehat{\mathcal{B}})$ together with all of its roll-ups. Since each of these biased graphs shares precisely the same set of unbalancing classes, we may write simply $\Sigma(u)/\sim$ for this set, when it is clear that we are considering a biased graph in the collection $\mathcal{R}_{(G,\mathcal{B})}$. Thus $|\mathcal{R}_{(G,\mathcal{B})}| = |\Sigma(u)/\sim| + 1$. Finally, observe that since $(\widehat{G}, \widehat{\mathcal{B}})$ has no pair of vertex-disjoint unbalanced cycles, $L(\widehat{G}, \widehat{\mathcal{B}}) = F(\widehat{G}, \widehat{\mathcal{B}})$.

There is a special case to consider if $(G, \mathcal{B})$ is balanced after removing its set of joints J. Let x be a new isolated vertex added to $V(G)$. Then J may be unrolled to any vertex of G, including x. In the case that J is unrolled to x, we obtain a balanced biased graph $(H, \mathcal{C}(H))$, so $F(G, \mathcal{B})$ is equal to the cycle matroid $M(H)$ of H. The reverse operation may be applied to any graph. Given a graph H and a vertex $x \in V(H)$, let $(G, \mathcal{B})$ be the rollup of the set of edges incident to x; that is, since the set of edges incident to x is a single unbalancing class, replace each edge xv with a joint incident to its endpoint v. Then $M(H) = F(G, \mathcal{B})$.

If a biased graph has two distinct balancing vertices, then it has the very restricted form described in Proposition 2.12.

Proposition 2.12 (Zaslavsky [14])**.** *Let $(G, \mathcal{B})$ be a 2-connected unbalanced biased graph with two distinct balancing vertices x and y. Then G is a union of subgraphs $G_1 \cup \cdots \cup G_m$ where for each pair $i \neq j$, $G_i \cap G_j = \{x, y\}$, and a cycle is in $\mathcal{B}$ if and only if it is contained in a single subgraph G_i. If $m \geq 3$ then x and y are the only balancing vertices of $(G, \mathcal{B})$.*

Observe that if a biased graph $(G, \mathcal{B})$ of the form described in Proposition 2.12 has a subgraph G_i with at least two edges, then $(E(G_i), E(G) - E(G_i))$ is a 2-separation of both $L(G, \mathcal{B})$ and $F(G, \mathcal{B})$.

2.6 Full-rank canonical lift-matrix representations

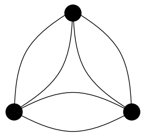
Let G be a graph and let γ be an $\mathbb{F}^+$-gain function on G, for some field $\mathbb{F}$. The canonical lift matrix $A_L(G,\gamma)$ consists of the oriented incidence matrix of the subgraph of G induced by its links together with a row v_0 of gains. It is sometimes inconvenient that this matrix is not of full rank. Choosing one vertex in each component of G, and deleting the rows of $A_L(G,\gamma)$ indexed by these vertices yields a matrix representation of $L(G,\mathcal{B}_\gamma)$ that is of full rank. In the case G is connected, just one row is removed, and the oriented incidence matrix of G is recovered from the resulting matrix by appending a row equal to the negation of the sum of all rows but v_0. When G is connected, let us denote by $A_L^{-v}(G,\gamma)$ the matrix obtained from $A_L(G,\gamma)$ by deleting the row indexed by the vertex $v \in V(G)$. Then $A_L^{-v}(G,\mathcal{B}_\gamma)$ is a full-rank matrix representing $L(G,\mathcal{B}_\gamma)$. Clearly, for any vertex $v \in V(G)$ the matrices $A_L^{-v}(G,\gamma)$ and $A_L(G,\gamma)$ are projectively equivalent, and for any pair of vertices $u, v \in V(G)$, the matrices $A_L^{-u}(G,\gamma)$ and $A_L^{-v}(G,\gamma)$ are projectively equivalent. We say $A_L^{-v}(G,\gamma)$ is a *full-rank canonical lift matrix representation* of $L(G,\mathcal{B}_\gamma)$. In the case that $(G,\mathcal{B}_\gamma)$ has a balancing vertex u after deleting its joints, it will be convenient to use $A_L^{-u}(G,\gamma)$ as the full-rank canonical lift matrix representation of $L(G,\mathcal{B}_\gamma)$.

3 Unavoidable minors

In this section we prove Theorems 4, 5, and 6. We show that there is a small collection of biased graphs, at least one of which must appear as a minor in every 2-connected biased graph. From this collection we obtain a slightly larger collection of biased graphs, and show that every 2-connected properly unbalanced biased graph contains a subdivision of at least one of these biased graphs. We prove an analogous result for 2-connected almost-balanced biased graphs, which we require for the proof of Theorem 5.5. Finally, we show that inequivalence of gain functions may always be found on a one of small number of unavoidable minors.

3.1 The minor-minimal, 2-connected, properly unbalanced biased graphs

Let $\mathcal{G}_0$ denote the set of minor-minimal biased graphs that are 2-connected and properly unbalanced. We first describe 13 biased graphs in $\mathcal{G}_0$, then show that these 13 biased graphs form the complete set. Recall that we denote the



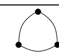
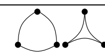
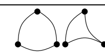
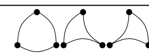
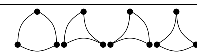
Name	Balanced cycles
T_0	none
T_1	
T_2	
T_2'	
T_3	
T_4	

Figure 3: The graph $2C_3$ and its six possible classes of balanced cycles not containing a cycle of length two.

graph obtained from a 3-cycle by replacing each edge with a pair of parallel edges by $2C_3$, and that we call the graph obtained from a 4-cycle by replacing each edge in a pair of non-adjacent edges with a pair of parallel edges the tube, and denote it by $2C_4''$. Six of the biased graphs in $\mathcal{G}_0$ have underlying graph $2C_3$, three have underlying graph $2C_4''$, and four have underlying graph K_4.

The set of cycles of the graph K_4 consists of four triangles and three quadrilaterals. We denote by $\mathsf{D}_{t,q} = (K_4, \mathcal{B}_{t,q})$ the biased K_4 with exactly t balanced triangles and q balanced 4-cycles. There are seven biased K_4's: $\mathsf{D}_{0,0}$, $\mathsf{D}_{0,1}$, $\mathsf{D}_{0,2}$, $\mathsf{D}_{0,3}$, $\mathsf{D}_{1,0}$, $\mathsf{D}_{2,1}$, and $\mathsf{D}_{4,2}$ [15]. A biased K_4 is properly unbalanced if and only if it does not contain a balanced triangle. Thus the properly unbalanced K_4's are $\mathsf{D}_{0,0}$, $\mathsf{D}_{0,1}$, $\mathsf{D}_{0,2}$, and $\mathsf{D}_{0,3}$.

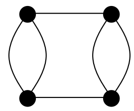
Proposition 3.1. *There are six unlabelled properly unbalanced biased $2C_3$'s.*

Proof. A biased graph $(2C_3, \mathcal{B})$ is properly unbalanced if and only $\mathcal{B}$ does not contain a 2-cycle. Thus by theta property, $\mathcal{B}$ is a collection of triangles pairwise intersecting in at most one edge. There are eight triangles in $2C_3$; any set of five contains a pair that intersect in more than one edge. Hence $\mathcal{B}$ contains at most 4 triangles. The possibilities are shown in Figure 3. □

A biased tube is properly unbalanced if and only it has no balanced 2-cycle. There are three such tubes, described in Figure 4.

We now show that the set of biased graphs consisting of the four biased K_4's with no balanced triangle, the six biased $2C_3$'s with no balanced 2-cycle, and the three biased tubes with no balanced 2-cycle,

$$\{\mathsf{D}_{0,0}, \mathsf{D}_{0,1}, \mathsf{D}_{0,2}, \mathsf{D}_{0,3}, \mathsf{T}_0, \mathsf{T}_1, \mathsf{T}_2, \mathsf{T}_2', \mathsf{T}_3, \mathsf{T}_4, \mathsf{B}_0, \mathsf{B}_1, \mathsf{B}_2\},$$



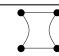
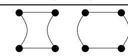
Name	Balanced cycles
B_0	none
B_1	
B_2	

Figure 4: The graph $2C_4''$ and its three possible classes of balanced cycles not containing a cycle of length two.

forms the complete collection $\mathcal{G}_0$ of minor-minimal, 2-connected, properly unbalanced biased graphs.

A *subdivision* of a biased graph $(G, \mathcal{B})$ is a biased graph $(H, \mathcal{S})$ in which H is a subdivision of G and a cycle C of H is in $\mathcal{S}$ if and only if its corresponding cycle C' of G is in $\mathcal{B}$. Proposition 3.2 follows immediately from Menger's Theorem.

Proposition 3.2. *Let $(G, \mathcal{B})$ be a 2-connected biased graph. If $(G, \mathcal{B})$ contains a vertex-disjoint pair of unbalanced cycles, neither of which is a loop, then $(G, \mathcal{B})$ contains a subdivision of B_0, B_1, or B_2.*

Thus to prove Theorem 4 it remains just to show that a properly unbalanced biased graph without two vertex-disjoint unbalanced cycles has a link minor from $\mathcal{G}_0$. A properly unbalanced biased graph with no two vertex-disjoint unbalanced cycles is *tangled*. The structure of tangled signed graphs was characterized by Slilaty [12] and the structure of tangled biased graphs in general was characterized by Chen and Pivotto [3]. The following theorem could be proven as a consequence of Chen and Pivotto's work in [3], but the direct proof we present here seems no more difficult.

Theorem 3.3. *Every tangled biased graph contains as a link minor either a biased $2C_3$ with no balanced 2-cycle or a biased K_4 with no balanced triangle.*

Lemma 3.4. *Let Ω be a tangled biased graph. Assume Ω contains an unbalanced cycle C and a pair of unbalanced cycles C_x and C_y such that $V(C) \cap V(C_x) = \{x\}$, $V(C) \cap V(C_y) = \{y\}$, and $x \neq y$. Then $C \cup C_x \cup C_y$ contains a biased $2C_3$ with no balanced 2-cycle as a link minor.*

Proof. Since Ω is tangled, $(C_x \cup C_y) - \{x, y\}$ is connected; furthermore, it is vertex-disjoint from C and so balanced. Let K be the edge set of $(C_x \cup C_y) - \{x, y\}$. Then $(C \cup C_x \cup C_y)/K$ is a link minor of $C \cup C_x \cup C_y$ and is a subdivision of a biased $2C_3$ with no balanced 2-cycle. The result follows. $\square$

Proof of Theorem 3.3. Let Ω be a link-minor-minimal counterexample. Then $|V(\Omega)| > 2$ and Ω has no joint, else Ω would not be tangled. If Ω has more than one unbalanced block but no two disjoint unbalanced cycles, then Ω must have a balancing vertex, a contradiction. Hence Ω has only one unbalanced block. Evidently our desired minor exists in Ω if and only if it exists in the unbalanced block of Ω. Hence by minimality Ω is 2-connected. By minimality we may also assume that Ω has no balanced 2-cycles.

Claim 1. The underlying graph of Ω is simple.

Proof of Claim: By way of contradiction assume that C is an unbalanced 2-cycle in Ω with vertices x and y. Thus $\Omega - x$ contains an unbalanced cycle C_y passing through y and $\Omega - y$ contains an unbalanced cycle C_x passing through x. By Lemma 3.4, $C_x \cup C_y \cup C$ contains a biased $2C_3$ without a balanced 2-cycle, a contradiction. ♣

If Ω has just three vertices, then the underlying graph of Ω is a triangle. This is not the case, as a biased triangle has a balancing vertex. If Ω has exactly four vertices, then $\Omega - v$ is an unbalanced triangle for each vertex v, else Ω has a balancing vertex. But then Ω is a biased K_4 without a balanced triangle, a contradiction. Thus $|V(\Omega)| \geq 5$.

Claim 2. For each vertex v, $\Omega - v$ is unbalanced and has a balancing vertex.

Proof of Claim: Minimality implies that for any vertex v in Ω, $\Omega - v$ is not tangled. Since $\Omega - v$ is unbalanced and has no two disjoint unbalanced cycles, it must have a balancing vertex. ♣

Given an edge e with endpoints x and y, we denote the vertex in Ω/e resulting from the identification of x and y by v_e or v_{xy}.

Claim 3. For each edge e, Ω/e has v_e as its unique balancing vertex.

Proof of Claim: By minimality, Ω/e is not tangled and has no two vertex-disjoint unbalanced cycles and so must therefore have a balancing vertex. If there is a balancing vertex $u \neq v_e$, then every unbalanced cycle of Ω/e passes through u, so every unbalanced cycle of Ω passes through u. But this implies that u is a balancing vertex of Ω, a contradiction. ♣

Claim 4. Ω does not have a vertical 2-separation (A, B) in which B is balanced.

Proof of Claim: Suppose, for a contradiction, that (A, B) is a vertical 2-separation in which B is balanced. Let $\{x, y\} = V(A) \cap V(B)$, and let e be an edge in B not incident to at least one of x and y. By Claim 3, Ω/e has balancing vertex v_e. By our choice of e, $(A, B \setminus e)$ is a 2-separation of Ω/e, and $V(A) \cap V(B \setminus e)$ is either $\{x, y\}$, $\{x, v_e\}$ or $\{v_e, y\}$. In any case, since $B \setminus e$ is balanced, every unbalanced cycle of Ω/e either does not intersect $B \setminus e$ or intersects $B\backslash e$ in the edges of a path linking the two vertices of $V(A) \cap V(B\backslash e)$. This implies that there is a vertex $v \in \{x, y\}$ such that every unbalanced cycle in Ω contains v. This means v a balancing vertex of Ω, a contradiction. ♣

Claim 5. Ω is 3-connected.

Proof of Claim: Suppose that Ω has a vertical 2-separation (A, B) and let $\{x, y\} = V(A) \cap V(B)$. By Claim 4, neither A nor B is balanced. Since Ω does not have a balancing vertex both $\Omega - x$ and $\Omega - y$ are unbalanced. Let C_x be an unbalanced cycle in $\Omega - x$ and C_y be an unbalanced cycle in $\Omega - y$. Without loss of generality, assume $E(C_x) \subseteq A$. Since Ω has no pair of vertex-disjoint unbalanced cycles and C_y does not contain y, this implies that also $E(C_y) \subseteq A$. Hence $E(C_x) \cup E(C_y) \subseteq A$. Since B is unbalanced, there is an unbalanced cycle C' in $\Omega[B]$; since C' is vertex-disjoint from neither C_x nor C_y, C' meets both vertices x and y. Thus by Lemma 3.4, $C_x \cup C_y \cup C'$ contains a biased $2C_3$ having no balanced 2-cycle, a contradiction. ♣

Now let $e = xy$ be an edge of Ω and let $E_1, \ldots, E_m$ be the unbalancing classes of edges incident to balancing vertex v_{xy} in Ω/e. Since Ω/e is unbalanced, $m \geq 2$. Let $E_{x,i}$ be the set of edges of E_i that are incident to x in Ω and let $E_{y,i}$ be the set of edges of E_i that are incident to y in Ω. Since $\Omega - y$ is unbalanced, at least two of $E_{x,1}, \ldots, E_{x,m}$ are nonempty; similarly, at least two of $E_{y,1}, \ldots, E_{y,m}$ are nonempty. Let X be the set of vertices in $\Omega - y$ adjacent to x, and let Y be the set of vertices in $\Omega - x$ that are adjacent to y. Since the underlying graph of Ω is simple, $|X| \geq 2$ and $|Y| \geq 2$. Now take $x_1, x_2 \in X$ so that edges xx_1 and xx_2 are in different sets $E_{x,1}, \ldots, E_{x,m}$; similarly, take $y_1, y_2 \in Y$ so that edges yy_1 and yy_2 are in different sets $E_{y,1}, \ldots, E_{y,m}$. Since the underlying graph of Ω is simple, $x_1 \neq x_2$ and $y_1 \neq y_2$.

Claim 6. Vertices x_1, x_2, y_1, y_2 cannot be chosen so that $\{x_1, x_2\} \cap \{y_1, y_2\} = \emptyset$.

Proof of Claim: Suppose to the contrary that $\{x_1, x_2\} \cap \{y_1, y_2\}$ is empty. Since Ω is 3-connected, there is an x_1-x_2 path P in $\Omega - \{x, y\}$. Because edges xx_1 and xx_2 are in different sets $E_{x,1}, \ldots, E_{x,m}$, the cycle xx_1Px_2x is unbalanced. For $i \in \{1, 2\}$ let e_i denote the yy_i-edge in Ω. Since v_{yy_i} is a balancing vertex in Ω/e_i (by Claim 3), the path P must contain y_1 and y_2. Hence there is a y_1-y_2 path P' properly contained in P that avoids both x_1 and x_2. Because edges yy_1 and yy_2 are in different sets $E_{y,1}, \ldots, E_{y,m}$, the cycle $C = yy_1P'y_2y$ is unbalanced. But C avoids x_1, x_2, and x, and so avoids the balancing vertex v_{xx_1} in Ω/xx_1, so this is a contradiction. ♣

Claim 7. Vertices x_1, x_2, y_1, y_2 cannot be chosen so that $|\{x_1, x_2\} \cap \{y_1, y_2\}| = 1$.

Proof of Claim: By way of contradiction assume that $|\{x_1, x_2\} \cap \{y_1, y_2\}| = 1$ where, without loss of generality, $x_2 = y_1$. As in the proof of Claim 6, any

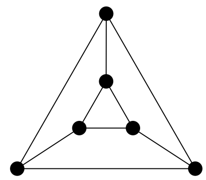


Figure 5: P has exactly two triangles, which comprise its set of balanced cycles.

x_1-x_2 path P in $\Omega - \{x, y\}$ must contain y_2. Thus there is a y_1-y_2 path P' properly contained in P and avoiding x_1, leading to the contradiction that there is an unbalanced cycle in Ω/xx_1 avoiding the balancing vertex v_{xx_1}. ♣

By Claims 6 and 7, every choice of x_1, x_2, y_1, y_2 has (without loss of generality) $x_1 = y_1$ and $x_2 = y_2$. This implies that $X = \{x_1, x_2\} = Y = \{y_1, y_2\}$, since otherwise we may choose a third vertex in X or Y so that $|\{x_1, x_2\} \cap \{y_1, y_2\}| \in \{0, 1\}$. Let e' and e'' be respectively the xx_1- and xx_2-edges in Ω and let f' and f'' be the yx_1- and yx_2-edges in Ω. It cannot be that e' and f' are in the same unbalancing equivalence class $E_j \in \{E_1, \ldots, E_m\}$ because then $\Omega - x_2$ would be balanced. Similarly e'' and f'' are not in the same equivalence class. Because Ω is 3-connected, there is an x_1x_2-path P in $\Omega - \{x, y\}$. The subgraph of Ω on edges $E(P) \cup \{e, e', e'', f', f''\}$ is a subdivision of K_4 without a balanced triangle, a contradiction. □

3.2 Unavoidable topological subgraphs

As is often the case with graphs, minors are harder to work with than subgraphs. In this section we prove Theorem 5, an analogue of Theorem 4 for biased topological subgraphs. We also prove a result on unavoidable biased topological subgraphs for almost-balanced biased graphs.

Let P denote the biased graph whose underlying graph is the triangular prism, with just its two triangles balanced (Figure 5). Let L denote the matching of three edges linking the two triangles of P. Observe that $\mathsf{P}/L \cong \mathsf{T}_2$. Let P_2 and P_1, respectively, be the biased graphs obtained from P by contracting 1 and 2 edges of L, respectively (so P_2 has two edges of L remaining, and P_1 has just one edge of L remaining from P). Set $\mathcal{T}_0 = \mathcal{G}_0 \cup \{\mathsf{P}, \mathsf{P}_1, \mathsf{P}_2\}$. Theorem 5 guarentees that at least one of the biased graphs in $\mathcal{T}_0$ is unavoidable as a biased topological subgraph in 2-connected properly unbalanced biased graphs.

Proof of Theorem 5. If Ω is not tangled, then the result follows from Proposition 3.2. So assume that Ω is tangled. By Theorem 3.3, Ω contains a link

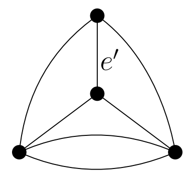


Figure 6: The graph G'.

minor $(G,\mathcal{B})$ that is either a biased K_4 with no balanced triangle or a biased $2C_3$ with no balanced 2-cycle. In the first case, since K_4 is 3-regular Ω contains a subdivision of $(G,\mathcal{B})$. In the second case, either Ω contains as a subgraph a subdivision of $(G,\mathcal{B})$ or Ω contains a link minor $(G',\mathcal{B}')$ that is 2-connected, has minimum degree 3, and contains an edge e' for which $(G',\mathcal{B}')/e' = (G,\mathcal{B})$. Since Ω is tangled, G' is as shown in Figure 6. It is straightforward to check that if $(G,\mathcal{B}) \not\cong \mathsf{T}_2$, then $(G',\mathcal{B}')$ contains a K_4 with no balanced triangle, and so our desired subdivision. So assume that $(G,\mathcal{B}) \cong \mathsf{T}_2$. Then either $\mathcal{B}'$ consists of a pair of edge-disjoint triangles both avoiding e' or $\mathcal{B}'$ consists of a pair of 4-cycles each of which contain e' but are otherwise edge-disjoint. In the latter case, we again have a biased K_4 with no balanced triangle as a subgraph, and so are done. In the former case, $(G',\mathcal{B}') \cong \mathsf{P}_1$. Thus either Ω contains a subdivision of P_1 or Ω contains as a link minor $(G'',\mathcal{B}'')$ which has minimum degree 3 and an edge e'' for which $(G'',\mathcal{B}'')/e'' \cong \mathsf{P}_1$. Either $\mathcal{B}''$ consists of a pair of 4-cycles sharing just e'' or $\mathcal{B}''$ consists of a pair of edge-disjoint triangles avoiding $\{e',e''\}$. Thus either $(G'',\mathcal{B}'')$ contains as a subdivision a biased K_4 with no balanced triangle, in which case we are done, or $(G'',\mathcal{B}'')$ is isomorphic to P_2. In the latter case, either Ω contains a subdivision of P_2 or Ω contains a link minor $(G''',\mathcal{B}''')$ with minimum degree 3 and an edge e''' for which $(G''',\mathcal{B}''')/e''' \cong \mathsf{P}_2$. Thus $\mathcal{B}'''$ either consists of a pair of disjoint triangles both avoiding $\{e',e'',e'''\}$ or a pair of 4-cycles sharing just e'''. But if $\mathcal{B}'''$ consists of a pair of 4-cycles, then $(G''',\mathcal{B}''')$ contains a pair of vertex-disjoint unbalanced triangles, contradicting the fact that Ω is tangled. Hence it must be the case the $\mathcal{B}'''$ consists of a pair of disjoint triangles, so $(G''',\mathcal{B}''') \cong \mathsf{P}$. $\square$

We now show that every almost-balanced biased graph containing a contrabalanced theta contains as a biased topological subgraph one of the biased graphs in following collection. Denote the graph obtained from $2C_3$ by deleting an edge by $2C_3\backslash e$. The graph $2C_3\backslash e$ is obtained from the tube $2C_4''$ by contracting one of its non-doubled links, so the cycles of $2C_3\backslash e$ are in bijective correspondence with the cycles of $2C_4''$. Thus there are exactly three biased graphs $(2C_3\backslash e,\mathcal{B})$ without a balanced 2-cycle, each obtained as a single-edge

Figure 7: Two contrabalanced biased graphs representing $U_{2,4}$.

contraction of B_0, B_1, or B_2 (Figure 4). We denote these biased graphs by B'_0, B'_1, and B'_2, respectively. Recall that $\mathsf{D}_{1,0} = (K_4, \mathcal{B})$ where $\mathcal{B}$ consists of exactly one balanced triangle; $D_{1,0}$ has a unique balancing vertex.

Proposition 3.5. *Let $(G, \mathcal{B})$ be a 2-connected biased graph that contains a contrabalanced theta subgraph and a unique balancing vertex after removing joints. Then $(G, \mathcal{B})$ contains a subdivision of $\mathsf{D}_{1,0}$, B'_0, B'_1, or B'_2.*

Proof. Denote by nK_2 the graph consisting of two vertices with n links between them. Since $(G, \mathcal{B})$ contains a contrabalanced theta subgraph, it contains a subdivision of $(nK_2, \emptyset)$ for some $n \geq 3$. Let K be such a subdivision in $(G, \mathcal{B})$ with n as large as possible. Let u be the balancing vertex of $(G, \mathcal{B})$. One of the two degree-n vertices of K is u; let v be the other degree-n vertex of K. Then K is the union of n internally disjoint u-v-paths $P_1, \ldots, P_n$. By assumption $(G, \mathcal{B})$ does not have the structure described in Proposition 2.12. Thus there is a path P in G internally disjoint from K with both its ends in K such that either

- both ends of P are internal vertices of two distinct paths P_i and P_j, or
- P has u as one end, an internal vertex of one of the paths P_i as its other end, and the cycle contained in $P \cup P_i$ is unbalanced.

In the first case, $P_i \cup P_j \cup P$ is a theta subgraph with its cycle $P_i \cup P_j$ unbalanced and its cycle avoiding u balanced. By the theta property the cycle in $P_i \cup P_j \cup P$ avoiding v is unbalanced. Thus $K \cup P$ contains a subdivision of $D_{1,0}$. In the second case, $K \cup P$ contains a subdivision of B'_0, B'_1, or B'_2. $\square$

3.3 Confining inequivalence to a small minor

We can now prove Theorem 6.

Two biased graphic representations of $U_{2,4}$ are U_2 and U_3, shown in Figure 7; all cycles in each are unbalanced. Theorem 6 localizes switching inequivalence of gain functions on a small minor: if not a biased graph in $\mathcal{G}_0$ then on both U_2 and U_3.

Proof of Theorem 6. By Theorem 5, $(G, \mathcal{B})$ has a biased subgraph $(G_0, \mathcal{B}_0)$ that is a subdivision of a member of $\mathcal{G}_0 \cup \{\mathsf{P}_1, \mathsf{P}_2, \mathsf{P}\}$. Since G is 2-connected and loopless, there is a sequence of 2-connected subgraphs $(G_0, \mathcal{B}_0), \ldots, (G_n, \mathcal{B}_n)$ such that $(G_n, \mathcal{B}_n) = (G, \mathcal{B})$ and $(G_{i+1}, \mathcal{B}_{i+1}) = (G_i, \mathcal{B}_i) \cup P_i$ for some path P_i that is internally disjoint from G_i. Let φ_i and ψ_i be the Γ-gain functions induced by φ and ψ on G_i. If φ_0 and ψ_0 are switching inequivalent (resp., switching-and-scaling inequivalent in the case that Γ is the additive group of a field), then the result follows by Proposition 2.3. Otherwise, there is an integer $t \in \{0, \ldots, n-1\}$ such that φ_i and ψ_i are switching (resp., switching-and-scaling) equivalent for $i \le t$ and φ_{t+1} and ψ_{t+1} are switching (resp., switching-and-scaling) inequivalent. Let e be an edge in path P_t. Since G_{t+1} is 2-connected, there is a spanning tree T_{t+1} of G_{t+1} that does not contain e. Normalize φ_{t+1} and ψ_{t+1} on T_{t+1}. Let T_t be T_{t+1} restricted to G_t. Then T_t is a spanning tree of G_t. Since φ_{t+1} and ψ_{t+1} are normalized on T_{t+1}, φ_t and ψ_t are normalized on T_t. Since φ_t and ψ_t are switching (-and-scaling) equivalent while φ_{t+1} and ψ_{t+1} are switching (-and-scaling) inequivalent, by Proposition 2.2 $\varphi_t = \psi_t$ (resp., $\varphi_t = a\psi_t$ for some scalar a) while φ_{t+1} and ψ_{t+1} are equal (resp., equal up to scaling) everywhere except at e. Because $(G_0, \mathcal{B}_0)$ is unbalanced, $(G_t, \mathcal{B}_t)$ is unbalanced, so neither φ_t nor ψ_t is trivial (that is, neither assigns the identity element of Γ to every edge). Hence for every cycle C of G_{t+1} containing path P_t, $\varphi_{t+1}(C) \neq \psi_{t+1}(C)$ (resp. if a is a scalar such that $\varphi_t = a\psi_t$ then $\varphi_{t+1}(C) \neq a\psi_{t+1}(C)$). Since φ_{t+1} and ψ_{t+1} are Γ-realizations of $(G_{t+1}, \mathcal{B}_{t+1})$, it must be that every such cycle C is unbalanced. Extend P_t in G_{t+1} to a path P that is internally disjoint from G_0 but whose endpoints are both on G_0. Let φ' and ψ' be φ_{t+1} and ψ_{t+1} restricted to the biased graph $(G_0, \mathcal{B}_0) \cup P$. Again, φ' and ψ' are equal (resp., equal up to scaling) on every edge of $(G_0, \mathcal{B}_0) \cup P$ save for the edge e and every cycle C in $(G_0, \mathcal{B}_0) \cup P$ containing e is therefore unbalanced. Now in $(G_0, \mathcal{B}_0) \cup P$ there is a link minor $\Omega = ((G_0, \mathcal{B}_0) \cup P)/K \backslash D$ for which $\Omega \backslash e$ is in $\mathcal{G}_0$ or $\Omega \backslash e/f$ is in $\mathcal{G}_0$ for some link f. The possibilities for Ω are as shown in Figure 8.

Re-normalize φ' and ψ' on a spanning tree of $(G_0, \mathcal{B}_0) \cup (P - e)$ that contains the contraction set K and let $\varphi|_\Omega$ and ψ_Ω be the induced gain functions on Ω. Again, $\varphi|_\Omega$ and $\psi|_\Omega$ are equal (equal up to scaling) on each edge of $(G_0, \mathcal{B}_0) \cup (P - e)$ but differ (differ even after scaling) on edge e, and every cycle containing e is unbalanced. The first outcome of the theorem holds in cases (i), (ii), (iv), and (v) of Figure 8 while the second outcome holds in cases (iii) and (vi). A single frame-type contraction of a joint is necessary in order to obtain U_2 in case (vi), but no other contraction of a joint is required. Thus all minors but this one are link minors. $\square$

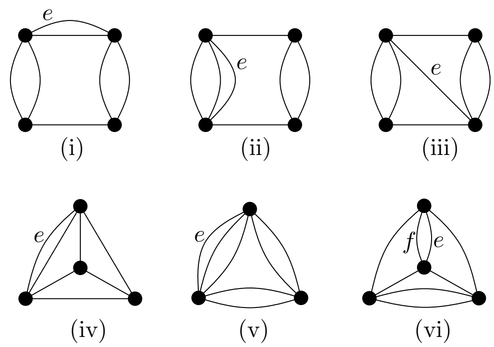


Figure 8: The possibilities for Ω.

The following observation will be used in the proof of Theorem 5.1.

Corollary 3.6. *If $(G, \mathcal{B})$ is tangled, then the first outcome of Theorem 6 holds.*

Proof. If the first outcome of Theorem 6 does not hold, then the biased graph Ω in the proof of Theorem 6 is that of either case (iii) or (vi) of Figure 8. Each of these contain a pair of vertex-disjoint unbalanced cycles, and so cannot occur if $(G, \mathcal{B})$ is tangled. $\square$

4 Representations of unavoidable minors

In this section we examine the relationship between gain functions on the biased graphs in $\mathcal{G}_0$ (along with a few other small biased graphs) and matrix representations of their associated frame and lift matroids. In Section 4.1 we show that for each biased graph $\Omega \in \mathcal{G}_0$, a pair of canonical representations of $F(\Omega)$ (resp., $L(\Omega)$) are projectively equivalent if and only if their associated gain functions are switching equivalent (resp., switching-and-scaling equivalent). In Section 4.2 we show that for each biased graph $\Omega \in \mathcal{G}_0$ every $\mathbb{F}$-representation of each of $F(\Omega)$ and $L(\Omega)$ is projectively equivalent to a canonical $\mathbb{F}$-representation particular to Ω.

4.1 Switching and projective equivalence

Let G be a graph and let $\mathbb{F}$ be a field. Recall that for an $\mathbb{F}^\times$-gain function φ and an $\mathbb{F}^+$-gain function ψ, we denote by $A_F(G, \varphi)$ and $A_L(G, \psi)$, respectively, the canonical frame and lift matrices defined by (G, φ) and (G, ψ), resp., as described in Section 1.1. Our starting point is the following result of Zaslavsky.

Proposition 4.1 (Zaslavsky [19])**.** *Let G be a graph and let $\mathbb{F}$ be a field. Let φ and ψ be $\mathbb{F}^\times$- (resp., $\mathbb{F}^+$-) gain functions on G. If φ and ψ are switching equivalent (resp., switching-and-scaling equivalent) then their canonical matrix representations are projectively equivalent.*

Proof. Suppose φ and ψ are $\mathbb{F}^\times$-gain functions on G and η is a switching function with $\varphi^\eta = \psi$. Let $V(G) = \{v_1, v_2, \ldots, v_{|V(G)|}\}$. Let T be the diagonal matrix with rows and columns indexed by $V(G)$ in which diagonal entry T_{ii} is $\eta(v_i)$, and let S be the $|E(G)| \times |E(G)|$ diagonal matrix with diagonal entries $S_{jj} = \eta(v_i)^{-1}$ if vertex v_i is the tail of edge e_j. Then $TA_F(G,\varphi)S = A_F(G,\psi)$.

Now suppose φ and ψ are $\mathbb{F}^+$-gain functions, η is a switching function, and that there is a scalar $s \in \mathbb{F}^\times$ so that $s\varphi^\eta = \psi$. Let T be the $(n+1)\times(n+1)$ matrix whose first row is

$$(s \ s\eta(v_1) \ s\eta(v_2) \ \cdots \ s\eta(v_n)),$$

first column is $(s \ 0 \ 0 \ \cdots 0)^T$, and with the $n \times n$ identity matrix as the submatrix consisting of its remaining rows and columns. Let S be the diagonal matrix with $s_{11} = 1/s$ and all other $s_{ii} = 1$. Then $TA_L(G,\varphi)S = A_L(G,\psi)$. □

The proof of Proposition 4.1 shows that if φ and ψ are switching equivalent $\mathbb{F}^\times$-gain functions, then a diagonal matrix T provides witness to the projective equivalence of $A_F(G,\varphi)$ and $A_F(G,\psi)$. The converse is also true. A similar statement holds for canonical lift matrices.

Lemma 4.2. *Let G be a loopless graph and let $\mathbb{F}$ be a field.*

(i) *Let φ and ψ be $\mathbb{F}^\times$-gain functions on G and assume $A_F(G,\varphi)$ and $A_F(G,\psi)$ are projectively equivalent. Then φ and ψ are switching equivalent if and only if there exists a nonsingular diagonal matrix T and a diagonal column-scaling matrix S such that $TA_F(G,\varphi)S = A_F(G,\psi)$.*

(ii) *Let φ and ψ be $\mathbb{F}^+$-gain functions on G and assume $A_L(G,\varphi)$ and $A_L(G,\psi)$ are projectively equivalent. Then φ and ψ are switching-and-scaling equivalent if and only if there exists a nonsingular matrix T and a diagonal column-scaling matrix S such that $TA_L(G,\varphi)S = A_L(G,\psi)$, where removing the row and column of T indexed by v_0 leaves an identity matrix, and all but the first entry of column v_0 consists of zeros.*

Proof. (i) Put $A = A_F(G,\varphi)$ and $B = A_F(G,\psi)$. If φ and ψ are switching equivalent, then the proof of Proposition 4.1 shows that A and B are projectively equivalent via diagonal nonsingular matrices T and S. Conversely, suppose that $B = TAS$ where T and S are both diagonal and nonsingular.

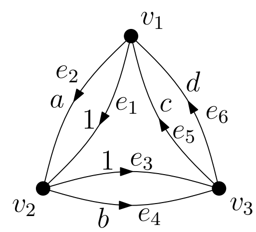


Figure 9: Gain function φ on $2C3$

Since T is diagonal, row i of TA is obtained by multiplying row v_i of A by T_{ii}. Since both A and B are canonical frame representations, both have 1 in position v_i of column e_j whenever vertex v_i is the tail of edge e_j. Hence the diagonal elements S_{jj} of S satisfy $S_{jj} = T_{ii}^{-1}$, where v_i is the tail of e_j. Thus φ and ψ are switching equivalent via $\eta(v_i) = T_{ii}$ for each $v_i \in V(G)$.

(ii) Put $A = A_L(G, \varphi)$ and $B = A_L(G, \psi)$. If φ and ψ are switching-and-scaling equivalent, then the proof of Proposition 4.1 shows that A and B are projectively equivalent via matrices T and S of the required forms. Conversely, suppose that $B = TAS$ where T and S are of the form in statement (ii). Put $n = |V(G)|$. Index the rows and columns of T by $\{v_0, v_1, \ldots, v_n\}$, where as usual row v_0 of A contains the gains assigned by φ and for $i \in \{1, \ldots, n\}$ row v_i corresponds to vertex v_i. Let $\eta : V(G) \to \mathbb{F}^{+}$ be the switching function defined by $\eta(v_i) = T_{v_0 v_i}$ for $i \in \{1, \ldots, n\}$. Then $(T_{v_0 v_0})\,\varphi^{\eta} = \psi$. $\square$

We now proceed with our examination of each biased graph in $\mathcal{G}_0$.

Lemma 4.3. *Let $\varphi \colon \vec{E}(2C_3) \to \mathbb{F}^{\times}$ be a gain function with $\mathcal{B}_{\varphi}$ containing no 2-cycle, and let $\psi \colon \vec{E}(2C_3) \to \mathbb{F}^{\times}$ be another gain function. Then φ and ψ are switching equivalent if and only if $A_F(2C_3, \varphi)$ and $A_F(2C_3, \psi)$ are projectively equivalent.*

Proof. If φ and ψ are switching equivalent, then $A_F(2C_3, \varphi)$ and $A_F(2C_3, \psi)$ are projectively equivalent by Proposition 4.1. To prove the converse, let $A = A_F(2C_3, \varphi)$ and $B = A_F(2C_3, \psi)$ be a pair of projectively equivalent canonical frame matrices. By normalizing on the spanning tree with edge set $\{e_1, e_3\}$, by Proposition 4.1 we may assume that φ assigns gains to $E(2C_3)$ as shown in Figure 9. Then

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & -c & -d \\ -1 & -a & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -b & 1 & 1 \end{pmatrix} \end{array} \tag{4.1}$$

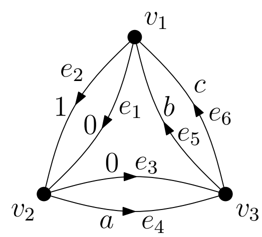


Figure 10: Gain function $\varphi\colon \vec{E}(2C_3) \to \mathbb{F}^+$.

Since $\mathcal{B}_\varphi$ contains no 2-cycle, $a \neq 1 \neq b$ and $c \neq d$. Since A and B are projectively equivalent, there is a nonsingular matrix T and a diagonal matrix S so that $TA = BS$. Denote by t_i row i of T, and by e_j column j of A. Entry $(BS)_{ij} = 0$ if and only if entry $(TA)_{ij} = 0$; consider the dot products $t_i \cdot e_j = 0$, where $(i,j) \in \{(3,1),(3,2),(1,3),(1,4),(2,5),(2,6)\}$. The product $t_3 \cdot e_1 = 0$ implies $T_{31} = T_{32}$, and $t_3 \cdot e_2 = 0$ implies $T_{31} = aT_{32}$. Together these imply (since $a \neq 1$) that $T_{32} = T_{31} = 0$. Similarly, $t_1 \cdot e_3 = t_1 \cdot e_4 = 0$ imply $T_{13} = T_{12} = 0$, and $t_2 \cdot e_5 = t_2 \cdot e_6 = 0$ imply $T_{21} = T_{23} = 0$. Hence T is diagonal, and so by Lemma 4.2, φ and ψ are switching equivalent. $\square$

Lemma 4.4. *Let $\varphi\colon \vec{E}(2C_3) \to \mathbb{F}^+$ be a gain function with $\mathcal{B}_\varphi$ containing no 2-cycle and let $\psi\colon \vec{E}(2C_3) \to \mathbb{F}^+$ be another gain function. Then φ and ψ are switching-and-scaling equivalent if and only if $A_L(2C_3,\varphi)$ and $A_L(2C_3,\psi)$ are projectively equivalent.*

Proof. If φ and ψ are switching-and-scaling equivalent, then their associated lift matrices are projectively equivalent by Proposition 4.1. Conversely, let $A = A_L(2C_3,\varphi)$ and $B = A_L(2C_3,\psi)$ be a pair of projectively equivalent canonical lift matrices. Normalizing on spanning tree $\{e_1, e_2\}$, and scaling if necessary, we may assume that φ assigns gains to $2C_3$ as shown in Figure 10. Then

$$A = \begin{array}{c} \\ v_0 \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & a & b & c \\ 1 & 1 & 0 & 0 & -1 & -1 \\ -1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 1 \end{pmatrix} \end{array} \tag{4.2}$$

Since φ has no balanced cycles of length 2, $a \neq 0$ and $b \neq c$. There is a non-singular matrix T and a diagonal matrix S so that $TA = BS$. As with φ, by switching and scaling we may assume ψ assigns gains to $\vec{E}(2C_3)$ as in Figure 10, replacing a, b, and c with x, y, and z, respectively. Then, denoting elements S_{ii} of S by s_i, we have

$$BS = \begin{pmatrix} 0 & s_2 & 0 & s_4x & s_5y & s_6z \\ s_1 & s_2 & 0 & 0 & -s_5 & -s_6 \\ -s_1 & -s_2 & s_3 & s_4 & 0 & 0 \\ 0 & 0 & -s_3 & -s_4 & s_5 & s_6 \end{pmatrix}$$

This gives 24 relations among the members of T, one for each dot product $t_i \cdot e_j$, where t_i is the ith column of T and e_j is the jth column of A. The eight relations $t_i \cdot e_j = 0$ yield $T_{12} = T_{13} = T_{14}$, $T_{21} = T_{31} = T_{41} = 0$, $T_{23} = T_{24}$, and $T_{42} = T_{43}$. Now after establishing these relations, $t_3 \cdot e_5 = 0$ yields $T_{32} = T_{34}$ and so

$$T = \begin{pmatrix} T_{11} & T_{12} & T_{12} & T_{12} \\ 0 & T_{22} & T_{23} & T_{23} \\ 0 & T_{32} & T_{33} & T_{32} \\ 0 & T_{42} & T_{42} & T_{44} \end{pmatrix}$$

Now the relations $s_1 = t_2 \cdot e_1$, $s_2 = t_2 \cdot e_2$, $s_3 = t_3 \cdot e_3$, $s_4 = t_3 \cdot e_4$, $s_5 = t_4 \cdot e_5$, $s_6 = t_4 \cdot e_6$, $-s_2 = t_3 \cdot e_2$, and $-s_3 = t_4 \cdot e_3$ yield $s_1 = s_2 = s_3 = s_4 = s_5 = s_6$. Hence the relation $s_2 = t_1 \cdot e_2$ yields $T_{11} = s_1$. Now the relations $t_1 \cdot e_4 = s_1 x$, $t_1 \cdot e_5 = s_1 y$, and $t_1 \cdot e_6 = s_1 z$ yield $a = x$, $b = y$ and $c = z$. This implies that φ and ψ are switching equivalent after scaling. □

Lemma 4.5. *Let $\varphi\colon \vec{E}(2C_3) \to \mathbb{F}^\times$ and $\psi\colon \vec{E}(2C_3) \to \mathbb{F}^+$ be gain functions, neither of which yield a balanced 2-cycle. Then $A_F(2C_3, \varphi)$ and $A_L(2C_3, \psi)$ are not projectively equivalent.*

Proof. As in previous cases, without loss of generality we may assume that φ assigns gains to $2C_3$ as in Figure 9, and ψ as in Figure 10, replacing a with x, b with y, and c with z. Then $A_F(2C_3, \varphi)$ is the matrix of (4.1), and $A_L(2C_3, \psi)$ is the matrix of (4.2) with a, b, and c replaced by x, y, and z respectively. Recall (Section 2.6) that $A_L(2C_3, \psi)$ is projectively equivalent to the matrix $A_L^{-v}(2C_3, \psi)$ obtained by deleting the row indexed by any vertex $v \in V(2C_3)$ from $A_L(2C_3, \psi)$. Let $B = A_L^{-v_3}(2C_3, \psi)$. Then B is a full-rank canonical lift matrix projectively equivalent to $A_L(G, \psi)$; in particular, A and B both have three rows. Now suppose for a contradiction that there exists a non-singular matrix T and a diagonal matrix S so that $TA = BS$. Writing $S_{ii} = s_i$, and denoting row i of T by t_i and column j of A by e_j, we have $t_2 \cdot e_1 = s_1$, $t_2 \cdot e_2 = s_2$, $t_2 \cdot e_3 = 0$, and $t_2 \cdot e_4 = 0$. Together these imply that $T_{22} = T_{23} = 0$ and that $T_{21} = s_1 = s_2$. Moreover, we have $t_3 \cdot e_1 = -s_1$, $t_3 \cdot e_2 = -s_2$, $t_3 \cdot e_5 = 0$, and $t_3 \cdot e_6 = 0$. Since $s_1 = s_2$, $a \neq 0, 1$, and $c \neq d$, these imply that $T_{31} = T_{32} = T_{33} = 0$, which implies that T is singular, a contradiction. □

Lemma 4.6. *Let $\varphi\colon \vec{E}(K_4) \to \mathbb{F}^\times$ be a gain function with $\mathcal{B}_\varphi$ containing no 3-cycle, and let $\psi\colon \vec{E}(K_4) \to \mathbb{F}^\times$ be another gain function. Then φ and ψ are switching equivalent if and only if $A_F(K_4, \varphi)$ and $A_F(K_4, \psi)$ are projectively equivalent.*

Proof. By switching we may assume that φ and ψ are both equal to the identity on a $K_{1,3}$-subgraph Y. This allows us to consider φ and ψ as gain functions on $\nabla_Y K_4 = 2C_3$. Now $\nabla_Y(K_4, \mathcal{B}_\varphi) = (2C_3, \mathcal{B}_\varphi)$ and $\nabla_Y(K_4, \mathcal{B}_\psi) = (2C_3, \mathcal{B}_\psi)$. Since φ has no balanced triangles in K_4, neither has it any balanced 2-cycles in $2C_3$. Thus by Lemma 4.3, φ and ψ are switching equivalent if and only if $A_F(2C_3, \varphi)$ and $A_F(2C_3, \psi)$ are projectively equivalent. Thus Propositions 2.9 and 2.10 imply that φ and ψ are switching equivalent if and only if $A_F(K_4, \varphi)$ and $A_F(K_4, \psi)$ are projectively equivalent. □

Using Y-Δ exchanges as in the proof of Lemma 4.6 yields Lemmas 4.7 and 4.8.

Lemma 4.7. *Let $\varphi\colon \vec{E}(K_4) \to \mathbb{F}^+$ be a gain function with $\mathcal{B}_\varphi$ containing no 3-cycle and let $\psi\colon \vec{E}(K_4) \to \mathbb{F}^+$ be another gain function. Then φ and ψ are switching-and-scalaing equivalent if and only if $A_L(K_4, \varphi)$ and $A_L(K_4, \psi)$ are projectively equivalent.*

Lemma 4.8. *Let $\varphi\colon \vec{E}(K_4) \to \mathbb{F}^\times$ and $\psi\colon \vec{E}(K_4) \to \mathbb{F}^+$ be gain functions neither of which yield a balanced 3-cycle. Then $A_F(K_4, \varphi)$ and $A_L(K_4, \psi)$ are not projectively equivalent.*

Lemma 4.9. *Let $\varphi\colon \vec{E}(2C_4'') \to \mathbb{F}^\times$ be a gain function with $\mathcal{B}_\varphi$ containing no 2-cycle, and let $\psi\colon \vec{E}(2C_4'') \to \mathbb{F}^\times$ be another gain function. Then φ and ψ are switching equivalent if and only if $A_F(2C_4'', \varphi)$ and $A_F(2C_4'', \psi)$ are projectively equivalent.*

Proof. Let $A = A_F(2C_4'', \varphi)$ and $B = A_F(2C_4'', \psi)$. If φ and ψ are switching equivalent, then A and B are projectively equivalent by Proposition 4.1. To prove the converse, let T and S be matrices with $TA = BS$, where T is nonsingular and S is a diagonal matrix scaling the columns of B. We may assume without loss of generality that the edge orientations chosen to define B are the same as those chosen to define A; by normalizing on the spanning tree with edges e_3, e_4, e_5, we may assume without loss of generality that φ assigns gains to $\vec{E}(K_4)$ as shown in Figure 11 where $a \neq b$ and $c \neq 1$. Thus

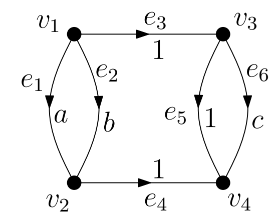


Figure 11: Labeled $2C_4''$ with a normalized gain function.

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \!\!\begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -a & -b & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & -1 & -c \end{pmatrix} \end{array}.$$

Now, each entry B_{ij} of B is zero if and only if entry $(TA)_{ij} = 0$. Now for a fixed row i, there are three distinct j such that $t_i \cdot e_j = 0$. It is a straightforward check that for each i, that these three relations yield $T_{ij} \neq 0$ if and only if $i = j$. For $i = 1$, the relations are $0 = t_1 \cdot e_4$, which implies $T_{12} = T_{14}$; $0 = t_1 \cdot e_5$, which implies $T_{13} = T_{14}$; and $0 = t_1 \cdot e_5$, which implies $T_{13} = cT_{14}$. Since $c \neq 1$, this implies $T_{12} = T_{13} = T_{14} = 0$. Similarly, the entries of T off its main diagonal in rows 2, 3, and 4 are all zero. Thus T is diagonal. By Lemma 4.2, φ and ψ are switching equivalent. □

Lemma 4.10. *Let $\varphi\colon \vec{E}(2C_4'') \to \mathbb{F}^+$ be a gain function with $\mathcal{B}_\varphi$ containing no 2-cycle, and let $\psi\colon \vec{E}(2C_4'') \to \mathbb{F}^+$ be another gain function. Then φ and ψ are switching-and-scaling equivalent if and only if $A_L(2C_4'', \varphi)$ and $A_L(2C_4'', \psi)$ are projectively equivalent.*

Proof. The "only if" statement again follows from Proposition 4.1. For the converse, without loss of generality assume that $\varphi(e_1) = \psi(e_1) = 1$, $\varphi(e_2) = a$, $\psi(e_2) = x$, $\varphi(e_3) = \psi(e_3) = 0$, $\varphi(e_4) = \psi(e_4) = 0$, $\varphi(e_5) = \psi(e_4) = 0$, $\varphi(e_6) = b$, and $\psi(e_6) = y$ (where $2C_4''$ has edges and orientations as in Figure 11) such that neither a nor x is 1 and neither b nor y is 0. Let $A = A_L(2C_4'', \varphi)$ and $B = A_L(2C_4'', \psi)$, and let T and S be matrices with $TA = BS$, where S is diagonal (with $s_i = S_{ii}$) scaling the columns of B. Denoting row i of T by t_i and column j of A by e_j we have $t_i \cdot e_j = 0$ for 15 pairs (i, j), three pairs for

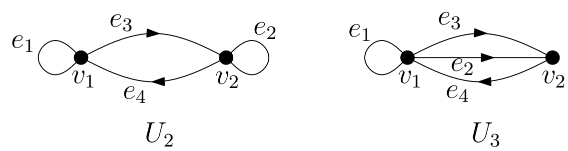


Figure 12

each row. It is straightforward to deduce that these relations imply

$$T = \begin{pmatrix} T_{11} & T_{12} & T_{12} & T_{12} & T_{12} \\ 0 & T_{22} & T_{23} & T_{23} & T_{23} \\ 0 & T_{32} & T_{33} & T_{32} & T_{32} \\ 0 & T_{42} & T_{42} & T_{44} & T_{42} \\ 0 & T_{52} & T_{52} & T_{52} & T_{55} \end{pmatrix}.$$

Now each column e_j has $i, k \geq 2$ such that $t_i \cdot e_j = s_j$ and $t_k \cdot e_j = -s_j$. These 12 relations yield $s_1 = s_2 = s_3 = s_4 = s_5 = s_6$. The relation $t_1 \cdot e_1 = s_1$ yields $T_{11} = s_1$. Now the relation $t_1 \cdot e_2 = s_1 x$ yields $a = x$ and the relation $t_1 \cdot e_6 = s_1 y$ yields $b = y$. Thus A and B are switching-and-scaling equivalent. □

Two biased graphic representations of $U_{2,4}$ are U_2 and U_3, shown in Figure 7, where all cycles in each are unbalanced. Denote the underlying graphs of U_2 and U_3 by U_2 and U_3, respectively.

Lemma 4.11. *Let φ and ψ be $\mathbb{F}^\times$-realizations of U_2. Then $A_F(U_2, \varphi)$ and $A_F(U_2, \psi)$ are projectively equivalent if and only if $\varphi(e_3 e_4) = \psi(e_3 e_4)$.*

Proof. We may assume U_2 is labeled with edge orientations as in Figure 12. Matrices $A_F(U_2, \varphi)$ and $A_F(U_2, \psi)$ are of the form

$$\begin{pmatrix} 1 & 0 & 1 & -g \\ 0 & 1 & -1 & 1 \end{pmatrix} \tag{4.3}$$

up to scaling columns e_1 and e_2. These are in standard form relative to the basis $\{e_1, e_2\}$ and so are projectively equivalent if and only if entry g is the same for both $A_F(U_2, \varphi)$ and $A_F(U_2, \psi)$. The result follows. □

Lemma 4.12. *Let φ and ψ be $\mathbb{F}^+$-realizations of U_3. Then $A_L(U_3, \varphi)$ and $A_L(U_3, \psi)$ are projectively equivalent if and only if $\varphi|_{\{e_2, e_3, e_4\}}$ and $\psi|_{\{e_2, e_3, e_4\}}$ are switching-and-scaling equivalent.*

Proof. We may assume that U_3 is labelled as in Figure 12. If $A = A_L(U_3, \varphi)$ and $B = A_L(U_3, \psi)$ are projectively equivalent, then there is an invertible matrix T and diagonal matrix S such that $TA = BS$. By switching and scaling we may assume that $\varphi(e_1) = \psi(e_1) = 1$, $\varphi(e_2) = \psi(e_2) = 0$, $\varphi(e_3) = \psi(e_3) = 1$, $\varphi(e_4) = a$, and $\psi(e_4) = x$. Again writing s_i for entry S_{ii},

$$\begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & a \\ 0 & 1 & 1 & -1 \\ 0 & -1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} s_1 & 0 & s_3 & s_4 x \\ 0 & s_2 & s_3 & -s_4 \\ 0 & -s_2 & -s_3 & s_4 \end{pmatrix}.$$

This yields $T_{11} = s_1$, $T_{21} = T_{31} = 0$, and $T_{12} = T_{13}$. That is,

$$\begin{pmatrix} s_1 & T_{12} & T_{12} \\ 0 & T_{22} & T_{23} \\ 0 & T_{32} & T_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & a \\ 0 & 1 & 1 & -1 \\ 0 & -1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} s_1 & 0 & s_3 & s_4 x \\ 0 & s_2 & s_3 & -s_4 \\ 0 & -s_2 & -s_3 & s_4 \end{pmatrix},$$

which yields $s_1 = s_2 = s_3 = s_4$. Thus $a = x$, and so φ and ψ are switching-and-scaling equivalent. □

4.2 All $\mathbb{F}$-representations are canonical

Let $\mathbb{F}$ be a field. In this section we show that every $\mathbb{F}$-matrix representation of a frame or lift matroid arising from a biased graph in $\mathcal{T}_0$ is projectively equivalent to a canonical representation particular to that biased graph. Recall that when $(G, \mathcal{B})$ is a biased graph with no two vertex-disjoint unbalanced cycles, $F(G, \mathcal{B}) = L(G, \mathcal{B})$, and we denote this common matroid by $M(G, \mathcal{B})$.

Lemma 4.13. *Let $(2C_3, \mathcal{B})$ be a biased graph with no balanced 2-cycle and let A be an $\mathbb{F}$-matrix representing $M(2C_3, \mathcal{B})$. Then A is projectively equivalent to a canonical lift matrix particular to $(2C_3, \mathcal{B})$ or to a canonical frame matrix particular to $(2C_3, \mathcal{B})$, but not both.*

Proof. We may assume that $2C_3$ is labeled and has edge orientations as shown in Figure 13. Let A be a matrix over $\mathbb{F}$ representing $M(2C_3, \mathcal{B})$. If $(2C_3, \mathcal{B}) \cong \mathsf{T}_4$ then $M(2C_3, \mathcal{B})$ is isomorphic to the cycle matroid of K_4, and so has a projectively unique representation over every field. Thus if the characteristic of $\mathbb{F}$ is two then A is projectively equivalent to the canonical lift matrix

$$B = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}$$

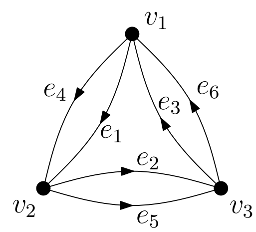


Figure 13

and if the characteristic of $\mathbb{F}$ is not two then A is projectively equivalent to the canonical frame matrix

$$C = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & -1 \\ 1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 1 \end{pmatrix}.$$

We now claim that there is no canonical frame matrix particular to T_4 over any field of characteristic two, and that neither is there a canonical lift matrix particular to T_4 in any field of characteristic different from two. For, toward a contradiction, suppose D is a canonical frame matrix particular to T_4 over a field $\mathbb{F}$ of characteristic two. Assume the collection of balanced cycles of T_4 is $\{e_1e_2e_6, e_1e_3e_5, e_2e_3e_4, e_4e_5e_6\}$. We may assume

$$D = \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{pmatrix} 1 & 0 & c & 1 & 0 & 1 \\ a & 1 & 0 & 1 & 1 & 0 \\ 0 & b & 1 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

where $a, b, c \in \mathbb{F}^{\times}$ (and we omit the customary negative signs as redundent). Since $e_1e_2e_6$ is balanced, $ab = 1$; since $e_1e_3e_5$ is balanced, $ac = 1$; and because $e_2e_3e_4$ is balanced, $bc = 1$. These relations imply that $a = b = c$ and so that $a^2 = 1$. But this implies $a = 1$, and so D does not represent $M(\mathsf{T}_4)$, a contradiction.

Similarly, suppose for a contradiction that D is a canonical lift matrix particular to T_4 over a field $\mathbb{F}$ of characteristic different from two. Then we may assume

$$D = \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & a & b & c \\ 1 & 0 & -1 & 1 & 0 & -1 \\ -1 & 1 & 0 & -1 & 1 & 0 \end{pmatrix} \end{array}$$

for some nonzero elements $a, b, c \in \mathbb{F}$, where the second and third rows are the oriented incidence matrix of $2C_3$ with its row corresponding to v_3 removed. Because $e_4e_5e_6$ is balanced, $a + b + c = 0$. Since $e_1e_2e_6$ is balanced, $c = 0$; since $e_1e_3e_5$ is balanced, $1 + b = 0$; and because $e_2e_3e_4$ is balanced, $1 + a = 0$. These relations imply that $a = b = -1$ and so that $a + b + c = -2 \neq 0$, a contradiction. This completes the proof in the case that $(2C_3, \mathcal{B}) \cong \mathsf{T}_4$.

Now assume $(2C_3, \mathcal{B}) \ncong \mathsf{T}_4$. By Proposition 3.1 we may assume that the triangles $\{e_1, e_2, e_3\}$ and $\{e_2, e_3, e_4\}$ are both unbalanced. Since the only form a 3-circuit takes in $(2C_3, \mathcal{B})$ is a balanced triangle and neither e_5 nor e_6 forms a triangle with $\{e_2, e_3\}$, A is projectively equivalent to the matrix

$$\begin{array}{c} \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & a & c \\ 0 & 0 & 1 & 1 & b & d \end{pmatrix} \end{array}.$$

Hence:

(i) Neither b nor c is 0. If $b = 0$ then $\{e_1, e_2, e_5\}$ is a circuit; if $c = 0$ then $\{e_1, e_3, e_6\}$ is a circuit: both contradictions.

(ii) $a \neq b$: If so then $a \neq 1$, as then e_4 and e_5 would form a parallel pair, a contradiction. But then $\{e_1, e_4, e_5\}$ is a circuit, also a contradiction.

(iii) $b \neq 1$: If so, then $\{e_2, e_4, e_5\}$ is a circuit, a contradiction.

(iv) $c \neq d$: If so, then $c \neq 1$ as e_4 and e_6 are not a parallel pair. But then $\{e_1, e_4, e_6\}$ is a circuit, a contradiction.

(v) $c \neq 1$: If so, $\{e_3, e_4, e_6\}$ is a circuit, a contradiction.

(vi) $a \neq c$: If so, then $d \neq b$ since e_5 and e_6 are not a parallel pair. But then $\{e_3, e_5, e_6\}$ is a circuit, a contradiction.

(vii) $b \neq d$: If so, then $a \neq c$ since e_5 and e_6 are not a parallel pair. But then $\{e_2, e_5, e_6\}$ is a circuit, a contradiction.

Suppose there are nonsingular matrices T and S such that TAS is a canonical frame matrix particular to $(2C_3, \mathcal{B})$, where S is diagonal column-scaling matrix. Then we may assume

$$TAS = \begin{array}{c} \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{matrix} \\ \begin{pmatrix} 1 & 0 & -g_1 & 1 & 0 & -g_4 \\ -1 & 1 & 0 & -g_2 & 1 & 0 \\ 0 & -1 & 1 & 0 & -g_3 & 1 \end{pmatrix} \end{array}$$

for some elements $g_1, \ldots, g_4 \in \mathbb{F} - \{0,1\}$. Let us denote row i of T by t_i and column j of A by e_j. Consider the products $t_i \cdot e_j = (TA)_{ij}$ for $1 \le i \le 3$, $1 \le j \le 6$. The products $t_3 \cdot e_1 = 0$, $t_1 \cdot e_2 = 0$, and $t_2 \cdot e_3 = 0$ imply $T_{31} = 0$, $T_{12} = 0$, and $T_{23} = 0$, respectively. Since $t_1 \cdot e_1 = -(t_2 \cdot e_1)$, $T_{21} = -T_{11}$. Similarly, $t_2 \cdot e_2 = -(t_3 \cdot e_2)$ implies $T_{32} = -T_{22}$. Now $t_3 \cdot e_4 = 0$ implies $T_{33} = -T_{32}$; $t_1 \cdot e_5 = 0$ implies $T_{13} = -T_{11}/b$; and finally, $t_2 \cdot e_6 = 0$ implies that $T_{22} = -T_{21}/c$. Thus T is the matrix

$$\begin{pmatrix} t & 0 & -t/b \\ -t & t/c & 0 \\ 0 & -t/c & t/c \end{pmatrix} = t \begin{pmatrix} 1 & 0 & -1/b \\ -1 & 1/c & 0 \\ 0 & -1/c & 1/c \end{pmatrix}$$

for some nonzero $t \in \mathbb{F}$. Since T has determinant $t^3(1/c^2 - 1/bc)$, T is nonsingular if and only if $b \neq c$. Assuming $b \neq c$, and taking $t = 1$,

$$TA = \begin{pmatrix} 1 & 0 & -1/b & (b-1)/b & 0 & (b-d)/b \\ -1 & 1/c & 0 & (1-c)/c & (a-c)/c & 0 \\ 0 & -1/c & 1/c & 0 & (b-a)/c & (d-c)/c \end{pmatrix}.$$

By claims 1-7 above TA has exactly two nonzero entries in each column, so scaling the columns of TA appropriately yields a canonical frame matrix. Thus A is projectively equivalent to a canonical frame matrix particular to $(2C_3, \mathcal{B})$ if and only if $b \neq c$.

Now let T be a nonsingular matrix such that TAS is a canonical lift matrix particular to $(2C_3, \mathcal{B})$, for some diagonal column-scaling matrix S. We may assume that TAS is of the form

$$TAS = \begin{pmatrix} 0 & 0 & 1 & g_1 & g_2 & g_3 \\ 1 & 0 & -1 & 1 & 0 & -1 \\ -1 & 1 & 0 & -1 & 1 & 0 \end{pmatrix}$$

for some nonzero $g_1, g_2, g_3 \in \mathbb{F}^+$, where row 1 is indexed by v_0 and rows 2 and 3 are the oriented incidence matrix of $2C_3$ with its row corresponding to v_3 removed. Consider the products $t_i \cdot e_j = (TA)_{ij}$. The products $t_1 \cdot e_1 = 0$, $t_1 \cdot e_2 = 0$, $t_2 \cdot e_2 = 0$, and $t_3 \cdot e_3 = 0$ imply $T_{11} = 0$, $T_{12} = 0$, $T_{22} = 0$, and $T_{33} = 0$, respectively. Thus $t_2 \cdot e_1 = -(t_3 \cdot e_1)$ implies $T_{21} = -T_{31}$ and $t_1 \cdot e_3 = -(t_2 \cdot e_3)$ implies $T_{13} = -T_{23}$; $t_2 \cdot e_5 = 0$ implies $T_{21} = -bT_{23}$ and $t_3 \cdot e_6 = 0$ implies $T_{31} = -cT_{32}$. Now $t_2 \cdot e_4 = -(t_3 \cdot e_4)$ yields $T_{23} = -T_{32}$, which the preceding relations imply is equivalent to the statement $T_{31}/b = T_{31}/c$. This holds if and only if either $T_{31} = 0$ or $b = c$. Hence if $b \neq c$, $T_{31} = 0$. Then the preceding relations imply that $T_{32} = 0$. Since $T_{33} = 0$, this implies T is singular, a contradiction. Thus in the case that $b \neq c$, A is not projectively equivalent to a canonical lift matrix particular to $(2C_3, \mathcal{B})$.

So assume T_{31} is nonzero and $b = c$. Then the relations above imply T is the matrix

$$\begin{pmatrix} 0 & 0 & t \\ bt & 0 & -t \\ -bt & t & 0 \end{pmatrix} = t \begin{pmatrix} 0 & 0 & 1 \\ b & 0 & -1 \\ -b & 1 & 0 \end{pmatrix}$$

for some nonzero $t \in \mathbb{F}$. Matrix T is non-singular; taking $t = 1$ we have

$$TA = \begin{pmatrix} 0 & 0 & 1 & 1 & b & d \\ -b & 0 & -1 & b-1 & 0 & b-d \\ b & 1 & 0 & 1-b & a-b & 0 \end{pmatrix}.$$

By claims 1, 2, 3, and 7, none of b, $b-1$, $a-b$, nor $b-d$ is zero. By scaling columns so that all nonzero entries in rows 2 and 3 are ± 1 (and appending a fourth row obtained by negating the sum of rows 2 and 3 if desired), we obtain a canonical lift matrix particular to $(2C_3, \mathcal{B})$. Thus A is projectively equivalent to a canonical lift matrix particular to $(2C_3, \mathcal{B})$ if and only if $b = c$. $\square$

Recall that P is the triangular prism with just its two triangles balanced, and that P_1 and P_2 are obtained from P by contracting 2 and 1 of the edges of the matching between the two triangles, respectively (Figure 5).

Lemma 4.14. *Let $\Omega \in \{\mathsf{P}, \mathsf{P}_1, \mathsf{P}_2\}$ and let A be an $\mathbb{F}$-matrix representing $M(\Omega)$. Then A is projectively equivalent to a canonical lift matrix particular to Ω or to a canonical frame matrix particular to Ω, but not both.*

Proof. First, consider P_1. Let A be an $\mathbb{F}$-representation of $M(\mathsf{P}_1)$. Let Y be a $K_{1,3}$-subgraph of P_1. Then $\nabla_Y \mathsf{P}_1$ is obtained from T_2 by the addition of an edge e that creates a balanced 2-cycle. Hence e is parallel with an element of $M(\nabla_Y \mathsf{P}_1)$. By Proposition 2.8 $M(\nabla_Y \mathsf{P}_1) = \nabla_Y M(\mathsf{P}_1)$. Thus by Lemma 4.13 every $\mathbb{F}$-representation of $\nabla_Y M(\mathsf{P}_1)$ is projectively equivalent to a canonical representation particular to $\nabla_Y \mathsf{P}_1$. In particular, $\nabla_Y A$ is projectively equivalent to a canonical representation particular to $\nabla_Y \mathsf{P}_1$. Thus by Proposition 2.11 A is projectively equivalent to a canonical representation particular to P_1.

Now consider P_2. Since $\nabla_Y \mathsf{P}_2$ is obtained from P_1 by the addition of an edge that creates a balanced 2-cycle, by the argument analogous to that of the previous paragraph every $\mathbb{F}$-representation of $M(\mathsf{P}_2)$ is projectively equivalent to a canonical representation particular to P_2. Finally, the observation that $\nabla_Y \mathsf{P}$ is obtained from P_2 by the addition of an edge that creates a balanced 2-cycle, along with the argument analogous to that of the previous paragraph, establishes the statement for $M(\mathsf{P})$.

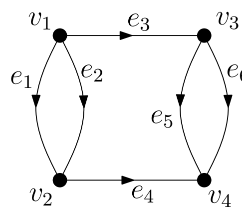


Figure 14

By Proposition 2.11 and Lemma 4.13, in no case may A be projectively equivalent to both a canonical lift and a canonical frame matrix. $\square$

Lemma 4.15. *Let $(K_4, \mathcal{B})$ be a biased graph with no balanced 3-cycle and let A be an $\mathbb{F}$-matrix representing $M(K_4, \mathcal{B})$. Then A is projectively equivalent to a canonical lift matrix particular to $(K_4, \mathcal{B})$ or to a canonical frame matrix particular to $(K_4, \mathcal{B})$, but not both.*

Proof. The biased graph $(K_4, \mathcal{B})$ is isomorphic to $\mathsf{D}_{0,i}$ for some $i \in \{0, 1, 2, 3\}$. There is a $K_{1,3}$-subgraph Y of $(K_4, \mathcal{B})$ so that $\nabla_Y(K_4, \mathcal{B}) \cong \mathsf{T}_{i+1}$. Since $\nabla_Y M(K_4, \mathcal{B}) = M(\nabla_Y(K_4, \mathcal{B}))$, the result follows by Lemma 4.13 and Proposition 2.11. $\square$

Lemma 4.16. *Let $(2C_4'', \mathcal{B})$ be a biased graph with no balanced 2-cycle and let A be an $\mathbb{F}$-matrix representing $F(2C_4'', \mathcal{B})$. Then A is projectively equivalent to a canonical frame matrix particular to $(2C_4'', \mathcal{B})$.*

Proof. Without loss of generality we may assume that $2C_4''$ is labelled as shown in Figure 14. There are three possibilities for $\mathcal{B}$: $|\mathcal{B}| \in \{0, 1, 2\}$.

Assume first that $|\mathcal{B}|$ is 0 or 1; *i.e.* either $\mathcal{B} = \emptyset$ or, without loss of generality, $\mathcal{B} = \{e_1e_3e_4e_6\}$. Then A is projectively equivalent to the matrix

$$A' = \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & a \\ 0 & 0 & 1 & 0 & 1 & b \\ 0 & 0 & 0 & 1 & 1 & c \end{pmatrix} \end{array}$$

where $a, b, c \in \mathbb{F}$ are distinct, neither of b nor c is 0, and none of a, b, or c are 1; in the case that $\mathcal{B} = \emptyset$, $a \neq 0$, while if $\mathcal{B} = \{e_1e_3e_4e_5\}$ then $a = 0$. Let

$$T = \begin{pmatrix} b-a & 1-b & a-1 & 0 \\ a-c & c-1 & 0 & 1-a \\ 0 & 0 & 1-a & 0 \\ 0 & 0 & 0 & a-1 \end{pmatrix}.$$

Then $\det(T) = (a-1)^3(c-b)$ so T is nonsingular, and

$$TA' = \begin{pmatrix} b-a & 1-b & a-1 & 0 & 0 & 0 \\ a-c & c-1 & 0 & 1-a & 0 & 0 \\ 0 & 0 & 1-a & 0 & 1-a & (1-a)b \\ 0 & 0 & 0 & a-1 & a-1 & (a-1)c \end{pmatrix}$$

which has the desired canonical form after column scaling.

So assume $|\mathcal{B}| = 2$. Without loss of generality, $\mathcal{B} = \{e_1e_3e_4e_6, e_2e_3e_4e_5\}$. Then A is projectively equivalent to

$$A' = \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & b \\ 0 & 0 & 0 & 1 & 1 & c \end{pmatrix} \end{array}$$

where b and c are nonzero, distinct, and not equal to 1. Let

$$T = \begin{pmatrix} -b & -1 & 1 & 0 \\ c & 1 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The determinant of T is $b-c$, so T is nonsingular. Now

$$TA' = \begin{pmatrix} -b & -1 & 1 & 0 & 0 & 0 \\ c & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & -b \\ 0 & 0 & 0 & 1 & 1 & c \end{pmatrix}$$

which has the desired canonical form after column scaling. □

Lemma 4.17. *Let $(2C_4'', \mathcal{B})$ be a biased graph with no balanced 2-cycle and let A be an $\mathbb{F}$-matrix representing $L(2C_4'', \mathcal{B})$. Then A is projectively equivalent to a canonical lift matrix particular to $(2C_4'', \mathcal{B})$.*

Proof. We may assume the edges of $2C_4''$ are labeled as in Figure 14. If $|\mathcal{B}| = 2$, then there is a $\mathrm{GF}(2)^+$-gain function γ realizing $(2C_4'', \mathcal{B})$. Thus $L(2C_4'', \mathcal{B})$ is binary, represented by $A_L(2C_4'', \gamma)$, so $L(2C_4'', \mathcal{B})$ has a projectively unique representation over every field, and the result follows. So now assume that either $\mathcal{B} = \emptyset$ or, without loss of generality, $\mathcal{B} = \{e_1e_3e_4e_6\}$. Since $\{e_1, e_2, e_5, e_6\}$ is a circuit, A is projectively equivalent to the matrix

$$A' = \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & a \\ 0 & 0 & 1 & 0 & 1 & b \\ 0 & 0 & 0 & 1 & 1 & b \end{pmatrix} \end{array}$$

for some $a, b \in \mathbb{F}$, where a and b are distinct, neither a nor b is 1, $b \neq 0$, and $a = 0$ if and only if $|\mathcal{B}| = 1$. Let

$$T = \begin{pmatrix} 0 & 1-b & 0 & 0 \\ b-a & 1-b & a-1 & 0 \\ a-b & b-1 & 0 & 1-a \\ 0 & 0 & 1-a & 0 \end{pmatrix}.$$

Then $\det(T) = (a-1)^2(a-b)(b-1) \neq 0$, so T is nonsingular, and

$$TA' = \begin{pmatrix} 0 & 1-b & 0 & 0 & 1-b & a-ab \\ b-a & 1-b & a-1 & 0 & 0 & 0 \\ a-b & b-1 & 0 & 1-a & 0 & 0 \\ 0 & 0 & 1-a & 0 & 1-a & b-ab \end{pmatrix}.$$

After scaling columns appropriately (and appending a fifth row obtained by negating the sum of rows 2, 3, and 4, if desired) this is a canonical lift matrix particular to $(2C_4'', \mathcal{B})$. □

For the almost-balanced case of Theorem 2 we need the result analogous to the previous lemmas for one more biased graph. Recall that we denote the graph obtained from $2C_3$ by deleting an edge by $2C_3\backslash e$.

Lemma 4.18. *Let $(2C_3\backslash e, \mathcal{B})$ be a biased graph with no balanced 2-cycle and let A be an $\mathbb{F}$-matrix representing $M(2C_3\backslash e, \mathcal{B})$. Then A is projectively equivalent to a canonical lift matrix particular to $(2C_3\backslash e, \mathcal{B})$ and A is projectively equivalent to a canonical frame matrix particular to $(2C_3\backslash e, \mathcal{B})$ or to a roll-up of $(2C_3\backslash e, \mathcal{B})$.*

Proof. Assume $2C_3\backslash e$ is labeled as in Figure 15. Then $\{e_1, e_2, e_3\}$ is a basis, so we may assume the first three columns of A are labelled e_1, e_2, e_3, and that

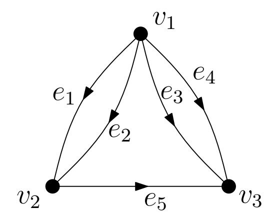


Figure 15: $2C_3\backslash e$.

these columns form an identity matrix. Hence A is projectively equivalent to the matrix

$$A' = \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & a \\ 0 & 0 & 1 & 1 & b \end{pmatrix} \end{array}$$

for some $a, b \in \mathbb{F}$. Since $\{e_1, e_2, e_5\}$ is not a circuit, $b \neq 0$, and since $\{e_3, e_4, e_5\}$ is not a circuit, $a \neq 1$. Choose an element $t \neq -1$ and let

$$T = \begin{pmatrix} t & 1 & -(a+t)/b \\ 1 & -1 & 0 \\ 0 & 0 & (a-1)/b \end{pmatrix}.$$

The determinant of T is $(1-a)(t+1)/b$, so T is nonsingular, and

$$TA' = \begin{pmatrix} t & 1 & -(a+t)/b & (b-a+t(b-1))/b & 0 \\ 1 & -1 & 0 & 0 & 1-a \\ 0 & 0 & (a-1)/b & (a-1)/b & a-1 \end{pmatrix}.$$

After scaling columns so that every entry in rows 2 and 3 is ± 1, this is a canonical lift matrix particular to $(2C_3\backslash e, \mathcal{B})$, where the first row is the "gains row" indexed by v_0 and rows 2 and 3 are indexed by vertices v_2 and v_3, respectively. Now re-index the first row as v_1. Taking $t = 0$ yields a canonical frame matrix particular to a roll-up of $(2C_3\backslash e, \mathcal{B})$. If $\mathbb{F} - \{0, -1, -a, (b-a)/(1-b)\}$ is nonempty, then choosing an element from this set for t yields a canonical frame matrix particular to $(2C_3\backslash e, \mathcal{B})$. $\square$

5 Matrix representations correspond to gain graphs

5.1 Projectively equivalent canonical representations arise from switching equivalent gain graphs

A biased graph representing a 3-connected matroid is 2-connected and has no 2-separation with one side inducing a balanced subgraph. Thus Theorem 1 follows immediately from Theorems 5.1 and 5.4 below.

5.1.1 Properly unbalanced representations

Theorem 5.1. *Let $(G, \mathcal{B})$ be a loopless, 2-connected, properly unbalanced biased graph. Let $\mathbb{F}$ be a field.*

(i) *The canonical frame matrices given by two $\mathbb{F}^\times$-gain functions φ and ψ realizing $(G, \mathcal{B})$ are projectively equivalent if and only if φ and ψ are switching equivalent.*

(ii) *The canonical lift matrices given by two $\mathbb{F}^+$-gain functions φ and ψ realizing $(G, \mathcal{B})$ are projectively equivalent if and only if φ and ψ are switching-and-scaling equivalent.*

(iii) *Let φ be an $\mathbb{F}^\times$-realization of $(G, \mathcal{B})$ and let ψ be an $\mathbb{F}^+$-realization of $(G, \mathcal{B})$. Then $A_F(G, \varphi)$ and $A_L(G, \psi)$ are not projectively equivalent.*

Proof. For statement (i) (respectively, statement (ii)), if φ and ψ are switching (resp., switching-and-scaling) equivalent then $A_F(G, \varphi)$ and $A_F(G, \psi)$ (resp., $A_L(G, \varphi)$ and $A_L(G, \psi)$) are projectively equivalent by Proposition 4.1.

For the converse of statement (i) (resp., statement (ii)) assume that φ and ψ are $\mathbb{F}^\times$-realizations (resp., $\mathbb{F}^+$-realizations) that are not switching (resp., switching-and-scaling) equivalent. By Theorem 6, there is a minor $(H, \mathcal{S})$ of $(G, \mathcal{B})$ such that either $(H, \mathcal{S}) \in \mathcal{G}_0$ with $\varphi|_H$ and $\psi|_H$ switching inequivalent or $(H, \mathcal{S}) \cong \mathsf{U}_2$ (resp., $(H, \mathcal{S}) \cong \mathsf{U}_3$) with $\varphi|_H$ and $\psi|_H$ switching (resp., switching-and-scaling) inequivalent on the 2-cycle of U_2 (resp., on the theta subgraph of U_3). By Lemmas 4.3, 4.6, 4.9, and 4.11, $A_F(H, \varphi|_H)$ and $A_F(H, \psi|_H)$ are not projectively equivalent (resp., by Lemmas 4.4, 4.7, 4.10, and 4.12, $A_L(H, \varphi|_H)$ and $A_L(H, \psi|_H)$ are not projectively equivalent). Thus $A_F(G, \varphi)$ and $A_F(G, \psi)$ (resp., $A_L(G, \varphi)$ and $A_L(G, \psi)$) are not projectively equivalent.

For statement (iii), first observe that if $(G, \mathcal{B})$ is not tangled, then $F(G, \mathcal{B}) \neq L(G, \mathcal{B})$ so $A_F(G, \varphi)$ and $A_L(G, \psi)$ are certainly not projectively equivalent. So assume $(G, \mathcal{B})$ is tangled. Then by Corollary 3.6 $(G, \mathcal{B})$ has a minor $(H, \mathcal{S}) \in \mathcal{G}_0$

where H is either $2C_3$ or K_4. Thus by Lemmas 4.5 and 4.8 $A_F(G,\varphi)$ and $A_L(G,\psi)$ are not projectively equivalent. $\square$

5.1.2 Almost balanced representations

Derived gain functions. Recall that when $(G,\mathcal{B})$ has no two vertex-disjoint unbalanced cycles, $F(G,\mathcal{B}) = L(G,\mathcal{B})$, and we write $M(G,\mathcal{B})$ to denote this matroid. Recall also that for an $\mathbb{F}^+$-gain function φ, in the case that G is connected, the matrix $A_L(G,\varphi)$ is projectively equivalent to the full-rank matrix $A_L^{-v}(G,\varphi)$, for any vertex $v \in V(G)$ (as described in Section 2.6). In particular, if $(G,\mathcal{B}_\varphi)$ is connected and has a balancing vertex u after deleting all joints, then $A_L(G,\varphi)$ is projectively equivalent to $A_L^{-u}(G,\varphi)$. Recall also that for every almost-balanced biased graph $(G,\mathcal{B})$, there is a family of biased graphs $\mathcal{R}_{(G,\mathcal{B})}$, each member of which represents $M(G,\mathcal{B})$ as a frame matroid, and there is a uniquely chosen member $(\widehat{G},\widehat{\mathcal{B}})$ of $\mathcal{R}_{(G,\mathcal{B})}$ such that all other members of $\mathcal{R}_{(G,\mathcal{B})}$ are obtained from $(\widehat{G},\widehat{\mathcal{B}})$ as roll-ups (Section 2.5).

Let $(G,\mathcal{B})$ be a biased graph with a joint. No $\mathrm{GF}(2)^\times$-gain function can realize $(G,\mathcal{B})$, for the trivial reason that $\mathrm{GF}(2)^\times$ has no non-identity element. Aside from those over GF(2), there is a very close relationship between gain functions from the additive and multiplicative groups of a field realizing almost-balanced biased graphs. Let $(G,\mathcal{B})$ be a connected almost-balanced biased graph, with balancing vertex u after deleting its set J of joints. Every gain function realizing $(G,\mathcal{B})$ is switching equivalent to a gain function assigning the group identity element to each link not incident u, obtained by normalizing on a spanning tree of $G-u$. So let T be a spanning tree of $G-u$, and let φ be a T-normalized $\mathbb{F}^\times$-gain function realizing $(G,\mathcal{B})$. Then there is a T-normalized $\mathbb{F}^+$-gain function φ^+ realizing $(\widehat{G},\widehat{\mathcal{B}})$ obtained from φ up to loops by simply replacing the multiplicative identity with the additive identity. That is, set $\varphi^+(e) = 0$ if e is a link not incident to u and $\varphi^+(e) = \varphi(e)$ if e is a link incident to u. To complete the definition of φ^+, simply set $\varphi^+(e) = 0$ if e is a joint not incident to u or a balanced loop, and $\varphi^+(e) = -1$ if e is a joint incident to u. Call φ^+ the $\mathbb{F}^+$-gain function *derived from* φ.

For each unbalancing class U of $\Sigma(u)$, we denote by $(G_U,\mathcal{B}_U)$ the roll-up of $(\widehat{G},\widehat{\mathcal{B}})$ in which U is a set of joints. Let T_U be a spanning tree of $\widehat{G}$ containing an edge in U, and let T be the spanning tree of $G-u$ obtained from T_U by deleting its edge incident to u. Let ψ be a T_U-normalized $\mathbb{F}^+$-gain function realizing $(\widehat{G},\widehat{\mathcal{B}})$. Assume $\mathbb{F}$ is not GF(2), and choose an element $a \in \mathbb{F}^\times$ that is not 1. Then there is a T-normalized $\mathbb{F}^\times$-gain function $\psi^\times$ realizing $(G_U,\mathcal{B}_U)$ obtained from ψ up to loops by simply replacing the additive identity with the multiplicative identity. That is, set $\psi^\times(e) = 1$ if e is a link not incident to u and $\psi^\times(e) = \psi(e)$ if e is a link incident to u. Every link e incident to u in G_U satisfies $\psi(e) \neq 0$, and so satisfies $\psi(e) \in \mathbb{F}^\times$. Complete the definition of $\psi^\times$ by simply setting $\psi^\times(e) = 1$ if e is a balanced loop and $\psi^\times(e) = a$ if e is a joint. Call $\psi^\times$ the $\mathbb{F}^\times$-gain function *derived from* ψ.

Lemma 5.2. *Let $(G, \mathcal{B})$ be a 2-connected almost-balanced biased graph and let $\mathbb{F}$ be a field other than* GF(2)*. Let J be the set of joints of $(G, \mathcal{B})$. Assume $(G, \mathcal{B})\backslash J$ has a unique balancing vertex u, and let T be a spanning tree of $G - u$.*

(i) *Let φ be a T-normalized $\mathbb{F}^\times$-gain function realizing a roll-up $(G_U, \mathcal{B}_U)$ of $(\widehat{G}, \widehat{\mathcal{B}})$. Then $A_L^{-u}(\widehat{G}, \varphi^+)$ is obtained from $A_F(G_U, \varphi)$ by scaling columns.*

(ii) *Let ψ be a T-normalized $\mathbb{F}^+$-gain function realizing $(\widehat{G}, \widehat{\mathcal{B}})$. Let U be the (possibly empty) unbalancing class of $\Sigma(u)$ for which $\psi(U) = \{0\}$. Then $A_F(G_U, \psi^\times)$ is obtained from $A_L^{-u}(\widehat{G}, \psi)$ by scaling columns.*

Proof. (i) Since φ is T-normalized, $\varphi(e) = 1$ for each link e that is not incident to u. We may assume that all edges incident to u are directed into u. Let A be the matrix obtained from $A_F(G_U, \varphi)$ by re-indexing row u as v_0 (the "gains" row) and scaling columns so that each column that is nonzero in just one row has that nonzero entry equal to -1, and columns with a nonzero entry in row v_0 have either all other entries 0 or have other nonzero entry equal to -1. Because φ assigns 1 to every link not incident to u, A is a full-rank canonical lift matrix (with row u removed) particular to $(\widehat{G}, \widehat{\mathcal{B}})$, where each edge in $\Sigma(u)$ is a link directed out from u. Moreover, for each edge $e \in E(G)$, the entry in row v_0, column e of A is equal to $\varphi^+(e)$, so $A = A_L^{-u}(\widehat{G}, \varphi^+)$.

(ii) Since ψ is T-normalized, $\psi(e) = 0$ for each edge e that is not incident to u. We may assume that all links incident to u in G are directed out from u. Since the rows of $A_L^{-u}(\widehat{G}, \psi)$ are indexed by v_0 (the "gains row") and $V(\widehat{G}) - u$, every column of $A_L^{-u}(\widehat{G}, \psi)$ has at most two nonzero entries. Let A be the matrix obtained from $A_L^{-u}(\widehat{G}, \psi)$ by re-indexing row v_0 as row u and scaling columns so that each column that has exactly two nonzero entries with a nonzero entry in row u has its other nonzero entry equal to 1, and every column with just one nonzero entry has its nonzero entry equal to $1 - a$, where a is the chosen element of $\mathbb{F}^\times$ different than 1 that $\psi^\times$ assigns to joints. Since $(\widehat{G}, \widehat{\mathcal{B}})$ has u as a balancing vertex, each column e of A with 0 in row u has either exactly two nonzero entries, which are 1 and -1, and these appear in rows v, w when e has endpoints v, w and neither v nor w is equal to u, or column e at most one nonzero entry, which is $1 - a$, appearing in row v when $e \in U$ with endpoints u, v in $\widehat{G}$. Thus A is a canonical frame matrix particular to $(G_U,\mathcal{B}_U)$ where each edge in $\Sigma(u)-U$ is a link directed into u and each element in U is a joint. Moreover, by definition the gain function $\psi^\times$ realizes $(G_U,\mathcal{B}_U)$ and $A=A_F(G_U,\psi^\times)$. $\square$

Let $(G,\mathcal{B})$ be a connected almost-balanced biased graph with a unique balancing vertex u after deleting its joints. Since we may always assume that all links incident to u are directed either into or out from u, and every gain function realizing $(G,\mathcal{B})$ is switching equivalent to a gain function assigning the identity element to each link not incident to u, we may define a *derived gain function from* any $\mathbb{F}^\times$- or $\mathbb{F}^+$-gain function, by first switching appropriately. Further, for each $\mathbb{F}^+$-gain function realizing $(\widehat{G},\widehat{\mathcal{B}})$ and each unbalancing class $U\in\Sigma(u)$, we may always switch to obtain an $\mathbb{F}^+$-gain function φ realizing $(\widehat{G},\widehat{\mathcal{B}})$ with the property that for each edge $e\in U$, $\varphi(e)=0$. Thus, with this extension of the notion of a derived gain function, Proposition 4.1 and Lemma 5.2 immediately yield the following.

Corollary 5.3. *Let $(G,\mathcal{B})$ be a 2-connected almost-balanced biased graph with a unique balancing vertex u after deleting its set of joints, and let $\mathbb{F}$ be a field other than* GF(2)*.*

(i) *For every $\mathbb{F}^+$-gain function φ realizing $(\widehat{G},\widehat{\mathcal{B}})$, and every unbalancing class $U\subseteq\Sigma(u)$, there is a derived $\mathbb{F}^\times$-gain function $\varphi^\times$ realizing the roll-up $(G_U,\mathcal{B}_U)$ of $(\widehat{G},\widehat{\mathcal{B}})$ such that $A_L(\widehat{G},\varphi)$ and $A_F(G_U,\varphi^\times)$ are projectively equivalent.*

(ii) *For every $\mathbb{F}^\times$-gain function ψ realizing a roll-up $(G_U,\mathcal{B}_U)$ of $(\widehat{G},\widehat{\mathcal{B}})$, there is a derived $\mathbb{F}^+$-gain function ψ^+ realizing $(\widehat{G},\widehat{\mathcal{B}})$ for which $A_F(G_U,\psi)$ and $A_L(\widehat{G},\psi^+)$ are projectively equivalent.*

(iii) *There is an $\mathbb{F}^\times$-gain function realizing $(\widehat{G},\widehat{\mathcal{B}})$ if and only if there is an $\mathbb{F}^+$-gain function φ realizing $(\widehat{G},\widehat{\mathcal{B}})$ for which $\varphi(e)\neq 0$ for each edge in $\Sigma(u)$.*

Equipped with the above tool, we can now state and prove a result analogous to Theorem 5.1 for almost-balanced biased graphs.

Theorem 5.4. *Let $(G,\mathcal{B})$ be a 2-connected, almost-balanced biased graph with a unique balancing vertex u after deleting its joints, with no joint incident to u, and with no vertical 2-separation with one side balanced. Let $\mathbb{F}$ be a field.*

(i) *The canonical lift matrices given by two $\mathbb{F}^+$-gain functions φ and ψ realizing $(\widehat{G},\widehat{\mathcal{B}})$ are projectively equivalent if and only if φ and ψ are switching-and-scaling equivalent.*

(ii) *Let U and W be unbalancing classes of $\Sigma(u)$. Let φ and ψ be $\mathbb{F}^\times$-gain functions realizing $(G_U,\mathcal{B}_U)$ and $(G_W,\mathcal{B}_W)$, respectively. The canonical frame* *matrices given by φ and ψ are projectively equivalent if and only if their derived gain functions φ^+ and ψ^+ are switching-and-scaling equivalent.*

(iii) *Let U be an unbalancing class of $\Sigma(u)$. The canonical lift matrix given by an $\mathbb{F}^+$-gain function φ realizing $(\widehat{G}, \widehat{\mathcal{B}})$ and the canonical frame matrix given by an $\mathbb{F}^\times$-gain function ψ realizing $(G_U, \mathcal{B}_U)$ are projectively equivalent if and only if φ and the derived gain function ψ^+ are switching-and-scaling equivalent.*

(iv) *Let φ and ψ be $\mathbb{F}^+$- and $\mathbb{F}^\times$-gain functions, respectively, realizing $(\widehat{G}, \widehat{\mathcal{B}})$. The canonical lift matrix given by φ and the canonical frame matrix given by ψ are projectively equivalent if and only if φ and the derived gain function ψ^+ are switching-and-scaling equivalent.*

Proof. (i) Let φ and ψ be $\mathbb{F}^+$-gain functions realizing $(\widehat{G}, \widehat{\mathcal{B}})$. If φ and ψ are switching-and-scaling equivalent then by Proposition 4.1 $A_L(\widehat{G}, \varphi)$ and $A_L(\widehat{G}, \psi)$ are projectively equivalent.

Conversely, suppose for a contradiction that $A_L(\widehat{G}, \varphi)$ and $A_L(\widehat{G}, \psi)$ are projectively equivalent but that φ and ψ are not switching-and-scaling equivalent. We may assume that all edges incident to u have u as their tail. By switching we may assume that $\varphi(e) = \psi(e) = 0$ for each edge e not incident to u. Consider the full-rank canonical lift matrices $A_L^{-u}(\widehat{G}, \varphi)$ and $A_L^{-u}(\widehat{G}, \psi)$ with rows indexed by $v_0 \cup (V(G) - u)$, where $v_0 \notin V(G)$ is the "gains row". Put $A = A_L^{-u}(\widehat{G}, \varphi)$ and $B = A_L^{-u}(\widehat{G}, \psi)$, and suppose T is a nonsingular matrix such that $TAS = B$, where S is a nonsingular diagonal column-scaling matrix. Let the rows and columns of T be indexed by $v_0 \cup (V(G) - u)$ according to the rows of A. By statement (ii) of Lemma 4.2 either T has an entry on its main diagonal that is not 1 or T has a nonzero entry off its main diagonal, in either case in a row other than v_0. Suppose first T has all entries off its main diagonal equal to 0, aside from those in row v_0. Let U be the $|V(G)| \times |V(G)|$ diagonal matrix with rows and columns indexed by $v_0 \cup V(G) - u$ in which entry $U_{v_0 v_0} = 1$ and entry $U_{vv} = a^{-1}$ if entry $T_{vv} = a$. Since T is non-singular no such entry is 0. Removing the row and column of UT indexed by v_0 leaves an identity matrix and $(UT)AR = B$, where R is an appropriate diagonal matrix scaling the columns of $(UT)A$. Thus φ and ψ are switching-and-scaling equivalent by statement (ii) of Lemma 4.2, contrary to assumption.

So assume there is a nonzero element off the main diagonal of T in a row other than v_0. Suppose the entry in row x, column y is nonzero, where $x \neq y$ and $x \neq v_0$. Since G is 2-connected and has no vertical 2-separation with one side balanced, there is an unbalanced cycle C avoiding x while containing y. Let f, f' be the edges of C incident to u. Then $\varphi(f) \neq \varphi(f')$. Denote by T_x row x of T and by A_e column e of A. Consider the equations $T_x \cdot A_e = B_{xe}$ given by each of the dot products of row x of T with column e of A, for each

$e \in E(C)$. Since C avoids x, for each edge $e \in E(C)$ entry B_{xe} is zero. There are precisely two nonzero entries in each column e of A with $e \in E(C)$, and other than columns f and f' one of these two entries is 1 and the other is -1. Thus the system of equations $\{T_x \cdot A_e = 0 : e \in E(C)\}$ imply $\varphi(f) = \varphi(f')$, a contradiction.

(ii) If φ^+ and ψ^+ are switching-and-scaling equivalent, then by Proposition 4.1, $A_L(\widehat{G}, \varphi^+)$ and $A_L(\widehat{G}, \psi^+)$ are projectively equivalent. Hence by Corollary 5.3, $A_F(G_U, \varphi)$ and $A_F(G_W, \psi)$ are projectively equivalent. Conversely, suppose $A_F(G_U, \varphi)$ and $A_F(G_W, \psi)$ are projectively equivalent. Then by Corollary 5.3, $A_L(\widehat{G}, \varphi^+)$ and $A_L(\widehat{G}, \psi^+)$ are projectively equivalent, and so by statement (i), φ^+ and ψ^+ are switching-and-scaling equivalent.

The proofs of statements (iii) and (iv) are straightforward modifications of the proof of (ii). □

5.2 Matrix representations arise from biased graph representations

Theorem 2 is an immediate consequence of Theorem 5.5 below (together with a straightforward check for the case of rank 2).

Theorem 5.5. *Let M be a 3-connected matroid of rank greater than two, and let $\mathbb{F}$ be a field. Let A be a matrix over $\mathbb{F}$ representing M and let $(G, \mathcal{B})$ be a biased graph representing M. If $(G, \mathcal{B})$ is properly unbalanced then exactly one of the following holds.*

(i) *A is projectively equivalent to a canonical lift matrix particular to $(G, \mathcal{B})$, or*

(ii) *A is projectively equivalent to a canonical frame matrix particular to $(G, \mathcal{B})$.*

If $(G, \mathcal{B})$ is almost-balanced then each of the following hold, unless $\mathbb{F}$ is GF(2), *in which case* (i) *holds.*

(i) *A is projectively equivalent to a canonical lift matrix particular to $(\widehat{G}, \widehat{\mathcal{B}})$, and*

(ii) *A is projectively equivalent to a canonical frame matrix particular to each roll-up of $(\widehat{G}, \widehat{\mathcal{B}})$.*

Theorem 5.5 follows immediately from Theorems 5.6 and 5.9 below.

Theorem 5.6. *Let M be a matroid represented by a 2-connected, properly unbalanced biased graph $(G, \mathcal{B})$. Let $\mathbb{F}$ be a field and let A be a matrix over $\mathbb{F}$ representing M. Exactly one of the following holds:*

(i) *A is projectively equivalent to a canonical lift matrix particular to $(G, \mathcal{B})$, or*

(ii) *A is projectively equivalent to a canonical frame matrix particular to $(G, \mathcal{B})$.*

We will require the following fact on several occasions. It follows immediately from the fact that a pair of edges incident to a vertex of degree two form a series pair in the matroid.

Lemma 5.7. *Let $(H, \mathcal{S})$ be a subdivision of $(G, \mathcal{B})$. Let $\mathbb{F}$ be a field and let $\Gamma \in \{\mathbb{F}^\times, \mathbb{F}^+\}$.*

(i) *The $\mathbb{F}$-matrix representations of $F(H, \mathcal{S})$ are in one-to-one correspondence with the $\mathbb{F}$-matrix representations of $F(G, \mathcal{B})$ up to projective equivalence.*

(ii) *The $\mathbb{F}$-matrix representations of $L(H, \mathcal{S})$ are in one-to-one correspondence with the $\mathbb{F}$-matrix representations of $L(G, \mathcal{B})$ up to projective equivalence.*

(iii) *The Γ-realizations of $(H, \mathcal{S})$ are in one-to-one correspondence with the Γ-realizations of $(G, \mathcal{B})$ up to switching (resp., switching-and-scaling).*

We also need the following more technical fact to prove Theorem 5.6.

Lemma 5.8. *Let $(G, \mathcal{B})$ be a connected biased graph with a joint e such that $(G, \mathcal{B})\backslash e$ is a biased $2C_3$ with no balanced 2-cycle or a biased K_4 with no balanced triangle. Let $\mathbb{F}$ be a field, and let φ be an $\mathbb{F}^\times$- or $\mathbb{F}^+$-gain function on G.*

(i) *If $A_F(G\backslash e, \varphi)$ represents $M((G, \mathcal{B})\backslash e)$ then $A_F(G\backslash e, \varphi)$ does not extend to an $\mathbb{F}$-representation of $L(G, \mathcal{B})$.*

(ii) *If $A_L(G\backslash e, \varphi)$ represents $M((G, \mathcal{B})\backslash e)$ then $A_L(G\backslash e, \varphi)$ does not extend to an $\mathbb{F}$-representation of $F(G, \mathcal{B})$.*

Proof. We give a detailed proof for the case in which $(G, \mathcal{B})\backslash e$ is a biased $2C_3$. The case for which $(G, \mathcal{B})\backslash e$ is a biased K_4 follows from Δ-Y and Y-Δ exchanges, by Propositions 2.8 and 2.11.

(i) Suppose for a contradiction that there is a matrix A over $\mathbb{F}$ representing $L(G, \mathcal{B})$ such that removing column e from A yields the matrix $A_F(G\backslash e, \varphi)$. We may assume that A has full rank, and so has three rows. Since e is not a loop of $L(G, \mathcal{B})$, column e of A is nonzero. If column e has just one nonzero entry, then A is an $\mathbb{F}$-representation of a matroid $F(\Omega)$ where Ω is obtained from $(G,\mathcal{B})\backslash e$ by adding a joint to a vertex. But comparing circuits we see that $F(\Omega) \neq L(G,\mathcal{B})$, a contradiction. If column e has exactly two nonzero entries, then A is an $\mathbb{F}$-representation of a matroid $F(\Omega)$, where Ω is obtained from $(G,\mathcal{B})\backslash e$ by adding a link. But again comparing circuits we see that $F(\Omega) \neq L(G,\mathcal{B})$, a contradiction. So finally suppose column e has three nonzero entries. Let X be an unbalanced 2-cycle of Ω. Then $X \cup e$ is a circuit of $L(G,\mathcal{B})$ but the columns of A corresponding to $X \cup e$ are linearly independent, a contradiction.

(ii) Suppose for a contradiction that there is a matrix A representing $F(G,\mathcal{B})$ such that removing column e from A yields $A_L(G\backslash e,\varphi)$. Let $V(G) = \{v_1,v_2,v_3\}$ where e is incident to v_1. Then $A\backslash e = A_L(G\backslash e,\varphi)$ has rows indexed by $v_0 \cup V(G)$ where $v_0 \notin V(G)$ is the "gains row" and removing row v_0 from $A_L(G\backslash e,\varphi)$ leaves the oriented incidence matrix of $G\backslash e$. Since $F(G,\mathcal{B})$ has rank three while A has four rows, A is not of full rank. Since in $A\backslash e$ row v_0 is not in the span of $\{v_1,v_2,v_3\}$, neither is row v_0 in the span of rows $\{v_1,v_2,v_3\}$ in A. Thus in A row v_3 is in the span of rows v_1 and v_2. Since any linear combination of rows v_1, v_2, and v_3 in A yields a corresponding linear combination in $A\backslash e$, this implies that the entries of A in rows v_1, v_2, and v_3 of column e sum to zero. Put $A_{v_1e} = a$, $A_{v_2e} = b$; then $A_{v_3e} = -(a+b)$.

First suppose that $a = b = 0$. As e is not a loop of $F(G,\mathcal{B})$, column e is nonzero. Thus $A_{v_0e} \neq 0$. Let X be the unbalanced 2-cycle consisting of the pair of edges linking v_2 and v_3. Then $X \cup e$ is independent in $F(G,\mathcal{B})$ but the columns of A representing $X \cup e$ are linearly dependent, a contradiction. Next suppose $a = -b$. Let X be the unbalanced 2-cycle consisting of the edges linking v_1 and v_3. The set $X \cup e$ is a circuit in $F(G,\mathcal{B})$ but has columns linearly independent in A, a contradiction. Finally, suppose none of a, b, nor $-(a+b)$ are zero. Let X be the unbalanced 2-cycle consisting of the edges linking v_1 and v_2. Then $X \cup e$ is a circuit of $F(G,\mathcal{B})$ but has columns linearly independent in A, a contradiction. $\square$

Proof of Theorem 5.6. We show that if $M = F(G,\mathcal{B})$ then there is an $\mathbb{F}^\times$-gain function φ such that A is projectively equivalent to $A_F(G,\varphi)$, that if $M = L(G,\mathcal{B})$ then there is an $\mathbb{F}^+$-gain function ψ such that A is projectively equivalent to $A_L(G,\psi)$, and that if $(G,\mathcal{B})$ is tangled, so $F(G,\mathcal{B}) = L(G,\mathcal{B})$, then A is not projectively equivalent to both $A_F(G,\varphi)$ and $A_L(G,\psi)$.

By Theorem 5, $(G,\mathcal{B})$ contains a subgraph Ω_0 that is a subdivision of a biased graph in $\mathcal{T}_0$. Let $(G_0,\mathcal{B}_0)$ be the biased subgraph of $(G,\mathcal{B})$ induced by $E(\Omega_0)$, with $V(G_0) = V(G)$. Let A_0 be the submatrix of A consisting of the columns whose elements are in $E(\Omega_0)$. By Lemmas 4.13, 4.14, 4.15, 4.16, or 4.17, and Lemma 5.7, there exists an $\mathbb{F}^\times$-gain function φ_0 such that A_0 is projectively equivalent to $A_F(G_0, \varphi_0)$, or there exists an $\mathbb{F}^+$-gain function ψ_0 such that A_0 is projectively equivalent to $A_L(G_0, \psi_0)$, but not both.

Let J be the set of joints of $(G, \mathcal{B})$. Since both Ω_0 and G are 2-connected, there is a sequence of 2-connected biased subgraphs $\Omega_0 \subset \Omega_1 \subset \cdots \subset \Omega_n$ where $\Omega_n = (G, \mathcal{B})\backslash J$ such that for each $i \in \{0, \ldots, n-1\}$ there is a path P_i in G internally disjoint from Ω_i so that $\Omega_i \cup P_i = \Omega_{i+1}$. For each $i \in \{0, \ldots, n\}$, let $(G_i, \mathcal{B}_i)$ be the biased subgraph of $(G, \mathcal{B})$ induced by $E(\Omega_i)$ with $V(G_i) = V(G)$. Let A_i be the submatrix of A consisting of all rows of A and precisely those columns representing $E(\Omega_i)$. Thus for each i, in the case that $M = F(G, \mathcal{B})$, A_i represents $F(\Omega_i)$; in the case that $M = L(G, \mathcal{B})$, A_i represents $L(\Omega_i)$. Inductively assume that for some $i \geq 0$ there exist nonsingular matrices $T_0, \ldots, T_i$, nonsingular diagonal matrices $S_0, \ldots, S_i$, and gain functions $\varphi_0, \ldots, \varphi_i$, such that either

(1) $T_j A_j S_j = A_F(G_j, \varphi_j)$ for each $j \in \{0, \ldots, i\}$, or

(2) $T_j A_j S_j = A_L(G_j, \psi_j)$ for each $j \in \{0, \ldots, i\}$.

We will show the same projective equivalence for A_{i+1}. We first obtain this conclusion in the case that P_i consists of a single edge. Then if P_i has length greater than one, the conclusion follows from Lemma 5.7. So suppose P_i consists of a single edge e_i linking vertices $u_i, v_i \in V(\Omega_i)$. We consider cases (1) and (2) above separately.

(1) Consider the matrix $T_i A_{i+1}$. Matrix A_{i+1} has rows indexed by $V(G)$, according to the indexing of the corresponding rows of $T_i A_i S_i$, and columns indexed by $E(G_{i+1})$. We first show that column e_i of $T_i A_{i+1}$ is zero in every row aside from u_i and v_i. Suppose for a contradiction that e_i is nonzero in a row x of $T_i A_{i+1}$ differing from u_i and v_i. Since Ω_0 does not have a balancing vertex, neither does Ω_i. Thus $\Omega_i - x$ is unbalanced and connected. Hence there is a subset $U_i \subseteq E(\Omega_i - x)$ that induces a subgraph of Ω_i that is a spanning tree of $\Omega_i - x$ along with one additional edge whose fundamental cycle with respect to this tree is unbalanced. Contained in $U_i \cup e_i$ is a biased subgraph C whose edge set is a circuit of $F(\Omega_{i+1})$. The subgraph C contains e_i and does not contain x. Since $T_i A_i S_i = A_F(G_i, \varphi_i)$, the columns of $T_i A_{i+1}$ representing $E(C) - e_i$ are all zero in row x. Hence while C is a circuit of $F(\Omega_{i+1})$ the columns of $T_i A_{i+1}$ representing C are linearly independent, a contradiction.

We now show that both rows u_i and v_i in column e_i of $T_i A_{i+1}$ are nonzero. Since e_i is not a loop of M, at least one entry of column e is nonzero; without loss of generality assume its entry in row u_i is not zero. For a contradiction, suppose its entry in row v_i is zero. Let Q be a subgraph of $\Omega_i - v_i$ consisting of an unbalanced cycle and a path (possibly trivial) connecting this cycle to u_i. In $F(\Omega_{i+1})$, $E(Q) \cup e_i$ is independent, but the columns of $T_i A_{i+1}$ representing $E(Q) \cup e_i$ are linearly dependent, a contradiction. Thus column e_i of $T_i A_{i+1}$ is nonzero in precisely its rows u_i, v_i corresponding to the endpoints of edge e_i in Ω_i. Let $T_{i+1} = T_i$ and let S_{i+1} be the diagonal matrix obtained from S_i by adding a column to scale column e_i of $T_{i+1}A_{i+1}$ so that its entry in row u_i is 1. Extend the $\mathbb{F}^{\times}$-gain function φ_i by defining $\varphi_{i+1}(e_i)$ to be $-(T_{i+1}A_{i+1}S_{i+1})_{v_i e}$. Now $T_{i+1}A_{i+1}S_{i+1}$ is the canonical frame matrix $A_F(G_{i+1}, \varphi_{i+1})$. By induction, there is a nonsingular matrix T_n, a diagonal matrix S_n, and a gain function φ_n such that $T_n A_n S_n = A_F(G_n, \varphi_n)$.

If $(G, \mathcal{B})$ has no joints we are done. So assume J is nonempty. We now claim that $M \neq L(G, \mathcal{B})$. For suppose contrarily that $M = L(G, \mathcal{B})$. Since $T_n A_n S_n$ is a canonical frame representation of $M \backslash J$, it must be the case that Ω_n is tangled. By Theorem 3.3 Ω_n contains a link minor $(H, \mathcal{S})$ that is either a biased $2C_3$ with no balanced 2-cycle or a biased K_4 with no balanced triangle. Since J is nonempty $(G, \mathcal{B})$ has a link minor $(H', \mathcal{S}')$ where $(H', \mathcal{S}')$ is obtained by adding a joint e incident to a vertex of $(H, \mathcal{S})$. Since $T_n A S_n$ is a representation over $\mathbb{F}$ for $L(G, \mathcal{B})$ that agrees with $T_n A_n S_n$ on all elements aside from possibly those in J, and since $(H', \mathcal{S}')$ is a link minor of $(G, \mathcal{B})$, by Lemma 2.5 there is a matrix B over $\mathbb{F}$ representing $L(H', \mathcal{S}')$ with the property that $B \backslash e$ is a canonical frame matrix particular to $(H, \mathcal{S})$. But this is impossible by Lemma 5.8. Thus $M \neq L(G, \mathcal{B})$.

Finally, we show that each column e of $T_n A$ for which $e \in J$ has exactly one nonzero entry. Suppose $e \in J$ has endpoint u and that column e of $T_n A$ is nonzero in row $v \neq u$. Let C be the edge set of an unbalanced cycle in $\Omega_n - v$ together with a path linking this cycle and u. Then C is a circuit of $F(G, \mathcal{B})$ but its corresponding columns in $T_n A$ are linearly independent, a contradiction. Thus there is a diagonal matrix S scaling the columns of $T_n A$ such that $T_n A S$ is a canonical frame matrix particular to $(G, \mathcal{B})$.

(2) We proceed as in case (1), considering the matrix $T_i A_{i+1}$. Matrix A_{i+1} has rows indexed by $V(G) \cup v_0$, according to the indexing of the corresponding rows of $T_i A_i S_i$ where $v_0 \notin V(G)$ corresponds to the "gains row" of $T_i A_i S_i$, and columns indexed by $E(G_{i+1})$. We first show that column e_i of $T_i A_{i+1}$ is zero in every row aside from u_i, v_i, and v_0. Suppose for a contradiction that e_i is nonzero in a row $x \notin \{u_i, v_i, v_0\}$ of $T_i A_{i+1}$. As in case (1), let $U_i \subseteq E(\Omega_i - x)$ be a set of edges inducing a subgraph of Ω_i that is a spanning tree of $\Omega_i - x$ along with one additional edge whose fundamental cycle with respect to this tree is unbalanced. Contained in $U_i \cup e_i$ is a biased subgraph C whose edge set is a circuit of $L(\Omega_{i+1})$. The subgraph C contains e_i and does not contain x. Since $T_i A_i S_i = A_L(G_i, \psi_i)$, the columns of $T_i A_{i+1}$ representing $E(C) - e_i$ are

all zero in row x. Hence while C is a circuit of $L(\Omega_{i+1})$ the columns of T_iA_{i+1} representing C are linearly independent, a contradiction.

Since Ω_i is unbalanced and 2-connected and Ω_{i+1} is obtained from Ω_i by adding a single edge, $r(L(\Omega_{i+1})) = r(L(\Omega_i))$. Thus A_i and A_{i+1} have the same rank. Since row v_0 of $T_iA_iS_i$ is not in the span of the rows $V(G)$ neither is row v_0 of T_iA_{i+1} in the span of the rows in $V(G)$. But the sum of the rows in $V(G)$ of T_iA_i is zero, so likewise the sum of the rows in $V(G)$ of T_iA_{i+1} must be zero: otherwise the rank of T_iA_{i+1} would be greater than that of T_iA_i, a contradiction.

Suppose first that both entries of column e in rows u_i and v_i are zero. Element e_i is not a loop of M, so then its entry in row v_0 is nonzero. Let C be an unbalanced cycle in $\Omega_i - u_i$. Then $C \cup e$ is independent in $L(\Omega_{i+1})$ but the columns of T_iA_{i+1} representing $C \cup e$ are linearly dependent, a contradiction. Thus column e_i has entries a and $-a$ in rows u_i and v_i, where $a \neq 0$. Take $T_{i+1} = T_i$ and let S_{i+1} be the diagonal matrix obtained by adding a column to S_i to scale column e_i of $T_{i+1}A_{i+1}$ by a^{-1}. Extend the $\mathbb{F}^+$-gain function ψ_i to $E(G_{i+1})$ by defining $\psi(e_i)$ to be the entry in row v_0 of column e_i. Now $T_{i+1}A_{i+1}S_{i+1}$ is the canonical lift matrix $A_L(G_{i+1}, \psi_{i+1})$. By induction, there is a nonsingular matrix T_n, a diagonal matrix S_n, and a gain function ψ_n such that $T_nA_nS_n = A_L(G_n, \psi_n)$.

If $(G, \mathcal{B})$ has no joints we are done. So assume J is nonempty. Analogous to the situation in case (1), we now claim that $M \neq F(G, \mathcal{B})$. Suppose to the contrary that $M = F(G, \mathcal{B})$. Since $T_nA_nS_n$ is a canonical lift representation of $M \backslash J$, it must be the case that Ω_n is tangled. By Theorem 3.3, Ω_n contains a link minor $(H, \mathcal{S})$ that is either a biased $2C_3$ with no balanced 2-cycle or a biased K_4 with no balanced triangle. Since J is nonempty $(G, \mathcal{B})$ has a link minor $(H', \mathcal{S}')$ where $(H', \mathcal{S}')$ is obtained by adding a joint e incident to a vertex of $(H, \mathcal{S})$. Since T_nAS_n is a representation over $\mathbb{F}$ for $F(G, \mathcal{B})$ that agrees with $T_nA_nS_n$ on all elements aside from possibly those in J, and since $(H', \mathcal{S}')$ is a link minor of $(G, \mathcal{B})$, by Lemma 2.5 there is a matrix B over $\mathbb{F}$ representing $F(H', \mathcal{S}')$ with the property that $B \backslash e$ is a canonical lift matrix particular to $(H, \mathcal{S})$. This violates Lemma 5.8, so $M \neq F(G, \mathcal{B})$.

Finally, we show that each column e of T_nA for which $e \in J$ has a nonzero entry only in row v_0. Suppose for a contradiction that $e \in J$ with column e of T_nA nonzero in row $v \neq v_0$. Let C be the edge set of an unbalanced cycle in $\Omega_n - v$. Then $C \cup e$ is a circuit of $L(G, \mathcal{B})$ but its corresponding columns in T_nA are linearly independent, a contradiction. Thus T_nAS_n is a canonical lift matrix particular to $(G, \mathcal{B})$.

This completes the proof that at least one of statements (i) or (ii) of the theorem hold. But A_0 is not projectively equivalent to both a canonical frame matrix and a canonical lift matrix particular to $(G_0, \mathcal{B}_0)$. Thus neither is A projectively equivalent to both a canonical frame matrix and a canonical lift matrix particular to $(G, \mathcal{B})$. $\square$

The first statement of Theorem 5.5 follows immediately from Theorem 5.6. For the second statement of Theorem 5.5, we would like to show that for each 3-connected matroid M represented by an almost-balanced biased graph $(G, \mathcal{B})$, given any matrix A representing M, A is projectively equivalent to a canonical lift matrix particular to $(\widehat{G}, \widehat{\mathcal{B}})$. Unfortunately, this can fail in the case that M has rank 2. Let M be a 3-connected rank-2 matroid represented by the biased graph $(G, \emptyset)$ consisting of $E(M) - 1$ links between a pair of vertices and a single joint. Let $\mathbb{F}$ be a field and let A be a matrix over $\mathbb{F}$ representing M. Then A is projectively equivalent to the matrix

$$A' = \begin{pmatrix} 1 & 0 & 1 & 1 & \cdots & 1 \\ 0 & 1 & a_1 & a_2 & \cdots & a_{n-2} \end{pmatrix}$$

where $n = |E(M)|$, for each i, $a_i \notin \{0, 1\}$, and the a_i are distinct. The matrix A' is a canonical frame representation particular to a roll-up of $(G, \emptyset)$, and A' is a full-rank canonical lift matrix particular to $(G, \emptyset)$ (with its second row as the "gains row" indexed by v_0, and "missing" its row indexed by one of the vertices of G). It is straightforward to see that by elementary row operations and column scaling we may obtain from A' a canonical frame matrix particular to $(G, \emptyset)$. However, $(\widehat{G}, \widehat{\emptyset})$ is the loopless contrabalanced biased graph obtained from $(G, \emptyset)$ by unrolling its joint, and there is no guarantee that A need be projectively equivalent to a canonical matrix representation particular to $(\widehat{G}, \widehat{\emptyset})$. Indeed, there is no guarantee that such a representation exists. For instance, if $n = 6$ and $\mathbb{F}$ is GF(5), then the above argument shows that A is projectively equivalent to a canonical frame matrix particular to a roll-up of $(G, \emptyset)$, and to a canonical lift matrix particular to $(G, \emptyset)$, but a canonical lift representation particular to $(\widehat{G}, \widehat{\emptyset})$ requires six distinct elements of $\mathbb{F}^+$, so no such matrix exists. The problem is that the field is just too small by one element to permit a canonical lift representation of a such a rank-2 matroid, Fortunately, this is the only problem that may occur:

Theorem 5.9. *Let M be a matroid of rank greater than two represented by a 2-connected almost-balanced biased graph $(G, \mathcal{B})$ having no 2-separation with one side balanced. Let $\mathbb{F}$ be a field and let A be an $\mathbb{F}$-matrix representing M. Then*

(i) *A is projectively equivalent to a canonical lift matrix particular to $(\widehat{G}, \widehat{\mathcal{B}})$, and*

(ii) *A is projectively equivalent to a canonical frame matrix particular to each roll-up of $(\widehat{G}, \widehat{\mathcal{B}})$, unless $\mathbb{F}$ is* GF(2)*.*

Furthermore, A is projectively equivalent to a canonical frame matrix particular to $(\widehat{G}, \widehat{\mathcal{B}})$ if and only if whenever φ is an $\mathbb{F}^+$-gain function for which A and $A_L(\widehat{G}, \varphi)$ are projectively equivalent, φ is switching equivalent to a gain function assigning 0 *to e if and only if e is a link not incident to u or e is a loop of M.*

We will need the following straightforward fact.

Lemma 5.10. *Let C be the balanced triangle of $D_{1,0}$ and let Y be a $K_{1,3}$-subgraph of $D_{1,0}$ meeting C in exactly two edges. Then the simplification of $\nabla_Y \mathsf{D}_{1,0}$ is isomorphic to B'_1.*

Proof of Theorem 5.9. Suppose first that $(\widehat{G}, \widehat{\mathcal{B}})$ does not contain a contrabalanced theta subgraph. The only biased graph representing $U_{2,4}$ without a contrabalanced theta is shown at right in Figure 7. Moreover, F_7 is neither frame nor lifted-graphic, and the only biased graphs representing F_7^*, $M^*(K_5)$, and $M^*(K_{3,3})$ are all properly unbalanced [16].[1] Since none of these biased graphs can occur as a minor of $(\widehat{G}, \widehat{\mathcal{B}})$ and $(\widehat{G}, \widehat{\mathcal{B}})$ represents M, M contains none of $U_{2,4}$, F_7, F_7^*, $M^*(K_5)$, nor $M^*(K_{3,3})$ as a minor. Thus M is graphic. The first two statements follow. The third statement follows from Corollary 5.3.

So assume now that $(\widehat{G}, \widehat{\mathcal{B}})$ contains a contrabalanced theta subgraph. Let v be a balancing vertex of $(\widehat{G}, \widehat{\mathcal{B}})$ and let J be the set of joints of $(\widehat{G}, \widehat{\mathcal{B}})$. Since the set of joints not incident to v form an unbalancing class of $\Sigma(v)$ in $(G, \mathcal{B})$, and all of the unbalancing classes of $\Sigma(v)$ are unrolled in $(\widehat{G}, \widehat{\mathcal{B}})$, every joint of $(\widehat{G}, \widehat{\mathcal{B}})$ is incident to v. We claim that $(\widehat{G}, \widehat{\mathcal{B}}) \backslash J$ does not have another balancing vertex $u \neq v$. For suppose to the contrary that $(\widehat{G}, \widehat{\mathcal{B}}) \backslash J$ has a balancing vertex $u \neq v$. Then $(\widehat{G}, \widehat{\mathcal{B}}) \backslash J$ has the structure described in Proposition 2.12: there are graphs $G_1, \ldots, G_m$ such that $\widehat{G} = G_1 \cup \cdots \cup G_m$ and $G_j \cap G_k = \{u, v\}$ for each pair $j \neq k$ and a cycle is in $\widehat{\mathcal{B}}$ if and only if it is contained in a single graph G_j. Since $(\widehat{G}, \widehat{\mathcal{B}})$ contains a contrabalanced theta, $m \geq 3$. If there is a subgraph G_i with $E(G_i) \geq 2$, then $(E(G_i), E(G) \backslash E(G_i))$ is a 2-separation of $(G, \mathcal{B})$ with one side balanced, contrary to assumption. Thus for each i, $|E(G_i)| = 1$. Thus $G = mK_2$ and $\mathcal{B}$ is empty. But $M(mK_2, \emptyset)$ is isomorphic to the m-point line $U_{2,m}$, so $r(M) = 2$, contrary to assumption.

[1] The mistake in [16], corrected at `http://people.math.binghamton.edu/zaslav/Tpapers/index.html` does not affect this claim.

Thus v is the unique balancing vertex of $(\widehat{G}, \widehat{\mathcal{B}})\backslash J$. Since G is 2-connected, so is $\widehat{G}$. Thus by Proposition 3.5 $(\widehat{G}, \widehat{\mathcal{B}})$ contains a biased subgraph Ω_0 that is a subdivision of $\mathsf{D}_{1,0}$, B'_0, B'_1, or B'_2. Since both Ω_0 and $\widehat{G}$ are 2-connected, there is a sequence of 2-connected biased subgraphs $\Omega_0 \subset \cdots \subset \Omega_n$ where $\Omega_n = (\widehat{G}, \widehat{\mathcal{B}})\backslash J$, and for each $i \in \{0, \ldots, n-1\}$ there is a path P_i in G internally disjoint from Ω_i so that $\Omega_{i+1} = \Omega_i \cup P_i$. For each $i \in \{0, \ldots, n\}$, let $(G_i, \mathcal{B}_i)$ be the biased subgraph of $(\widehat{G}, \widehat{\mathcal{B}})$ induced by $E(\Omega_i)$ with $V(G_i) = V(\widehat{G})$ and let A_i be the submatrix of A consisting of all rows of A and precisely those columns representing $E(\Omega_i)$. Thus for each i, A_i represents $M(\Omega_i)$. By Lemmas 5.7 and 4.18, Proposition 2.9, and Lemma 5.10, A_0 is projectively equivalent to a canonical lift matrix particular to Ω_0. Inductively assume that there are sequences $T_0, \ldots, T_i$ and $S_0, \ldots, S_i$ of $\mathbb{F}$-matrices such that for each $k \in \{0, \ldots, i\}$ each matrix $T_k A_k S_k$ is a canonical lift matrix particular to Ω_k. We may assume that A is of full rank, and so that for each $k \in \{0, \ldots, i\}$ the rows of $T_k A_k S_k$ are indexed by $v_0 \cup (V(G_k) - v)$ (as described in Section 2.6). Since for each k, $V(G_k) = V(\widehat{G})$, for each k the rows of $T_k A_k S_k$ are indexed by $v_0 \cup (V(\widehat{G}) - v)$.

Consider $\Omega_{i+1} = \Omega_i \cup P_i$ and the matrix $T_i A_{i+1}$ representing $M(\Omega_{i+1})$. Let us assume P_i consists of a single edge e_i whose endpoints are $x_i, y_i \in V(\Omega_i)$. Let $x \in V(G_i) - \{x_i, y_i\}$. Since Ω_i is 2-connected, there is a spanning tree T of $\Omega_i - x$. If x is not v, then there is an edge f such that the fundamental cycle in $T \cup f$ is unbalanced in Ω_i: set $W = E(T) \cup f$; otherwise set $W = E(T)$. The subgraph $W \cup e_i \subseteq \Omega_{i+1}$ contains a subgraph C that is either a balanced cycle, a pair of unbalanced cycles sharing just vertex v, or a contrabalanced theta. Since $E(C)$ is a circuit of $M(\Omega_{i+1})$, this implies that in column e_i of $T_i A_{i+1}$, the entry in row x is 0. As long as neither endpoint of e_i is v, this also implies that the entries in rows x_i and y_i are nonzero. If e_i has endpoints v, y_i, for some vertex $y_i \neq v$, then the form of C in Ω_{i+1} implies that the entry in matrix $T_i A_{i+1}$ in column e_i, row y_i must be nonzero. Thus $T_i A_{i+1} S_{i+1}$, with rows indexed by $v_0 \cup \Big(V(\widehat{G}) - v\Big)$, is a canonical lift matrix for some appropriate column scaling matrix S_{i+1}. Hence by induction, there are matrices T_n and S_n such that $T_n A_n S_n$ is a canonical lift matrix particular to $\Omega_n = (\widehat{G}, \widehat{\mathcal{B}})\backslash J$.

Finally, consider the set of joints J. Let e be a joint. By assumption e is incident to v and every other joint is in parallel with e. Since v is the unique balancing vertex of $(\widehat{G}, \widehat{\mathcal{B}})$, for every vertex $x \neq v$, there is an unbalanced cycle C_x in $G - x$ of length > 1. Since $C_x \cup e$ is a circuit of M, row x of column e of $T_n A$ is zero. Thus every row of column e aside from row v_0 is zero. Since e is not a loop of M, the entry in row v_0 of column e must be nonzero. Since all joints of $(\widehat{G}, \widehat{\mathcal{B}})$ are in parallel with e, every column of A representing a joint is zero in all rows but v_0. Thus there is a diagonal matrix S scaling the columns of T_nA so that T_nAS is a canonical lift matrix particular to $(G, \mathcal{B})$. This completes the proof of statement (i).

(ii) Let $(H, \mathcal{S})$ be a roll-up of $(\widehat{G}, \widehat{\mathcal{B}})$, and let $U \subseteq \Sigma(v)$ be the unique unbalancing class of edges in $\Sigma(v)$ that are joints in $(H, \mathcal{S})$. By statement (i) there is a $\mathbb{F}^+$-gain function φ on $\widehat{G}$ realizing $\widehat{\mathcal{B}}$ for which $A = A_L(\widehat{G}, \varphi)$. Let T be a spanning tree of $\widehat{G}$ containing exactly one edge in U. Then the T-normalized gain function φ' obtained by switching on φ satisfies $\varphi'(e) = 0$ for all edges e not incident to v and $\varphi'(e) = 0$ for each edge in U. By Corollary 5.3, the derived $\mathbb{F}^\times$-gain function $\varphi'^\times$ realizes $(H, \mathcal{S})$. By Lemma 5.2, A and $A_F(H, \varphi'^\times)$ are projectively equivalent.

The final statement follows immediately from Corollary 5.3. □

5.3 Projective equivalence classes are in 1-1 correspondence with switching classes

Finally, we can prove Theorem 3.

Proof of Theorem 3. By Proposition 4.1, every switching class of gain functions is contained in a projective equivalence class of matrix representations. Thus we just need show that if A and B are projectively equivalent representations of M, then A and B are each projectively equivalent to canonical representations whose gain functions are contained in the same switching class. Because M is 3-connected, $(G, \mathcal{B})$ is 2-connected and has no 2-separation with one side balanced.

Assume first that $(G, \mathcal{B})$ is properly unbalanced and not tangled. Then either $M = F(G, \mathcal{B})$ or $M = L(G, \mathcal{B})$, but not both. Assume that $M = F(G, \mathcal{B})$. Let A and B be projectively equivalent $\mathbb{F}$-matrices representing M. By Theorem 5.5(1), each of A and B are projectively equivalent to a canonical frame matrix particular to $(G, \mathcal{B})$. Let φ and ψ be $\mathbb{F}^\times$-gain functions such that A is projectively equivalent to $A_F(G, \varphi)$ and B is projectively equivalent to $A_F(G, \psi)$. By Theorem 5.1(1), φ and ψ are switching equivalent. Similarly, if $M = L(G, \mathcal{B})$ and A and B are projectively equivalent $\mathbb{F}$-matrices representing M, then by Theorem 5.5(1), each of A and B are projectively equivalent to a canonical lift matrix particular to $(G, \mathcal{B})$. Let φ and ψ be $\mathbb{F}^+$-gain functions such that A is projectively equivalent to $A_L(G, \varphi)$ and B is projectively equivalent to $A_L(G, \psi)$. By Theorem 5.1(ii), φ and ψ are switching-and-scaling equivalent.

Now assume that $(G, \mathcal{B})$ is properly unbalanced but tangled. Then $L(G, \mathcal{B})$ and $F(G, \mathcal{B})$ coincide: $M = L(G, \mathcal{B}) = F(G, \mathcal{B})$. Let A and B be projectively equivalent $\mathbb{F}$-matrices representing M. By Theorem 5.5, each of A and B is projectively equivalent to a canonical lift matrix particular to $(G, \mathcal{B})$, or to a canonical frame matrix particular to $(G, \mathcal{B})$, but not both. By Theorem 5.1(iii), either A and B are both projectively equivalent to canonical lift matrices or both are projectively equivalent to canonical frame matrices. In either case, by statement (i) or (ii) of Theorem 5.1, the gain functions from which these canonical representations arise belong to the same switching class.

Finally, assume that $(G, \mathcal{B})$ is almost-balanced, and let A and B be projectively equivalent $\mathbb{F}$-matrices representing M. By Proposition 2.12, $(G, \mathcal{B})$ has a unique balancing vertex. By Theorem 5.5, A and B are each projectively equivalent to a canonical lift matrix particular to $(\widehat{G}, \widehat{\mathcal{B}})$, say, given by $\mathbb{F}^+$-gain functions φ and ψ respectively. By Theorem 5.4(i), φ and ψ belong to the same switching class. $\square$

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References

[1] Nathan Bowler, Daryl Funk, and Daniel Slilaty, *Describing quasi-graphic matroids*, European J. Combin. **85** (2020), 103062, 26. MR 4037634

[2] Tom Brylawski, *Modular constructions for combinatorial geometries*, Trans. Amer. Math. Soc. **203** (1975), 1–44.

[3] Rong Chen and Irene Pivotto, *Biased graphs with no two vertex-disjoint unbalanced cycles*, J. Combin. Theory Ser. B **130** (2018), 207–245. MR 3772740

[4] Henry H. Crapo, *Single-element extensions of matroids*, J. Res. Nat. Bur. Standards Sect. B **69B** (1965), 55–65. MR 0190045

[5] Matt DeVos and Daryl Funk, *Almost balanced biased graph representations of frame matroids*, Adv. in Appl. Math. **96** (2018), 139–175. MR 3767506

[6] Jim Geelen, *Some open problems on excluding a uniform matroid*, Adv. in Appl. Math. **41** (2008), no. 4, 628–637. MR 2459453 (2009k:05050)

[7] Jim Geelen, Bert Gerards, and Geoff Whittle, *Quasi-graphic matroids*, arXiv:1512.03005, 14pp.

[8] ———, *The highly connected matroids in minor-closed classes*, Ann. Comb. **19** (2015), no. 1, 107–123.

[9] ———, *Quasi-graphic matroids**, Journal of Graph Theory **87** (2018), no. 2, 253–264.

[10] J. Kahn and J. P. S. Kung, *Varieties of combinatorial geometries*, Trans. Amer. Math. Soc. **271** (1982), no. 2, 485–499. MR 654846 (84j:05043)

[11] James Oxley, *Matroid theory*, second ed., Oxford Graduate Texts in Mathematics, vol. 21, Oxford University Press, Oxford, 2011. MR 2849819 (2012k:05002)

[12] Daniel Slilaty, *Projective-planar signed graphs and tangled signed graphs*, J. Combin. Theory Ser. B **97** (2007), no. 5, 693–717. MR 2344133 (2008j:05161)

[13] Geoff Whittle, *Stabilizers of classes of representable matroids*, J. Combin. Theory Ser. B **77** (1999), no. 1, 39–72. MR 1710531 (2000f:05027)

[14] Thomas Zaslavsky, *Vertices of localized imbalance in a biased graph*, Proc. Amer. Math. Soc. **101** (1987), no. 1, 199–204. MR 897095 (88f:05103)

[15] ———, *Biased graphs. I. Bias, balance, and gains*, J. Combin. Theory Ser. B **47** (1989), no. 1, 32–52. MR 1007712 (90k:05138)

[16] ———, *Biased graphs whose matroids are special binary matroids*, Graphs Combin. **6** (1990), no. 1, 77–93. MR 1058551 (91f:05097)

[17] ———, *Biased graphs. II. The three matroids*, J. Combin. Theory Ser. B **51** (1991), no. 1, 46–72. MR 1088626 (91m:05056)

[18] ———, *Frame matroids and biased graphs*, European J. Combin. **15** (1994), no. 3, 303–307. MR 1273951 (95a:05021)

[19] ———, *Biased graphs. IV. Geometrical realizations*, J. Combin. Theory Ser. B **89** (2003), no. 2, 231–297. MR 2017726 (2005b:05057)