The family of bicircular matroids closed under duality

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Abstract

We characterize the 3-connected members of the intersection of the class of bicircular and cobicircular matroids. Aside from some exceptional matroids with rank and corank at most 5, this class consists of just the free swirls and their minors.

1 Introduction

Whitney showed that the intersection of the classes of graphic and cographic matroids is exactly the class of planar graphic matroids. Slilaty showed [16] that the intersection of the classes of connected cographic matroids and connected signed-graphic matroids is exactly the class of connected cographic matroids of projective-planar graphs. Carmesin [10] greatly extends these ideas by defining a class of r-locally planar graphs and a class of r-local matroids which describes the intersection of this class of matroids with the class of cographic matroids. Carmesin also extends these ideas to 2-complexes embedded in 3-space [5, 6, 7, 8, 9]. The intersections investigated in all these works are described in terms of topological embeddings.

Given a graph G, the *bicircular matroid* of G is denoted by $B(G)$. The formal definition of $B(G)$ is given at the beginning of Section 2. The definition we use is not exactly the one from Oxley [15, p.238] but rather a slight modification which accounts for matroid loops. We say that a matroid M is *bicircular* when $M = B(G)$ for some graph G, we say that M is *cobicircular* when $M^* = B(G)$ for some graph G, we say that M is *doubly bicircular* when M is both bicircular and cobicircular. In this paper, we determine the intersection of the classes of 3-connected bicircular matroids and cobicircular matroids; that is, we find all 3-connected doubly bicircular matroids. Unsurprisingly, we find that these matroids are not described in terms of topological embeddings. Aside from some exceptional matroids with rank and corank at most 5, this intersection consists solely of the free swirls and their 3-connected minors. The free swirl is the bicircular matroid $B(2C_n)$ in which $2C_n$ is the graph obtained from the cycle of length n by doubling each edge.

The class of bicircular matroids is a minor-closed class of matroids that is properly contained within the class of transversal matroids. Transversal matroids are neither closed under duality nor minors. In order to describe the minors and duals of transversal matroids, the more general class of gammoids is used. Other investigations of natural subclasses of transversal matroids have also found closure under both minors and duality to be a property of interest. Bonin and de Mier [1, 2] investigated

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the class of lattice-path matroids and found that it is both minor closed and duality closed. Bonin and Giménez [3] investigate the class of multi-path matroids which sits properly between the classes of lattice-path and transversal matroids and yet is still closed under both minors and duality. Las Vergnas [12] showed that the class of fundamental transversal matroids is closed under duality but not closed under minors. Brualdi [4] shows that Las Vergnas' result is a corollary of a more general phenomenon. Neudauer [14] characterizes the intersection of the classes of bicircular matroids and fundamental transversal matroids. Her result is thus related to ours. The intersection of the class of bicircular matroids with lattice-path matroids and multi-path matroids may be an interesting topic of investigation.

Our main result is Theorem 1.1. Theorem 1.1 is actually a corollary of a stronger statement (Theorem 4.2) which is stated and proven in the final section of the paper. Theorem 4.2 also contains information about excluded minors.

Theorem 1.1 (Main Result)**.** *If M is a 3-connected matroid, then M is doubly bicircular if and only if M is a minor of a free swirl or M is a minor of one of $B(K_4^{++})$, $B_{(}N_8)$, $B(O_8)$, $B(F_{10})$, $B(Z_8)$, and $B(Z_8^*)$; that is, the bicircluar matroid of one of the graphs in Figure 1.*

We remark that the minors mentioned in Theorem 1.1 are not all minors of free swirls and the bicircular matroids of the graphs in Figure 1, but only the 3-connected minors. It is well known that the free swirl of rank n is identically self dual and (as we remarked above) is bicircular. We leave it to the reader to check that the bicircular matroids of the graphs of Figure 1 are doubly bicircular. More specifically, $B(K_4^{++})$, $B(N_8)$, $B(O_8)$, and $B(F_{10})$ are all self dual and $B^*(Z_8) \cong B(Z_8^*)$. †

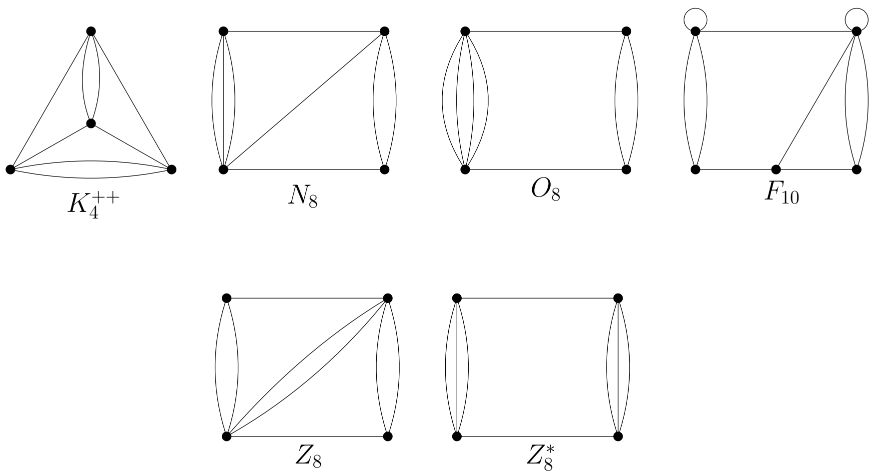

Figure 1: Some graphs for doubly bicircular matroids.

It may very well be possible to characterize the doubly bicircular matroids which are connected but not 3-connected; however, just accounting for parallel and coparallel elements in bicircular matroids already presents a non-trivial technical difficulty. A pair of elements e and f are parallel in bicircluar matroid $B(G)$ if and only if edges e and f are loops in graph G which are incident to the same vertex. Also, if links e and f in G are incident to a vertex v of degree 2, then e and f are coparallel elements

†This check can be done by hand or by using the SageMath software package. One way to represent the bicircular matroid of a graph G in SageMath is as follows. Define a $\mathbb{Q}$-matrix A whose rows are indexed by $V(G)$ and whose columns are indexed by $E(G)$. The column corresponding to a link e having endpoints in rows i and j should have a -1 in row i, a prime number p_e unique to e in row j, and zeros in all other rows. The column corresponding to a loop incident to vertex v should be the elementary column vector corresponding to the row for v. Now $M(A) = B(G)$.

in $B(G)$. So now if $B^*(G) = B(H)$, then subdividing links in G preserves bicircularity but can only preserve cobicircularity if we know all possibilities for H, which edges of H are loops, and if these loops in H correspond to the subdivided links in G.

In Section 2 we go over some basic definitions and observations. Theorem 1.1 is a corollary of Theorem 4.2. Theorem 4.2 Part (2) lists some excluded minors for the class of cobicircular matroids. In Section 3 we prove that these matroids are in fact excluded minors for the class of cobicircular matroids. In Section 4 we present Theorem 4.2 and its proof.

2 Preliminaries

We assume that the reader is familiar with matroid theory as in Oxley's book [15]. However, the definition of a bicircular matroid on Page 238 of [15] does not allow for matroid loops and so the class of matroids defined is not closed under taking minors. A simple modification of the definition will allow for loops and will define a class of matroids that is closed under taking minors.

A graph G consists of a vertex set $V(G)$ and an edge set $E(G)$ in which an edge $e \in E(G)$ is either a *link* connecting two distinct vertices, a *loop* on a single vertex, or a *free edge* which is not incident to any vertex. Given a graph G, the bicircular matroid $B(G)$ has element set $E(G)$ in which the free edges are matroid loops in $B(G)$ and every other circuit of $B(G)$ is the edge set of a subgraph of G which is a subdivision of one of the graphs shown in Figure 2. The following two observations are well worth noting. One, e is a loop of matroid $B(G)$ if and only if e is a free edge of G. Furthermore, loops in the graph G never correspond to matroid loops in $B(G)$. Two, e and f are parallel elements of $B(G)$ if and only if e and f are loops in G incident to the same vertex. For a matroid M, when $M = B(G)$ we say that G is a *bicircular representation* of M.

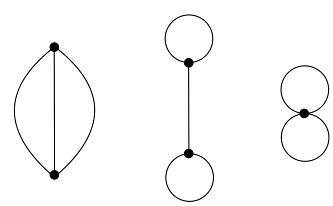

Figure 2: Graphs describing the circuits of $B(G)$.

This modified definition for bicircular matroids does now define a minor-closed class of matroids as follows. For any edge e, $B(G)\backslash e = B(G\backslash e)$. If e is a free edge, then $B(G)/e = B(G)\backslash e$. If e is a link, then $B(G)/e = B(G/e)$. If e is a loop which is incident to vertex v, then $B(G)/e = B(\tilde{G})$ in which $\tilde{G}$ is defined as follows: $V(\tilde{G}) = V(G) - v$, $E(\tilde{G}) = E(G) - e$, if $x \neq e$ is a loop in G that is incident to v then x becomes a free edge in $\tilde{G}$, if x is a link in G that is incident to v then x is a loop in $\tilde{G}$ which will be incident to its second endpoint from G, and all other $e \in E(G)$ remain as they are in G.

Another important fact about bicircular matroids and graphs is as follows. A graph H is a minor of a graph G when H is obtained from G by a sequence of edge deletions, link contractions, and deletion of isolated vertices. Therefore, by the previous paragraph, if a graph H is a minor of a graph G, then the bicircular matroid $B(H)$ is a minor of bicircular matroid $B(G)$; however, if bicircular matroid $B(H)$ is a minor of bicircular matroid $B(G)$, then it need not be the case that H is a minor of G.

We use Proposition 2.1 without further mention.

Proposition 2.1 (Wagner [18, Prop.2])**.** *If G is connected, has no free edges and has at least three vertices, then $B(G)$ is 3-connected if and only if G is 2-connected, has no degree-2 vertices, and has no two loops incident to the same vertex.*

As suggested by Proposition 2.1, studying 3-connected bicircular matroids requires a way of understanding 2-connected graphs. Towards this purpose we use the *canonical tree decomposition* of a non-separable graph G. (The reader may see [11], [15, pp.308–315], or [17] for a full discussion.) We will use this tree decomposition in the proof of Theorem 4.2. A graph G is *separable* if there exists subgraphs G_1 and G_2 of G with non-empty edge sets such that $G = G_1 \cup G_2$ but $G_1 \cap G_2$ is either empty or a single vertex. Thus a graph on at least three vertices is non-separable if and only if it is 2-connected and loopless. Given an integer $n \geq 2$, an *n-multilink* is the graph consisting of two vertices along with n links connecting them. The n-multilink is denoted by nK_2. Now if G is non-separable, then there is a unique labeled tree T satisfying the following.

- Each vertex v in T is labeled with either a 3-connected simple graph, a cycle of length at least three, or mK_2 for some $m \geq 3$.
- No two cycle-labeled vertices are adjacent in T and no two multilink-labeled vertices are adjacent in T.
- If e is an edge of T whose endpoints are labeled with graphs G_1 and G_2, then e corresponds to an edge e_1 in G_1 and an edge e_2 in G_2.
- G is obtained by executing the 2-sums indicated by the vertex labels of T along the edges indicated by the edges of T.

An important consequence of this tree decomposition is that if T_0 is a subtree of T, then the graph G_0 obtained by executing the 2-sums indicated in T_0 is a minor of G.

3 Excluded Minors

In this section we explore some minor-minimal bicircular matroids that are not cobicircular. These matroids are used in Theorem 4.2.

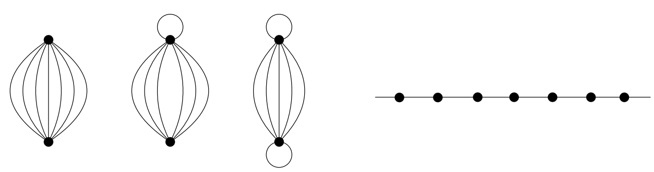

Figure 3: The matroid $U_{2,7}$ and its bicircular representations.

Proposition 3.1. *The matroid $U_{2,7}$ is bicircular but minimally not cobicircular. Figure 3 shows all possible bicircular representations of $U_{2,7}$.*

Proof. It is evident that the graphs of Figure 3 are all possible bicircular representations $U_{2,7}$. Theorem 3.8 from [13] determines exactly which uniform matroids are bicircular. It implies that $U_{5,7}$ is minimally not bicircular and so $U_{2,7}$ is minimally not cobicircular. □

Given an element e in a matroid M with a transitive symmetry group, let M' be the matroid obtained from M by replacing e with a pair of coparallel elements called e and e'. For our discussions, it is worth noting that $M' = (M^* \cup e')^*$ where e' is an element added to M^* which is in parallel to element e of M^*. When we write M'' we mean that M has a 2-transitive symmetry group and two distinct elements of M are replaced by coparallel pairs.

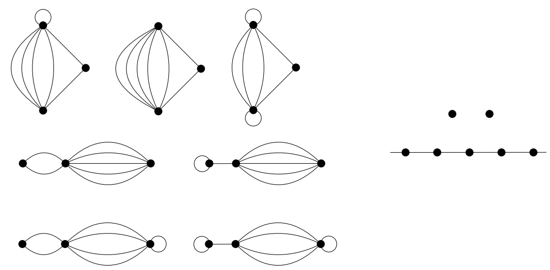

Figure 4: The matroid $U'_{2,6}$ along with its bicircular representations.

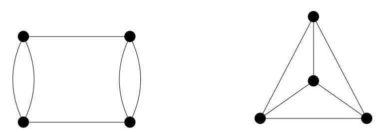

Figure 5: The graphs C_4^{++} and K_4 are the only bicircular representations of $U_{4,6}$.

Proposition 3.2. *Let B be a doubly bicircular matroid with a transitive symmetry group and let $e \in E(B)$. If for every graph G with $B(G) = B^*$ the edge e is a link in G, then B' is bicircular but not cobicircular.*

Proof. First, B' is bicircular because any bicircular representation of B yields a bicircular representation of B' by subdividing the edge e. Second, by way of contradiction, assume that B' is cobicircular. In particular, say that $B(H) = B^* \cup e'$. Thus $B(H \backslash e') = B^*$. However, since e is assumed to be a link in $H \backslash e'$, there is no way that e and e' could be parallel elements in $B(H)$, a contradiction. □

Proposition 3.3. *The matroid $U'_{2,6}$ is bicircular but minimally not cobicircular. Figure 4 shows all possible bicircular representations of $U'_{2,6}$ and Figure 5 shows all possible bicircular representations of $U_{4,6}$.*

Proof. That the graphs in Figure 4 are the complete list all possible bicircular representations of $U'_{2,6}$ is clear. A bicircular representation G of $U^*_{2,6} = U_{4,6}$ must have four vertices of degree at least 3 each. Since G has 6 edges, we now get that all vertices have degree 3. All possible 2-connected cubic graphs on four vertices are shown in Figure 4 and both are bicircular representations of $U_{4,6}$. Note that there can be no loops in a 2-connected 3-regular graph. Since there is no bicircular representation of $U_{4,6}$ which uses loops, Proposition 3.2 implies that $U'_{2,6}$ is not cobicircular. The reader can check that $U'_{2,6}$ is minimally not cobicircular. □

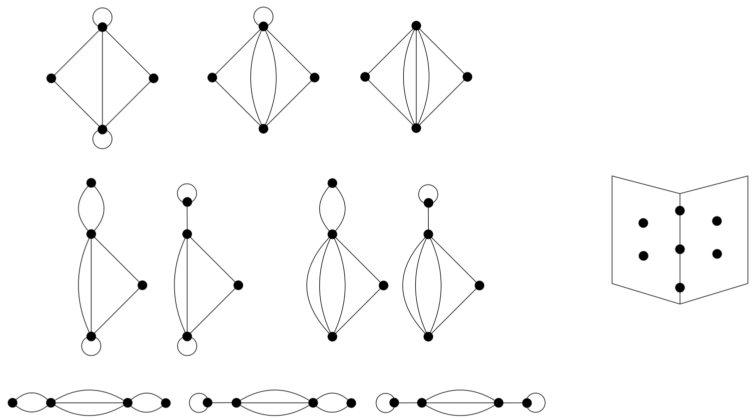

Figure 6: The matroid $U''_{2,5}$ and its bicircular representations.

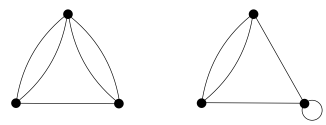

Figure 7: The bicircular representations of $U_{3,5}$.

Proposition 3.4. *The matroid $U''_{2,5}$ is bicircular but minimally not cobicircular. Figure 6 shows all possible bicircular representations of $U''_{2,5}$. Figure 7 shows all possible bicircular representations of $U_{3,5}$.*

Proof. That the graphs shown are all possible bicircular representations of $U''_{2,5}$ and $U_{3,5}$ can easily be checked by the reader. Since $U^*_{2,5} = U_{3,5}$ and there is at most one loop in a bicircular representation of $U_{3,5}$ it follows that $U''_{2,5}$ is not cobicircular as in the proof of Proposition 3.2. That $U''_{2,5}$ is minimally not cobicircular can be checked by the reader. □

Given positive integers a, b, and c, we write $T_{a,b,c}$ to denote that graph obtained from a triangle by replacing the three edges of the triangle with multilinks aK_2, bK_2, and cK_2 on the corresponding endpoints.

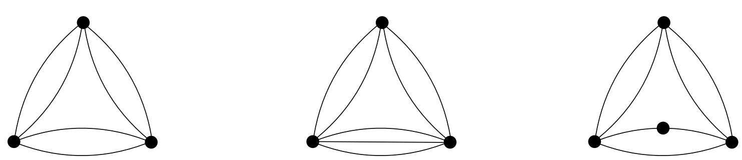

Figure 8: The only bicircular representation of $U_{3,6}$ is $T_{2,2,2}$ [19, Lemma 2.12]. The matroids $B(T_{3,2,2})$ and $B(T'_{2,2,2})$ are both minimally not cobicircular.

Proposition 3.5. *The only bicircular representation of $U_{3,6}$ is $T_{2,2,2}$. The matroids $B(T_{3,2,2})$ and $B(T'_{2,2,2})$ are both minimally not cobicircular.*

Proof. That $T_{2,2,2}$ is the only bicircular representation of $U_{3,6}$ was noted in [19, Lemma 2.12]. Since $U_{3,6}^* \cong U_{3,6}$, we get that $B(T'_{2,2,2})$ is not cobicircular by Proposition 3.2. The reader may check for minimality. It is evident that every single-element contraction and deletion of $B(T_{3,2,2})$ is cobicircular. Now consider an element e such that $B(T_{3,2,2})\backslash e = B(T_{2,2,2})$. So now $B^*(T_{3,2,2})/e \cong B(T_{2,2,2})$. Thus if we assume that $B(G) = B^*(T_{3,2,2})$, then G would have minimum degree 3 and then be forced to have two vertices of degree 3. These two degree-3 vertices in G form cotriangles in $B(G) = B^*(T_{3,2,2})$ and so form triangles in $B(T_{3,2,2})$; however, $B(T_{3,2,2})$ contains only one triangle, a contradiction. □

Given a vertex-transitive graph G, we let G^ℓ be the graph obtained from G by adding a loop to some vertex. If G is a loopless graph, then G° is the graph obtained from G by adding a loop at each vertex.

Proposition 3.6. *The only bicircular representation of $\mathcal{W}^3$ (i.e., the rank-3 whirl) is C_3°.*

Proof. Let $\{1,2,3,4,5,6\}$ be the groundset of $\mathcal{W}^3$ and let G be a bicircular representation of $\mathcal{W}^3$. Consider the two triangles $\{1,2,3\}$ and $\{1,4,5\}$ in $\mathcal{W}^3$. Evidently there are three possibilities for $G\backslash 6$. They are shown in the first row of Figure 9.

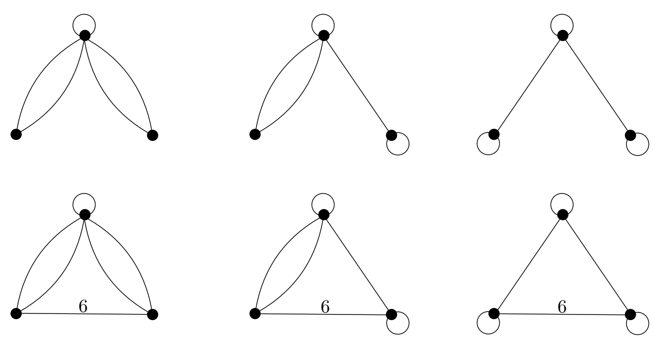

Figure 9: The sixth graph shown is the only bicircular representation of $\mathcal{W}^3$.

Since $\mathcal{W}^3$ is 3-connected, G must be 2-connected and so edge 6 must be a link bridging the cut vertex of $G\backslash 6$. Thus the three possibilities for G are shown in the second row of Figure 9; however, only the sixth graph is actually a bicircular representation for $\mathcal{W}^3$, as required. □

Proposition 3.7. *Let W_4 be the 4-wheel graph. The matroids $B(K_4^\ell)$ and $B(W_4)$ are minimally not cobicircular.*

Proof. Note that $B(K_4^\ell)/\ell$ is the rank-3 whirl. Since $(\mathcal{W}^3)^* \cong \mathcal{W}^3$, the only possible bicircular representation for $B^*(K_4^\ell)$ would be the graph G obtained from C_3° with ℓ added as a link. However, $B(G)$ has six triangles while $B(K_4^\ell)$ has only three cotriangles, a contradiction. We leave it to the reader to check minimality.

Let e and f be non-adjacent edges on the rim of W_4. Notice that $W_4/\{e,f\} \cong T_{2,2,2}$. Now by way of contradiction, assume that $B(G) = B^*(W_4)$. Thus $B(G\backslash\{e,f\}) = B(G)\backslash\{e,f\} = B^*(W_4)\backslash\{e,f\} = B^*(W_4/\{e,f\}) = B^*(T_{2,2,2}) \cong U_{3,6}^* \cong U_{3,6} \cong B(T_{2,2,2})$. Now since $T_{2,2,2}$ is the unique bicircular representation of $U_{3,6}$, we get that $G\backslash\{e,f\} = T_{2,2,2}$. Since $B(T_{3,2,2})$ is not cobicircular (Proposition 3.5) it must be that G is obtained from $T_{2,2,2}$ with e and f added as loops at two different vertices. Hence $G = 2C_5/\{x,y\}$ where x and y are a 2-edge matching. Thus $B^*(G) \cong B(2C_5\backslash\{x,y\}) \cong B(W_4)$; however, this contradicts the result of Wagner that the wheels W_n with $n \geq 4$ are the unique bicircular representations of their bicircular matroids [18, Proposition 5]. We leave it to the reader to check minimality. □

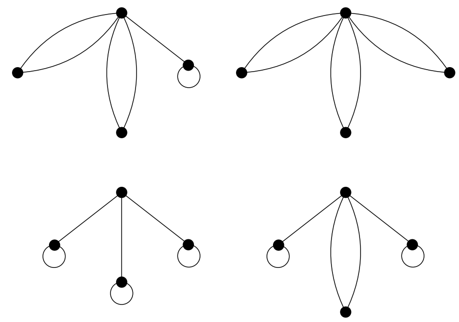

Figure 10: These four graphs are all possible bicircular representations of the graphic matroid $M(K_{2,3})$.

The proof of Proposition 3.8 is very similar to the others in this section and is left to the reader.

Proposition 3.8. *The only bicircular representations of the graphic matroid $M(K_{2,3})$ are those shown in Figure 10. The matroid $M(K_{2,3})$ is minimally not cobicircular.*

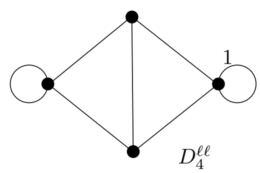

Figure 11: The bicircular matroid of the graph $D_4^{\ell\ell}$ is minimally not cobicircular.

Proposition 3.9. *The bicircular matroid $B(D_4^{\ell\ell})$ (see Figure 11) is minimally not cobicircular.*

Proof. Note that $B(D_4^{\ell\ell})/1 \cong \mathcal{W}^3$. If we assume by way of contradiction that H is a bicircular representation of $B^*(D_4^{\ell\ell})$, then $B(H\backslash 1) \cong \mathcal{W}^3$ and so $H\backslash 1 \cong C_3^\circ$ by Proposition 3.6. Furthermore, edge 1 cannot be added as a loop to C_3° because $B(D_4^{\ell\ell})$ is cosimple. Thus the only possibility for H is C_3° with one link doubled. Edge 1 is now in three triangles of $B(H)$; however, edge 1 is only in one cotriangle of $B(D_4^{\ell\ell})$, a contradiction. We leave it to the reader to check minimality. □

4 Proof of our main result

Proposition 4.1 implies that for a 3-connected graph G, $B(G)$ is cobicircular if and only if G is a minor of K_4^{++}. This proves our main result (Theorem 1.1) for the case in which G is a 3-connected graph.

Proposition 4.1. *If G is a 3-connected graph, then the following are equivalent.*

(1) $B(G)$ is cobicircular.

(2) G has no W_4-, K_4^ℓ-, or $T_{3,2,2}$-minor.

(3) G is a minor of K_4^{++}.

Proof. In the introduction we addressed the fact that $B(K_4^{++})$ is self dual and hence doubly bicircular. This proves $3 \to 1$. Propositions 3.5 and 3.7 prove $1 \to 2$. We now finish by proving $2 \to 3$.

Let G be a 3-connected graph which contains none of W_4, K_4^ℓ, and $T_{3,2,2}$ as a minor. By Tutte's Wheel Theorem and the fact that there is no K_4^ℓ-minor in G, G is obtained from K_4 by a sequence of adding and de-contracting links while maintaining graph 3-connectedness at each step; furthermore, the first step must be adding a link, call it e, to K_4 to obtain the graph K_4^+. The second step cannot be decontraction because the only 3-connected graph that is a decontraction of K_4^+ is W_4. Thus the second step is adding another link to K_4^+, call it f. If f is parallel to e, then G contains a $T_{3,2,2}$-minor, a contradiction. If f is adjacent to e but not parallel to e, then again G contains a $T_{3,2,2}$-minor, a contradiction. Thus adding f to K_4^+ yields K_4^{++}.

The third step cannot be a decontraction because, again, a 3-connected decontraction of K_4^{++} contains a W_4-minor. The third step also cannot be adding a link because wherever a link is added to K_4^{++} we obtain a graph with a $T_{3,2,2}$-minor, a contradiction. Thus G is a minor of K_4^{++}. □

The fact that Parts (1) and (3) in Theorem 4.2 are logically equivalent completes the proof of our main result (Theorem 1.1). Part (2) of Theorem 1.1 gives some information about the set of excluded minors for the class of doubly bicircular matroids; however, since our result is only for 3-connected matroids, we do not obtain the full list of excluded minors for the class of doubly bicircular matroids.

Theorem 4.2. *If $B(G)$ is 3-connected, then the following are equivalent.*

(1) $B(G)$ is cobicircular.

(2) G has no ordinary graph minor H which is a bicircular representation of $U_{2,7}$, $U'_{2,6}$, $U''_{2,5}$, $M(K_{2,3})$, $B(T_{3,2,2})$, $B(T'_{2,2,2})$, $B(W_4)$, $B(K_4^\ell)$, and $B(D_4^{\ell\ell})$.

(3) $B(G)$ is a minor of a free swirl or is a minor of one of $B(K_4^{++})$, $B(N_8)$, $B(O_8)$, $B(F_{10})$, $B(Z_8)$, and $B(Z_8^*)$; that is, the bicircular matroids of the graphs in Figure 1.

Lemma 4.3. *If $G\backslash e$ is one of the graphs in Figure 1 and $B(G)$ is 3-connected, then G contains one of the following graphs as a minor: $T_{3,2,2}$, $7K_2$, $(6K_2)^\ell$, and $(5K_2)^\circ$.*

Proof. Since $B(G)$ is 3-connected, then G is obtained from $G\backslash e$ by adding e as a link with both endpoints on the vertices of $G\backslash e$ or as a loop that is not on the same vertex as an existing loop. If $G\backslash e \cong K_4^{++}$ and e is a loop, then G has a $(6K_2)^\ell$-minor. If $G\backslash e \cong K_4^{++}$ and e is a link, then G has a $7K_2$-minor. If $G\backslash e \in \{N_8, O_8, Z_8, Z_8^*\}$ and e is a loop, then G contains a $(6K_2)^\ell$-minor. If $G\backslash e \in \{N_8, O_8, Z_8, Z_8^*\}$ and e is a link, then G either has a $T_{3,2,2}$- or $7K_2$-minor. If $G\backslash e \cong F_{10}$ and e is a loop, then G contains a $(5K_2)^\circ$-minor. If $G\backslash e \cong F_{10}$ and e is a link, then G either has a $T_{3,2,2}$- or $(6K_2)^\ell$-minor. □

Proof of Theorem 4.2. In the introduction we addressed the fact that the free swirls and the bicircular matroids of the graphs in Figure 1 are actually doubly bicircular. This proves $3 \to 1$. In Section 3 we showed that the nine bicircular matroids listed in (2) are not cobicircular. This proves $1 \to 2$. We now finish by proving $2 \to 3$.

Because $B(G)$ is 3-connected, G has at least two vertices. If G has exactly two vertices, then to avoid a $U_{2,7}$-minor in $B(G)$, G must be a subgraph of $6K_2$, $(5K_2)^\ell$, or $(4K_2)^\circ$. These graphs are, respectively, minors of O_8, F_{10}, and K_4^{++}, a desired outcome. For the remainder of the proof, we now assume that G has at least three vertices. Because $B(G)$ is 3-connected and G has at least three vertices, we now get that G is 2-connected, has no vertices of degree 2, has no free edges, and has no two loops incident to the same vertex. If G is 3-connected, then our result follows from Proposition 4.1. So for the remainder of the proof we may assume that G is 2-connected but not 3-connected.

Let $\hat{G}$ be the graph obtained from G by removing all of its loops. Let T be the canonical tree decomposition of $\hat{G}$. In Case 1 say that T has a 3-connected term, call it K, and in Case 2 that each term of T is either a cycle or a multi-link.

Case 1 By Proposition 4.1, $K \cong K_4$. If T has another 3-connected term $K' \cong K_4$, then G has a $K_4 \oplus_2 K_4$-minor which has a $T_{3,2,2}$-minor, a contradiction. Thus K is the only 3-connected term of T. Since G is not 3-connected, there must be a cycle term C in T. In Case 1.1 say that C is adjacent to K in T and in Case 1.2 that there is no cycle term adjacent to K in T.
Case 1.1 Since G has minimum degree 3, there is either a multi-edge term M adjacent to C in T or there is a loop in G incident to one of the internal vertices of C. Either possibility, however, creates a K_4^ℓ-minor in G, a contradiction.
Case 1.2 In this case K is adjacent to a multi-link term M which is then adjacent to a cycle term C. Now $K_4 \oplus_2 M \oplus_2 C$ is a minor of $\hat{G}$ and contains a degree-2 vertex coming from C. Since G has no degree-2 vertices, C must be adjacent in T to another multi-link term or this degree-2 vertex must support a loop. Either possibility yields a graph with a K_4^ℓ-minor, a contradiction.
Case 2 Because G has at least three vertices, T must have a cycle term, C_s which will denote a cycle of length $s \geq 3$. Choose C_s to be the longest such cycle label in T and make it the root of T. In Case 2.1 say that the vertex in T corresponding to C_s has degree at least three. In Case 2.2, say that the vertex in T corresponding to C_s has degree two. In Case 2.3, say that the vertex in T corresponding to C_s has degree zero or 1.
Case 2.1 If C_s is adjacent to three or more multi-edge terms, then each such multi-edge is $3K_2$ because otherwise we would produce a $T_{3,2,2}$-minor in G, a contradiction. Thus, if T has no cycle term at distance 2 from C_s, then $\hat{G}$ is a minor of $2C_n$ which makes G a minor of $2C_{2n}$ (a desired result). If T has a cycle term, call it C_t whose distance from C_s is 2, then C_t is adjacent in T to one of the multi-edge terms M in T which is adjacent to C_s. This however, would create a $T'_{2,2,2}$-minor in $\hat{G}$, a contradiction.
Case 2.2 Say that $M_1 \cong m_1K_2$ and $M_2 \cong m_2K_2$ are the two multi-edge terms whose corresponding vertices in T are adjacent to C_s. The two edges of C_s into which M_1 and M_2 are summed are either adjacent or non-adjacent edges. Let these be Cases 2.2.1 and Cases 2.2.2. In both cases $m_1, m_2 \geq 3$ and $m_1 + m_2 \leq 8$ because $\hat{G}$ has no $7K_2$-minor.
Case 2.2.1 In Case 2.2.1.1 say that $s = 3$ and in Case 2.2.1.2 say that $s \geq 4$.
Case 2.2.1.1 Since $m_1, m_2 \geq 3$ and $m_1 + m_2 \leq 8$, we have up to symmetry that (m_1, m_2) is one of $(4, 4)$, $(3, 5)$, $(3, 4)$, $(3, 3)$. Let these be, respectively, Cases 2.2.1.1.1–2.2.1.1.4.
Case 2.2.1.1.1 Here $(m_1, m_2) = (4, 4)$. First, we claim that T consists of C_s, M_1, and M_2, only. If by way of contradiction, there is another term in T, then without loss of generality it must be a cycle term, call it C, adjacent to M_1. If the vertex corresponding to C in T is a leaf of T, then G must have a loop incident to an internal vertex of C and so G has a $(6K_2)^\ell$-minor, a contradiction. If C is not a leaf of T, then it is adjacent to another multi-edge term, call it M. This, however, would create a $7K_2$-minor in $\hat{G}$, a contradiction. Thus T consists of C_s, M_1, and M_2, only and so $\hat{G} \cong T_{3,3,1}$; furthermore, if we add a loop anywhere to $\hat{G}$, then $B(G)$ would have a $(6K_2)^\ell$-minor, a contradiction. Thus $G \cong T_{3,3,1}$ which is a minor of Z_8^*, a desired result.
Case 2.2.1.1.2 Here $(m_1, m_2) = (3, 5)$. In a similar fashion as in Case 2.2.1.1.1, we get that T consists of C_s, M_1, and M_2, only, because otherwise we would be able to construct a $(6K_2)^\ell$- or $7K_2$-minor in G, a contradiction. Thus $\hat{G} \cong T_{1,2,4}$ and we cannot add a loop without creating a $(6K_2)^\ell$-minor. Thus $G \cong T_{1,2,4}$ and is a minor of Z_8, a desired result.
Case 2.2.1.1.3 Here $(m_1, m_2) = (3, 4)$. If $V(T) = \{C_s, M_1, M_2\}$, then $\hat{G} \cong T_{3,2,1}$. Since G has no $(5K_2)^\circ$-minor, G is therefore a subgraph of $T_{3,2,1}$ along with either a loop at each end of the undoubled edge or a loop at the vertex of degree 5. Both graphs are minors of F_{10}, a desired result.

If $|V(T)| \geq 4$, then T has a 3-cycle term C adjacent to either M_1 or M_2. If C is adjacent to M_2, then because G has minimum degree 3, G has as a minor one of graphs in the top row of Figure 12. The first two graphs have minors that are bicircular representations of $M(K_{2,3})$, a contradiction. The third graph is Z_8. By Lemma 4.3, Z_8 is maximal among 2-connected 4-vertex graphs which do not contain a minor from among $T_{3,2,2}$, $7K_2$, $(6K_2)^\ell$, and $(5K_2)^\circ$. Hence if G has four vertices, then

$\hat{G} \cong Z_8$. If G has five or more vertices, then the tree decomposition T of $\hat{G}$ must have another 3-cycle term. Again, recall that G has minimum degree 3. Hence, adding on the new 3-cycle term to the tree will replace one edge with at least three edges. This will yield a minor H in G with four vertices which properly contains Z_8, a contradiction by Lemma 4.3.

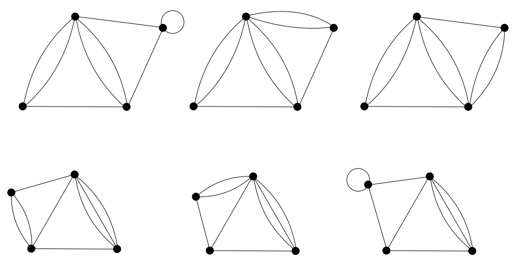

Figure 12: Graphs for the proof of Case 2.2.1.1.3

Finally, if C is adjacent to M_1, then G contains as a minor one of graphs on the bottom row of Figure 12. The fifth and sixth graphs of Figure 12 both contain minors representing $U''_{2,5}$, a contradiction. The fourth graph is N_8. If G has four vertices, then by Lemma 4.3, $G \cong N_8$. If G has at least five vertices, then the tree decomposition T of $\hat{G}$ must have another 3-cycle term. As before, adding in this new 3-cycle term will replace one edge of G with at least three edges and so yields a minor H in G with four vertices which properly contains N_8, a contradiction by Lemma 4.3.

Case 2.2.1.1.4 Here $(m_1, m_2) = (3, 3)$. If $V(T) = \{C_s, M_1, M_2\}$, then G is a minor of $2C_6$, a desired result. If T has more than three vertices, then vertices M_1 and M_2 together can have at most two children. If they had three or more, then G would contain as a minor the first graph of Figure 13. Contracting the two unsubdivided edges of this graph yields a minor representing $M(K_{2,3})$, a contradiction.

If we assume that M_1 and M_2 together have two children, then G contains as a subgraph a graph, call it K, obtained from $T_{2,2,1}$ by subdividing two of the four doubled edges. Since G has minimum degree 3, G has a minor K_0 which is obtained from K by attaching a link or loop to K to each degree-2 vertex in K with the second endpoint of a link adjacent in K to the first. There are 11 such graphs, all of which have a minor representing $M(K_{2,3})$, a contradiction.

Lastly, assume that M_1 and M_2 together have just one child. This must be a 3-cycle term, call it C, and say without loss of generality that C is adjacent to M_1 in T. If $V(T) = \{C_s, M_1, M_2, C\}$, then G has four vertices and must contain as a subgraph the middle graph of Figure 13. Furthermore, G must be obtainable from the middle graph of Figure 13 by adding loops. A loop added to vertex 1 would yield a graph with minor representing $M(K_{2,3})$, a contradiction. A loop added to vertex 2 would yield a graph with a $D_4^{\ell\ell}$-minor, a contradiction. Thus G is contained between the second and third graphs shown in Figure 13, the second graph is a minor of F_{10}, a desired outcome.

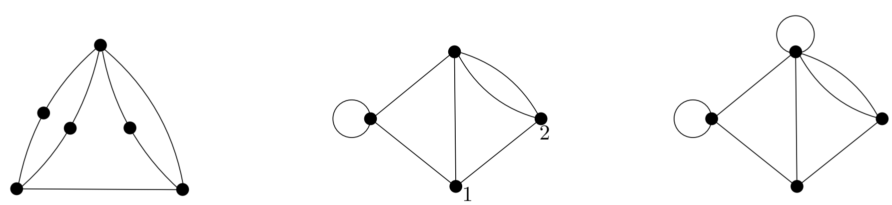

Figure 13: Graphs for the proof of Case 2.2.1.1.4

If T has exactly five vertices, say $V(T) = \{C_s, M_1, M_2, C, M\}$, then the fifth vertex M must be a multi-edge term adjacent to C. If $M = tK_2$ for $t \geq 4$, then $\hat{G}$ contains as a spanning subgraph one of the graphs in the first row of Figure 14

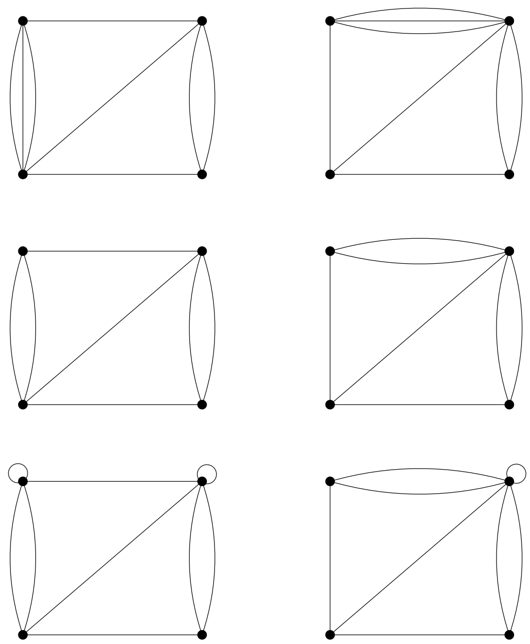

Figure 14: More graphs for the proof of Case 2.2.1.1.4

The second graph contains a minor representing $U''_{2,5}$, a contradiction. The first graph is N_8 and because G has four vertices we get that $G \cong N_8$ by Lemma 4.3, a desired result. Now we may assume that $M = 3K_2$. In this case, $\hat{G}$ is one of the two graphs in the second row of Figure 14. We then obtain G from $\hat{G}$ by adding loops. For the first graph, in order to avoid creating a $U_{2,7}$-minor, we can add at most two loops as shown in the third row of Figure 14. For the second graph, in order to avoid creating an $M(K_{2,3})$- or $U''_{2,5}$-minor, we can add at most one loop as shown in the third row of the figure. Both of the graphs in the third row are minors of F_{10}, a desired outcome. If T has a sixth vertex, call it X, then X is a multi-edge attached to C or is a 3-cycle term attached to M. In the former case, G would contain a $T_{3,2,2}$-minor, a contradiction, and so X is a 3-cycle term attached to M. In this case, G contains one of the graphs of Figure 15 as a minor; however, all six of these graphs have a minor representing $B(K_{2,3})$, a contradiction.

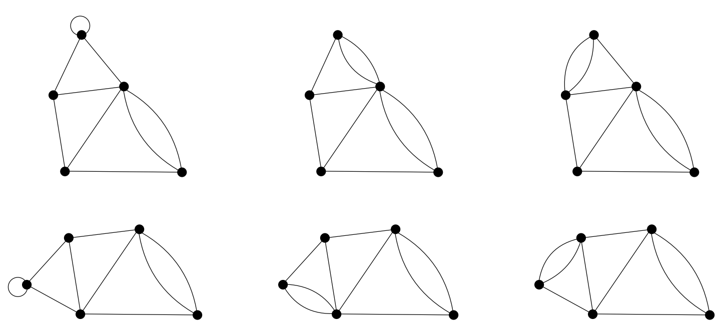

Figure 15: More graphs for the proof of Case 2.2.1.1.4

Case 2.2.1.2 Since M_1 and M_2 are summed into adjacent edges of C_s, any vertex of C_s not used by M_1 and M_2 (call these vertices $v_1, \ldots, v_{s-3}$) is a vertex of G and must have a loop incident to it. Again, $m_1, m_2 \geq 3$. If m_1 or $m_2 = 4$, then G contains the graph on the left of Figure 16 as a minor and this graph has a subgraph representing $U''_{2,5}$, a contradiction.

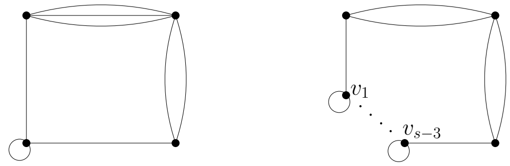

Figure 16: Graphs for the proof of Case 2.2.1.2.

Thus $m_1 = m_2 = 3$. If $V(T) = \{C_s, M_1, M_2\}$, then G is obtained from the graph on the right of Figure 16 by adding loops. Thus G is a minor of $2C_{2s}$, a desired result. If T has a fourth vertex, then it must be a cycle term, call it C, which is, without loss of generality, adjacent to M_1. Thus G contains one of the graphs of Figure 17 as a minor. Each of these graphs, however, contains a minor representing $M(K_{2,3})$, a contradiction.

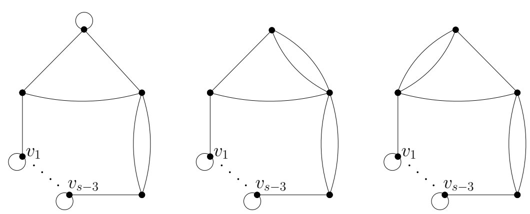

Figure 17: More graphs for the proof of Case 2.2.1.2.

Case 2.2.2 We cannot have that $s = 3$. If $s \geq 5$, then in a very similar fashion as in Case 2.2.1.2, we get that G is a minor of $2C_n$, a desired result. So it remains to consider the case in which $s = 4$. Again, we have that (m_1, m_2) is one of $(4,4)$, $(3,5)$, $(3,4)$, and $(3,3)$. Let these be, respectively Cases 2.2.2.1 – 2.2.2.4.
Case 2.2.2.1 Here $(m_1, m_2) = (4,4)$. Thus G contains Z_8^* as minor. If G contains a Z_8^*-subgraph, then again, $G \cong Z_8^*$ by Lemma 4.3. If G does not contain a Z_8^*-subgraph, then the tree decomposition of $\hat{G}$ has vertex set containing $\{C_s, M_2, M_2, C\}$ in which C is a cycle term summed onto M_1. Since G has no vertices of degree 2, then G contains a minor obtained from Z_8^* by adding an edge with both endpoints in Z_8^*. Again, Lemma 4.3 implies that $G \cong Z_8^*$.
Case 2.2.2.2 Say that $(m_1, m_2) = (3,5)$. In a similar fashion as with $(m_1, m_2) = (4,4)$ we get that $G \cong O_8$, a desired result.
Case 2.2.2.3 Say that $(m_1, m_2) = (3,4)$. If $V(T) = \{C_s, M_1, M_2\}$, then $\hat{G}$ is the first graph shown in Figure 18, call if O. If we add a loop to one of the top two vertices of O and to one of the bottom two vertices of O, then the resulting graph has a minor representing $U_{2,7}$, a contradiction. If we add loops to both of the top vertices, then we obtain the second graph of Figure 18 which is a minor of F_{10}, a desired result. If T has a fourth vertex, then it is a cycle term attached to either M_1 or M_2. Therefore G will contain as a minor one of the last four graphs of Figure 18. The third and fourth graphs both contain a bicircular representation of $U''_{2,5}$ as a minor, a contradiction. The fifth and sixth graphs both contain a bicircular representation of $M(K_{2,3})$ as a minor, again a contradiction.

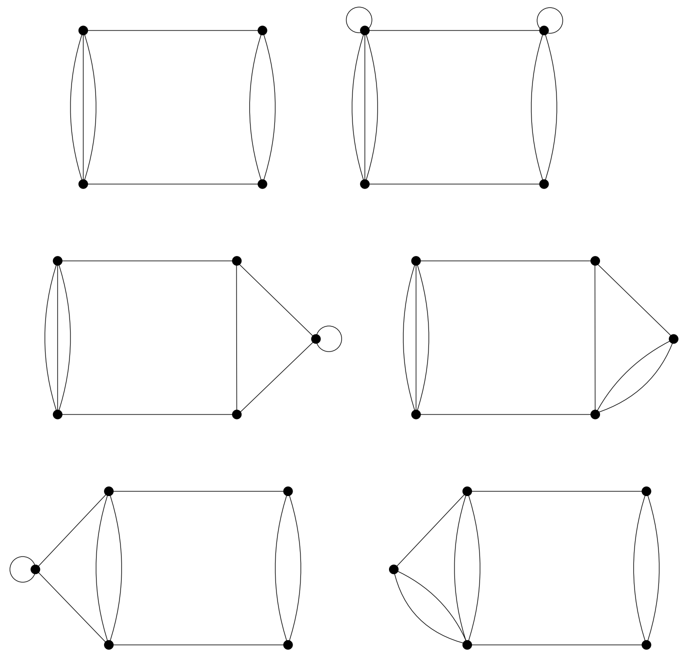

Figure 18: Graphs for the proof of Case 2.2.2.3

Case 2.2.2.4 Say that $(m_1, m_2) = (3, 3)$. If $V(T) = \{C_s, M_1, M_2\}$, then $\hat{G}$ is C_4^{++} and so G is a minor of $2C_8$, a desired result. If $V(T)$ has a fourth vertex, then it is a cycle term, call it C, and say without loss of generality that C is adjacent to M_1 in T. If C is a leaf of T, then G contains as a minor the first graph of Figure 19. This graph, however, contains a $D_4^{\ell\ell}$-minor, a contradiction. Thus $V(T)$ must contain a fifth vertex, call it X, that is adjacent to C, which makes X a multi-edge term. Suppose now that $V(T) = \{C_s, M_1, M_2, C, X\}$. If $X = tK_2$ for $t \geq 4$, then G contains as a minor the second graph of Figure 19 which contains a bicircular representation of $U''_{2,5}$ as a minor, a contradiction. Thus $X = 3K_2$ and so $\hat{G}$ is the third graph of Figure 19, call it F. If we add a loop to F at vertex 1, then the resulting graph has a $D_4^{\ell\ell}$-minor, a contradiction. If we add a loop to F at vertex 2, then the resulting graph contains a bicircular representation for $M(K_{2,3})$ as a minor, a contradiction. If we add a loop to vertex 3, then the resulting graph contains a bicircular representation of $U''_{2,5}$ as a minor, again a contradiction. Thus G is contained between the third graph of the figure and the graph obtained by adding loops to both of the unnumbered vertices. This latter graph is F_{10}, a desired outcome.

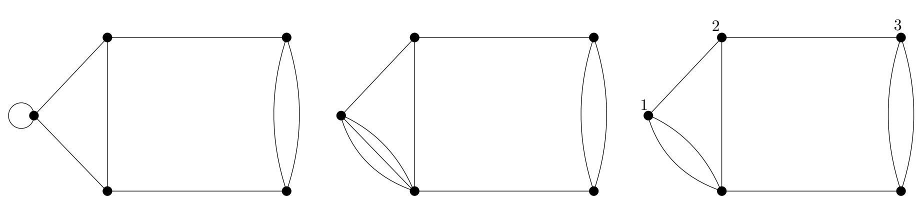

Figure 19: More graphs for the proof of Case 2.2.2.4

So now suppose that $V(T)$ properly contains $\{C_s, M_1, M_2, C, X\}$ and so has a sixth vertex, call it Y. The vertex Y is either a multi-edge term adjacent to C or a cycle term adjacent to one of M_1, M_2, and X. The reader can verify all of the outcomes in these four cases. One, if Y is adjacent to C, then G contains a $T_{3,2,2}$-minor, a contradiction. Two, if Y is adjacent to X, then G contains a representation of $M(K_{2,3})$, a contradiction. Three, if Y is adjacent to M_2, then G either contains a representation of $M(K_{2,3})$ or $U''_{2,5}$, a contradiction. Four, if Y is adjacent to M_1, then G contains a representation of $M(K_{2,3})$ as a minor, a contradiction.

Case 2.3 If C_s has degree zero in T, then $\hat{G} = C_s$ and so G is obtained from C_s by adding a loop to each vertex. Thus G is a minor of $2C_{2s}$, a desired outcome. If C_s has degree 1 in T, then we split the remainder of this case into two subcases. In Case 2.3.1, say that $s \geq 4$ and in Case 2.3.2 say $s = 3$. In each case, let M be the multi-edge term whose corresponding vertex in T is adjacent to C_s.

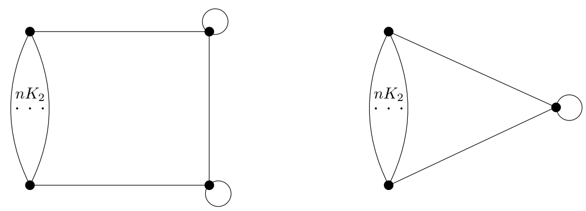

Figure 20: Graphs for the proof of Case 2.3.

Case 2.3.1 In this case G contains as a minor the first graph shown in Figure 20. If $n \geq 3$, then the graph shown contains as a minor a bicircular representation of $U''_{2,5}$, a contradiction. So now suppose that $n = 2$. If $V(T) = \{C_s, M\}$, then G is a minor of $2C_{2s}$, a desired outcome. If T contains a third vertex, then this is a cycle term, call it C, which is adjacent to M. In this case G contains as a minor one of the two graphs shown in Figure 21. Both of these graphs, however, contain a bicircular representation of $M(K_{2,3})$, a contradiction.

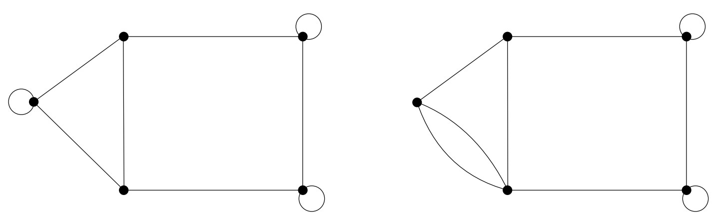

Figure 21: Graphs for the proof of Case 2.3.1.

Case 2.3.2 If $V(T) = \{C_s, M\}$, then G is obtained from the second graph of Figure 20 by adding loops. If $n = 2$, then G is a minor of $2C_6$, a desired outcome. If $n = 3$, then the second graph of the figure with one additional loop added is a minor of F_{10}, a desired outcome. If a third loop is added, then the graph obtained contains a representation of $U'_{2,6}$ as a minor, a contradiction. If $n = 4$, then the second graph of the Figure 20 is a minor of O_8, a desired outcome; however, if another loop is added then we would obtain a graph which has a minor representing $U_{2,7}$, a contradiction. It cannot be that $n \geq 5$ because, again, the graph would contain a bicircular representation of $U_{2,7}$ as a minor.

Now suppose that $|V(T)| \geq 3$. This implies that $V(T) \supseteq \{C_s, M, C\}$ in which C is a cycle term adjacent to M. Recall that $M \cong tK_2$ for some $t \geq 3$. If $t \geq 4$, then G contains one of the graphs of Figure 22 as a minor. The first graph contains graph contains $D_4^{\ell\ell}$ as a subgraph and the second graph contains a bicircular representation of $M(K_{2,3})$ as a minor, both are contradictions.

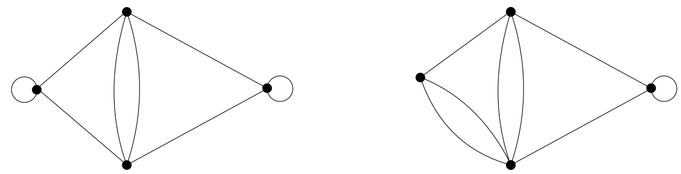

Figure 22: Graphs for the proof of Case 2.3.2.

Therefore $t = 3$. It cannot be that $V(T) = \{C_s, M, C\}$, because then G would again contain $D_4^{\ell\ell}$ as a subgraph. Thus $V(T) \subseteq \{C_s, M, C, M'\}$ in which M' must be a multi-edge sK_2 with $s \geq 3$. If $s \geq 4$, then the reader may check that G contains a bicircular representation of $U''_{2,5}$ as a minor, a contradiction. Therefore $s = 3$. If $V(T) = \{C_s, M, C, M'\}$, then G is obtained from the first graph of Figure 23 by adding loops. We cannot add a loop to vertex 1 because we would obtain a graph containing a representation of $U''_{2,5}$. We cannot add a loop to vertex 2 because we would obtain a graph containing a representation of $M(K_{2,3})$. Thus G is a subgraph of the second graph in Figure 23 which is a minor of F_{10}, a desired result.

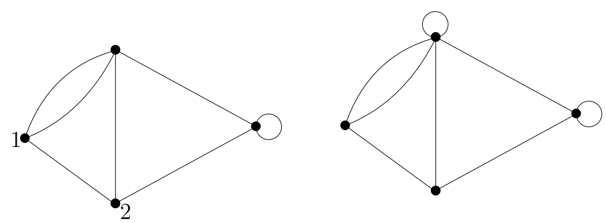

Figure 23: More graphs for the proof of Case 2.3.2.

If $|V(T)| \geq 4$, then $V(T) \subseteq \{C_s, M, C, M', X\}$ in which X is either a cycle attached to M, multi-edge attached to C, or cycle attached to M'. We leave it to the reader to check that the following outcomes hold, all of which are contradictions: in the first and third cases G would contain a representation of $M(K_{2,3})$ and in the second case G would contain $T'_{2,2,2}$ as a minor. □

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