Bicircular matroids representable over $GF(4)$ or $GF(5)$

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Abstract

Given a bicircular matroid $B(G)$ and $q \in \{4, 5\}$, we characterize when the bicircular matroid $B(G)$ is $GF(q)$-representable by precisely describing the structure of G. These descriptions yield polynomial-time algorithms with input G to certify if $B(G)$ is or is not $GF(q)$-representable.

1   Introduction

We will assume that the reader is familiar with matroid theory as presented by Oxley in [13]. The graph theory terminology used is standard and is explicitly defined in Section 2. A *frame matroid* is a matroid M which has an extension M_J satisfying the following.

- $r(M) = r(M_J)$

- $E(M_J) = E(M) \cup J$ and J is a basis of M_J.

- For every nonloop $e \in E(M)$ that is not parallel to some $b \in J$, there is a unique $\{b_1, b_2\} \subseteq J$ such that $e \in \mathrm{cl}_{M_J}\{b_1, b_2\}$.

The elements of J are usually referred to as *joints*. A fundamental example of frame matroids is the following. Given a field $\mathbb{F}$, an $\mathbb{F}$-*frame matrix* is an $\mathbb{F}$-matrix for which every column has at most two nonzero components. The vector matroid of a frame matrix is a frame matroid where the joints are the columns of an identity matrix prepended to the frame matrix. An $\mathbb{F}$-*frame matroid* is a matroid that is representable by a frame matrix over $\mathbb{F}$. Not every frame matroid that is $\mathbb{F}$-representable is an $\mathbb{F}$-frame matroid, however. For example, $U_{3,6}$ is $GF(4)$-representable but has no frame-matrix representation over $GF(4)$ (see Theorem 10).

Any frame matroid has a graphical structure (called a *biased graph*) that was first explored by Zaslavsky [17–19]. *Gain graphs* (sometimes called *voltage graphs*, especially in the area of topological graph theory) are a specific type of biased graph that are especially useful for studying Dowling geometries and their minors.

The relative importance of frame matroids among all matroids was first displayed by Kahn and Kung [9]. They found that there are only two classes of matroid varieties that can contain 3-connected matroids: simple matroids representable over $GF(q)$ and Dowling geometries and their simple minors (which are frame matroids). More recently the matroid-minors project of Geelen, Gerards, and Whittle has found the following far-reaching generalization of Seymour's decomposition theorem for regular matroids [6, Theorem 3.1]. If $\mathcal{M}$ is a proper minor-closed class of the class of $GF(q)$-representable matroids, then any member of $\mathcal{M}$ of

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sufficiently high vertical connectivity is either a bounded-rank perturbation of a frame matroid, the dual of a bounded-rank perturbation of a frame matroid, or is representable over some subfield of $GF(q)$.

Prominent within the class of frame matroids are the *bicircular matroids*. Given a graph G the bicircular matroid $B(G)$ has elements $E(G)$ and circuits that are edge sets of subdivisions of the graphs shown in Figure 1. The joints of $B(G)$ are loops attached to the vertices of G.
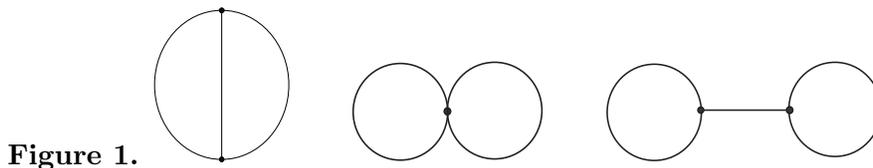

Figure 1.

Representability of bicircular matroids over finite fields is a natural topic of investigation, yet not much is known. Some immediate results are the following. Proposition 2 and facts similar to it have been discovered independently over the years (in particular, see [2,20]). We will see how it is true later in this paper through the usage of antivoltages.

Proposition 2. *If G is a graph, then there is n_0 such that for any $q \geq n_0$, $B(G)$ has a $GF(q)$-frame-matrix representation.*

Proposition 3. *Consider a finite field $GF(q)$ and a positive integer $n \geq 2$. The following are true when referring to graphs without multiple loops at a vertex.*

(1) *There are only finitely many vertically 3-connected graphs G for which $B(G)$ has no $U_{2,n}$-minor.*

(2) *There are only finitely many vertically 3-connected graphs G for which $B(G)$ is $GF(q)$-representable.*

(3) *There are only finitely many internally (or vertically) 4-connected bicircular matroids $B(G)$ for which $B(G)$ has no $U_{2,n}$-minor.*

(4) *There are only finitely many internally (or vertically) 4-connected bicircular matroids $B(G)$ for which $B(G)$ is $GF(q)$-representable.*

Proof. The graph nK_2 consists of two vertices with n parallel links. Note that a graph on v vertices without multiple loops at any vertex and without an nK_2-subgraph has at most $(n-1)\binom{v}{2} + v$ edges. So since $B(nK_2) \cong U_{2,n}$, there are only finitely many graphs on v vertices whose bicircular matroids are $U_{2,n}$-free. Now for any positive integer n, there is an n_0 such that any vertically 3-connected graph G with at least $v \geq n_0$ vertices contains either a $K_{3,n}$-minor or a W_n-minor [11]. Both of these graphs contain an nK_2-minor and so we have (1). Part (2) is a corollary of Part (1). For Parts (3) and (4), note that if $B(G)$ is internally (or vertically) 4-connected, then G is vertically 3-connected save perhaps for vertical 2-separations (A, B) with $|A| = 3$. We can now prove Parts (3) and (4) using the same argument as before save for a slight modification which accounts for these vertical 2-separations with a small side. $\square$

A structural characterization of graphs G for which $B(G)$ is binary was given by Matthews [10] and for when $B(G)$ is ternary was given by Sivaraman [14]. More generally (since bicircular matroids are also examples of transversal matroids) Sousa and Welsh [4] determined which transversal matroids are binary and Oxley [12] determined which gammoids (minors of transversal matroids) are ternary. The succinctly stated result of [12] is that a 3-connected gammoid is ternary iff it is a whirl of any rank.

In this paper we determine the structure of G when $B(G)$ is $GF(4)$-representable and when $B(G)$ is $GF(5)$-representable. Our main result for $GF(5)$-representability is Theorem 4. If G is a graph, then by G° we mean the graph obtained from G by adding a loop to each vertex; the other notation and terminology used in these theorems will be defined specifically in Section 2. We also obtain Theorem 7. For $GF(4)$-representability we have Proposition 8 as a corollary of the main result of Geelen, Gerards, and Kapoor [5]. We also present Theorem 10.

Theorem 4. *If G has no isolated vertices and $B(G)$ is connected, simple, and $GF(5)$-representable, then one of the following holds.*

(1) *G is a subdivision of a minor of $6K_2$ or W_4.*

(2) *G is a subdivision of a minor of the graph in Figure 5.*

(3) *G is a subgraph of $H°$ where H is a subdivision of a minor of K_4 or $2C_n$.*

(4) *G is not vertically 2-connected and $B(G) = B(G_1) \oplus_2 B(G_2)$ is a cut-vertex 2-sum or $B(G) = B(G_1) \|_2 B(G_2)$ is a cut-vertex parallel connection.*

Conversely, if G is constructed by cut-vertex 2-sums and cut-vertex parallel connections of graphs from (2) and (3), then $B(G)$ is $GF(5)$-representable.
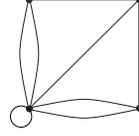

Figure 5.

Some rank-3 excluded minors for the class of $GF(5)$-representable matroids are the following, taken from Betten, Kingan, and Kingan [1].
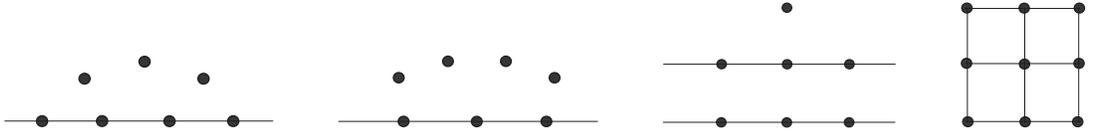

Figure 6.
The rank-3 matroids $\mathcal{B}_{7,1}$, $\mathcal{B}_{7,2}$, $\mathcal{B}_{7,3}$, and $B^*(K_{3,3})$. The latter matroid is also known as the *tic-tac-toe* matroid, denoted by M_3 in [1].

Theorem 7. *The excluded minors for $GF(5)$-representability within the class of bicircular matroids are $U_{2,7}$, $\mathcal{B}_{7,1}$, $\mathcal{B}_{7,1}^*$ $\mathcal{B}_{7,2}$, $\mathcal{B}_{7,3}$, $\mathcal{B}_{7,3}^*$, and $B(K_{3,3})$.*

Proposition 8. *The excluded minors for $GF(4)$-representability within the class of bicircular matroids are $U_{2,6}$, $U_{4,6}$, and P_6 and if G is a graph without isolated vertices such that $B(G) \cong U_{2,6}$, $U_{4,6}$, or P_6, then G is isomorphic to $6K_2$, $5K_2^°\backslash\ell$ where ℓ is a loop, $4K_2^°$, K_4, or one of the graphs shown.*
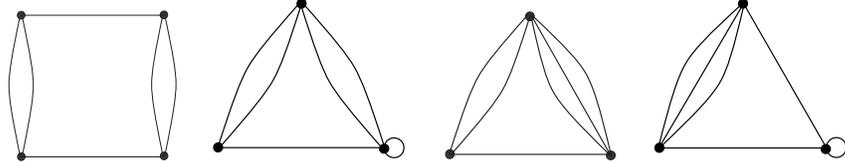

Figure 9.

Proof. The excluded minors for $GF(4)$-representability [5] are $U_{2,6}$, $U_{4,6}$, P_6, F_7^-, $(F_7^-)^*$, P_8, and P_8''. (See [5] and/or [13] for definitions and properties of these matroids.) The matroids P_8 and P_8'' both are not transversal matroids while bicircular matroids are transversal [10]. The matroids F_7^- and $(F_7^-)^*$ both have an $M(K_4)$-minor but $M(K_4)$ is not a bicircular matroid. That the graphs indicated are the only ones whose bicircular matroids are isomorphic to one of $U_{2,6}$, $U_{4,6}$, and P_6 is easy to check. Thus $U_{2,6}$, $U_{4,6}$, and P_6 are the excluded minors for $GF(4)$-representability within the class of bicircular matroids. □

Theorem 10. *If G has no isolated vertices and $B(G)$ is connected, simple, and $GF(4)$-representable, then one of the following holds.*

(1) *G is a subdivision of a minor of $5K_2$ or $2C_3$.*

(2) *G is a subdivision of a minor of the graph in Figure 11.*

(3) *G is a subgraph of $H°$ where H is a subdivision of $3K_2$.*

(4) *G is not vertically 2-connected and $B(G) = B(G_1) \oplus_2 B(G_2)$ is a cut-vertex 2-sum or $B(G) = B(G_1) \|_2 B(G_2)$ is a cut-vertex parallel connection.*

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Conversely, if G is constructed by cut-vertex 2-sums and cut-vertex parallel connections of graphs from (2) and (3), then B(G) is GF(4)-representable.
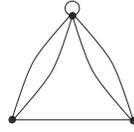

Figure 11.

In [7] it is shown that if $\mathcal{M}$ is a minor-closed class of matroids not containing $U_{2,n}$, then there exists a constant c such that for all simple matroids $M \in \mathcal{M}$ either

(i) $|E(M)| \leq c\, r(M)$,

(ii) $\mathcal{M}$ contains all graphic matroids and $|E(M)| \leq c(r(M))^2$, or

(iii) $\mathcal{M}$ contains all $GF(q)$-representable matroids for some q and $|E(M)| \leq c\, q^{r(M)}$.

It is worth noting the the class of graphs $\mathcal{G}_q$ whose bicircular matroids are $GF(q)$-representable falls into the linear-growth category. This along with the finite number of vertically 3-connected members of $\mathcal{G}_q$ give a "sparse" feel to $\mathcal{G}_q$ which is particularly evident for $q \leq 5$.

2 Preliminaries

Graphs   Graphs have two types of edges, links and loops: a link has its two ends attached to distinct vertices and a loop has its two ends attached to the same vertex. Given a graph G, by G° we mean G with a loop added to each vertex. By nG we mean the graph obtained from G by replacing each of the links of G with n parallel links on the same two vertices. We call nK_2 an *n-multilink* or just a *multilink*. The n-vertex cycle is denoted by C_n and the n-spoked wheel is denoted by W_n. Given integers $a, b, c \geq 1$, the (a, b, c)-*fat-triangle* $T_{a,b,c}$ is the triangle with the three links replaced by a-, b-, and c-multilinks.

A connected graph G is said to be *vertically k-connected*, when either $|V(G)| = k+1$ and the simplification of G is K_n or $|V(G)| \geq k+2$ and there is no set of $t < k$ vertices in G whose removal from G disconnects the graph. In most of graph theory "vertically k-connected" is usually called just "k-connected", but we use the former term to avoid confusion with k-connectivity of $M(G)$. Of course $M(G)$ is k-connected when G is vertically k-connected and has girth at least k.

Bicircular matroids   Given a graph G and $X \subseteq E(G)$, let v_X be the number of vertices incident to edges in X, let $G{:}X$ be the subgraph of G with edge set X and vertices incident to edges in X, and let t_X be the number of acyclic connected components of $G{:}X$. The *bicircular matroid* $B(G)$ has elements $E(G)$ and rank function $r(X) = v_X - t_X$. The circuits of $B(G)$ are the minimal edge sets X satisfying $|X| = v_X + 1$, i.e., $G{:}X$ is a subdivision of one of the graphs shown in Figure 1. A cocircuit of $B(G)$ is a minimal edge set $X = K \cup D$ where either $K = \emptyset$ or K is an edge cut and D is the complement of a spanning tree of one of the components of $G \setminus K$. An independent set of $B(G)$ is an edge set X for which no connected component contains more than a single cycle.

If e is a link in G, then $B(G)\backslash e = B(G\backslash e)$ and $B(G)/e = B(G/e)$. These almost show that the class of bicircular matroids is a minor-closed class of matroids. Graph loops and matroid loops present a slight technical challenge. If G is a graph, then $B(G)$ is loopless. Matroid loops in bicircular matroids are just noted separately from the graphical representation. If e is a loop in G, then $B(G)\backslash e = B(G\backslash e)$ but $B(G)/e = B(G')$ where G' is obtained from G by removing the endpoint of e, replacing the links incident to e with loops incident to their other endpoints, and replacing the other graph loops incident to e with matroid loops. We now have that the class of bicircular matroids is a minor-closed class of matroids.

Proposition 12 contains some basic connectivity results for bicircular matroids. All of these facts are quite easily verified, save perhaps Part (6) which is from [16, Proposition 2].

Proposition 12. *Let G be a graph with no isolated vertices.*

(1) e and f are parallel in $B(G)$ iff e and f are loops with the same endpoint.

(2) If $|V(G)| \geq 2$, then $B(G)$ is connected iff G is connected, G has minimum degree 2, and $G \not\cong C_n$.

(3) If G has a vertex with exactly two edges e and f incident to it, then e and f are coparallel in $B(G)$.

(4) If G is connected and $G \not\cong C_n$, then e is a coloop of $B(G)$ iff e is an isthmus of G with the subgraph on at least one side of the isthmus being acyclic. If $G \cong C_n$, then every element of $B(G)$ is a coloop.

(5) If G is connected, contains more than one cycle, and $B(G)$ is not connected, then G has a maximal subgraph G' for which $B(G')$ is connected and G is obtained from G' by attaching trees to any of the vertices of G'. Furthermore, any edge in one of these attached trees is a coloop of $B(G)$.

(6) If $|V(G)| \geq 3$, then $B(G)$ is 3-connected iff G is vertically 2-connected, has minimum degree at least 3, and has no two loops attached to the same vertex.

If G has no isolated vertices, $B(G)$ is connected and simple, but G is not vertically 2-connected, then there is a cut vertex v in G such that $G = G'_1 \cup G'_2$ where each G'_i is connected and contains a cycle and G'_1 and G'_2 intersect at v only. If G'_i has a loop at v, then let $G_i = G'_i$. If G'_i does not have a loop at v, then let G_i be G'_i with a new loop attached at v. If G does not have a loop at v, then the reader can check that $B(G) = B(G_1) \oplus_2 B(G_2)$ where the 2-sum is along the graphical loops at v in each term. Call this 2-sum a *cut-vertex 2-sum* at v. If G does have a loop at v, then, again, the reader may check that $B(G)$ is the parallel connection of $B(G_1)$ and $B(G_2)$ along the graphical loops at v in each term. Call this a *cut-vertex parallel connection* at v and denote it by $B(G_1) \parallel_2 B(G_2)$.

For any connected matroids M and N and any field $\mathbb{F}$ it is well known that $M \oplus_2 N$ is $\mathbb{F}$-representable iff each of M and N are $\mathbb{F}$-representable iff $M \parallel_2 N$ is $\mathbb{F}$-representable. Thus our characterizations of when $B(G)$ is connected along with our cut-vertex operations of the previous paragraph enable us to to characterize $\mathbb{F}$-representability of $B(G)$ by focusing on the cases where G is vertically 2-connected or when G has only two vertices.

Some well-known matroids represented as bicircular matroids are as follows: $B(C_n^{\circ})$ is the rank-n whirl matroid and $B(2C_n^{\circ})$ for $n \geq 3$ is the rank-n free swirl matroid with joints. If G has no isolated vertices, then $B(G) \cong U_{3,6}$ iff $G \cong T_{2,2,2}$ [18, 2.12] and $B(G) \cong U_{2,n}$ iff $G \cong nK_2$, $(n-2)K_2^{\circ}$, or $(n-1)K_2^{\circ}\backslash \ell$ where ℓ is a loop.

Gain functions and antivoltages   Given a graph G, an *oriented edge* e is an edge of G along with a direction along that edge. The edge with the reverse direction is denoted by $-e$ and the set of oriented edges of G is denoted by $\vec{E}(G)$. A *walk* in G can now be expressed as a sequence of oriented edges $e_1, \ldots, e_n$ where the head of e_i is the tail of e_{i+1}. Given an abelian group Γ (which we will always consider to be additive) a Γ-*gain function* is a function $\varphi \colon \vec{E}(G) \to \Gamma$ satisfying $\varphi(-e) = -\varphi(e)$. A gain function extends to any walk $w = e_1, \ldots, e_n$ in G by setting $\varphi(w) = \sum_i \varphi(e_i)$. (Gain graphs can be defined for nonabelian groups with just a few added details [17]; however, we will not need nonabelian groups in this paper.) We say that φ is a Γ-*antivoltage* when for any cycle C in G and any closed Eulerian walk w_C on C we have $\varphi(w_C) \neq 0$. (Of course, if w_1 and w_2 are two closed Eulerian walks along C, then $\varphi(w_1) = \pm\varphi(w_2)$.) Antivoltages were defined and studied by Zaslavsky in [21]. If φ is a Γ-antivoltage for G, then φ trivially extends to a Γ-antivoltage on H° where H is any subdivision of G. Also if H is any minor of G and φ is a Γ-antivoltage on G, then there is an induced Γ-antivoltage on H [17, 5.4]. This latter fact yields Proposition 13.

Proposition 13. *Let Γ be an abelian group of order n. If φ is a Γ-antivoltage on G, then $B(G)$ is $U_{2,n+3}$-free.*

For cyclic groups, a $\mathbb{Z}_n$-gain function φ on G is depicted as follows. For a link e, if $\varphi(e) = k \in \{0, \ldots, n-1\}$, then k directional arrows are placed on e in the appropriate direction. For example, a $\mathbb{Z}_4$-antivoltage for $2C_n$ can be constructed for any n in the analogous fashion as is shown in Figure 14 for $n = 6$.
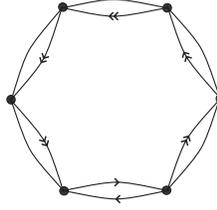
Figure 14.

Graphically encoding frame-matrix representations   If φ is a $\mathbb{Z}_{q-1}$-antivoltage on G, then $B(G)$ has the following frame representation over $GF(q)$. Write $V(G) = \{v_1, \ldots, v_n\}$ and let $\vec{v}_i$ be the elementary column vector over $GF(q)$ with a 1 in row i and zeros everywhere else. Let ψ be an isomorphism from $\mathbb{Z}_{q-1}$ to the multiplicative subgroup of $GF(q)$. Create a frame matrix $F(G, \varphi)$ with rows indexed by $V(G)$ and columns indexed by $E(G)$ where the column corresponding to link e oriented from v_i to v_j is $\vec{v}_i - \psi(\varphi(e))\vec{v}_j$. Of course if we construct this column using $-e$ instead e then the resulting column vector is obtained from the column vector for e by multiplying by $-\psi(\varphi(e))^{-1} = -\psi(\varphi(-e))$. The column corresponding to a loop at vertex v_i is just the vector $\vec{v}_i$.

Proposition 15 (Zaslavsky [20, 21]).

(1) *If φ is a $\mathbb{Z}_{q-1}$-antivoltage on G and φ° is an associated $\mathbb{Z}_{q-1}$-antivoltage on H° where H is any subdivision of G, then the matroid for the $GF(q)$-matrix $F(H^\circ, \varphi^\circ)$ is $B(H^\circ)$.*

(2) *If $B(G)$ has a frame-matrix representation F over $GF(q)$, then there is G' with $B(G) = B(G')$ for which G' has a $\mathbb{Z}_{q-1}$-antivoltage φ such that $F(G', \varphi) = F$ up to column scaling.*

(3) *If $B(G^\circ)$ has a matrix representation A over $GF(q)$, then there is a $\mathbb{Z}_{q-1}$-antivoltage φ on G° such that $F(G^\circ, \varphi)$ is equivalent to A.*

Proposition 16. *If H is a subdivision of a minor of K_4 or $2C_n$, then $B(H^\circ)$ is $GF(5)$-representable.*

Proof. The multiplicative subgroup of $GF(5)$ is isomorphic to $\mathbb{Z}_4$ and a $\mathbb{Z}_4$-antivoltages for K_4 is easy to obtain. A $\mathbb{Z}_4$-antivoltage for $2C_n$ is described in Figure 14. So now $B(H^\circ)$ is $GF(5)$-representable by Proposition 15. ☐

The lift matroids and matrix representations   Given a graph G, the Higgs lift $L(G)$ of the graphic matroid $M(G)$ has circuits that are unions of modular pairs of circuits in $M(G)$. That is, edge sets X such that $G{:}X$ is a subdivision of the one of the graphs in Figure 17. Denote by $L_0(G)$, the free coextension of $M(G)$ by new element e_0. Thus $L(G) = L_0(G)\backslash e_0$ (see [8] or [13, p.289]). One can check that $L_0(G) = L(G_0)$ where G_0 is obtained from G by adding a loop e_0 to any vertex of G or to a new vertex outside of G. Proposition 18 is immediate.
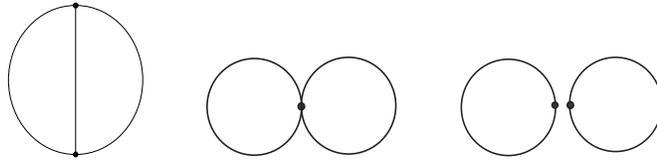

Figure 17.

Proposition 18.

- $B(G) = L(G)$ *iff G has no two vertex-disjoint cycles.*

- $B(G_0) = L(G_0) = L_0(G)$ *iff $G\backslash v$ is a forest and G_0 has its new loop attached to v.*

If φ is a $(\mathbb{Z}_p)^n$-antivoltage on G, then we construct a $GF(p^n)$-matrix $A(G, \varphi)$ as follows. Write $V(G) = \{v_1, \ldots, v_n\}$ and let $\vec{v}_i$ for $i \in \{0, 1, \ldots, n\}$ be the elementary column vector over $GF(q)$ with a 1 in row i and zeros everywhere else. Let ψ be an isomorphism from $(\mathbb{Z}_p)^n$ to the additive subgroup of $GF(p^n)$. The rows of $A(G, \varphi)$ are indexed by $V(G) \cup \{v_0\}$ and the columns of $A(G, \varphi)$ by $E(G)$ where the column corresponding

to an edge e oriented from v_i to v_j is $\vec{v}_i - \vec{v}_j + \psi(\varphi(e))\vec{v}_0$. If we construct this column using $-e$ instead of e then the resulting column vector is the negative of the column vector for e. The column corresponding to a loop at vertex v_i is just the vector $\vec{v}_i$. Define the matrix $A_0(G, \varphi) = A(G_0, \varphi)$.

Proposition 19 (Zaslavsky [20, 21]). *If φ is a $(\mathbb{Z}_p)^n$-antivoltage on G, then the matroid for matrix $A_0(G, \varphi)$ is $L_0(G)$.*
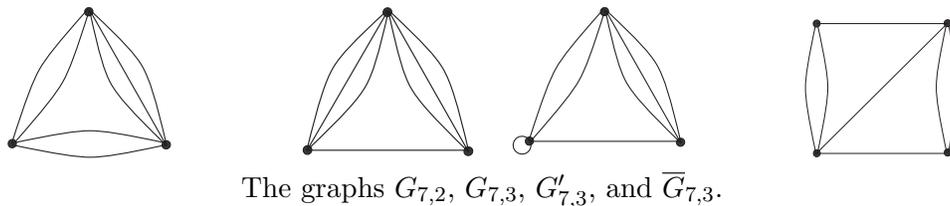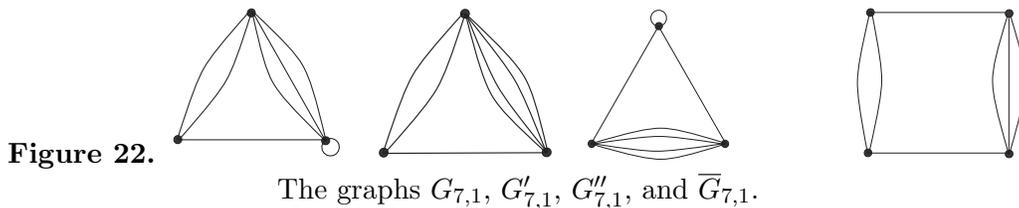
Proposition 20. *If H is a subdivision of a minor of $6K_2$, W_4, or the graph of Figure 5, then $B(H)$ is $GF(5)$-representable.*

Proof. If G is one of the three graphs in the statement of the proposition, then G has no two vertex-disjoint cycles and so $B(G) = L(G)$. We know that $B(6K_2) \cong U_{2,6}$ and so is $GF(5)$-representable. For the other two graphs, the additive subgroup of $GF(5)$ is $\mathbb{Z}_5$ and finding a $\mathbb{Z}_5$-antivoltage is routine. Thus $B(H) = L(H)$ is $GF(5)$-representable by Proposition 19. $\qquad\square$

3 $GF(5)$-representable bicircular matroids

Proposition 21. *Let G be a connected graph.*

- $B(G) \cong U_{2,7}$ iff $G \cong 7K_2$, $6K_2^\circ \setminus \ell$ where ℓ is a loop, or $5K_2^\circ$.
- $B(G) \cong \mathcal{B}_{7,1}$ iff $G \cong G_{7,1}$, $G'_{7,1}$, or $G''_{7,1}$.
- $B(G) \cong \mathcal{B}_{7,1}^*$ iff $G \cong \overline{G}_{7,1}$.
- $B(G) \cong \mathcal{B}_{7,2}$ iff $G \cong G_{7,2}$.
- $B(G) \cong \mathcal{B}_{7,3}$ iff $G \cong G_{7,3}$ or $G'_{7,3}$.
- $B(G) \cong \mathcal{B}_{7,3}^*$ iff $G \cong \overline{G}_{7,3}$.
- $B(G) \cong B(K_{3,3})$ iff $G \cong K_{3,3}$.

Figure 22.

The graphs $G_{7,1}$, $G'_{7,1}$, $G''_{7,1}$, and $\overline{G}_{7,1}$.

The graphs $G_{7,2}$, $G_{7,3}$, $G'_{7,3}$, and $\overline{G}_{7,3}$.

Sketch of Proof for Proposition 21. We will provide sketches for the last two parts, the rest are routine. Let x be the element of $B_{7,3}$ that is not on a triangle and so $B_{7,3}^* \setminus x \cong U_{4,6}$. One can check that the only graphs whose bicircular matroids are $U_{4,6}$ are K_4 and $2C_4 \setminus \{e, f\}$ where e and f are the edges of a matching. The only way to add an element to these graphs to obtain the matroid $B_{7,3}^*$ is to make $\overline{G}_{7,3}$.

The rank of $B(K_{3,3})$ is 6 with 9 elements. So if $B(G) \cong B(K_{3,3})$, then G must have average degree 3. Since $B(K_{3,3})$ is cosimple, the minimum degree of G must be 3 and so G is 3-regular. It cannot be that G has loops, because a loop at vertex of degree 3 will yield coparallel elements in $B(G)$. It cannot be that G has a 3-multilink because $B(K_{3,3})$ has no triangles. Thus G is either simple or simple up to multilinks of size two. The only 3-regular simple graph on 6 vertices other than $K_{3,3}$ is the triangular prism graph P (i.e., the planar dual graph of $K_5 \setminus e$). The elements of $B(K_{3,3})$ are a disjoint union of 3 cocircuits of size 3; however,

the only cocircuits of size 3 in $B(P)$ are the vertex cocircuits and these do not have have this property. Thus G has a 2-multilink on vertices u and v with edges e and f. If e' is the other edge incident to u and f' is the other edge incident to v, then $\{e, e', f, f'\}$ is a 4-element coline of $B(G)$; however $B(K_{3,3})$ has no 4-element colines. $\qquad\square$
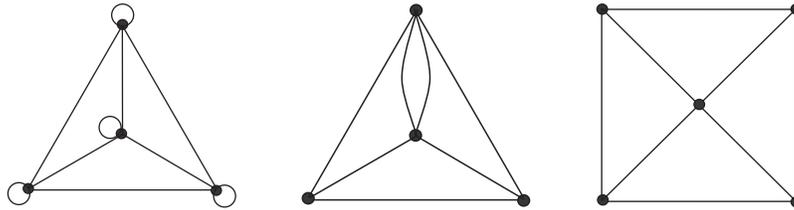

Figure 23.

The graphs K_4°, $K_4^\| = K_4 \oplus_2 3K_2$, and W_4.

Proposition 24.

1. *The matroid $B(W_4)$ is a splitter for the class of $GF(5)$-representable matroids.*

2. *If G has no isolated vertices, $B(G)$ is 3-connected and $GF(5)$-representable, and one of G/e and $G\backslash e$ is $K_4^\|$, then $G \cong W_4$.*

Proof. For Part 1, first notice that $B(W_4)$ is $GF(5)$-representable because of Proposition 19 and the fact that W_4 has a $\mathbb{Z}_5$-antivoltage. We will now show that every 3-connected $B(G)$ that is a single-element extension or coextension of $B(W_4)$ is not $GF(5)$-representable by finding a minor in G from among the graphs of Proposition 21. By [16, Proposition 5] a connected graph G has $B(G) \cong B(W_4)$ iff $G \cong W_4$. Now if $B(G)$ is 3-connected and a single-element extension or coextensions of $B(W_4)$, then G is vertically 2-connected, has minimum degree 3, and is obtained from W_4 by the addition or decontraction of an edge. If a parallel edge is added to the rim of W_4, then a $G_{7,3}$-minor is created. If a parallel edge is added to the spokes of W_4, then again a $G_{7,3}$-minor is created. If a missing link is added to W_4, then a $G'_{7,1}$-minor is created. If a loop is added anywhere to W_4, then a $G'_{7,1}$-minor is created. If $G/e \cong W_4$, then G is $K_{3,3}$ or a triangular prism (i.e., two vertex-disjoint triangles with three connecting links): $B(K_{3,3})$ is not $GF(5)$-representable and the triangular prism contains a $\overline{G}_{7,3}$-minor.

For Part 2, one can check that $B(K_4^\|) \cong B(2C_4\backslash e)$ and so $B(K_4^\|)$ has a $GF(5)$-frame-matrix representation as given in Figure 14 and Proposition 15. Now we will show that every graph G that is vertically 2-connected, has minimum degree 3, and is a single-edge extension or coextension of $K_4^\|$ is not $GF(5)$-representable aside from $G \cong W_4$ (again by finding a minor in G from among the graphs of Proposition 21). For extensions, if e is a link, then $K_4^\|\cup e$ contains a $G_{7,2}$- or $G_{7,3}$-minor. If e is a loop, then $K_4^\|\cup e$ contains a $G_{7,1}$- or $G'_{7,3}$-minor. If $G/e \cong K_4^\|$, G has no vertex of degree 2, and $G \not\cong W_4$, then G is obtained from $K_4^\|$ by pulling the digon away from a vertex of degree 4. The graph G has a $\overline{G}_{7,3}$-minor. $\qquad\square$

Proposition 25.

1. *The graph K_4 has a $\mathbb{Z}_4$-antivoltage while neither $K_4^\|$ nor W_4 has a $\mathbb{Z}_4$-antivoltage.*

2. *Both $B(K_4^\circ)$ an $B(K_4^\|)$ are $GF(5)$-frame matroids while $B(W_4)$ is not a $GF(5)$-frame matroid.*

Proof. Partition the edges of K_4 in T' and T where T' forms a triad and T a triangle. A $\mathbb{Z}_4$-antivoltage for K_4 is given by placing a gain of 0 on the edges of T' and a gain of 1 on the edges of T all in the same direction along the triangle. Neither $K_4^\|$ nor W_4 have a $\mathbb{Z}_4$-antivoltage because if $G \in \{K_4^\|, W_4\}$ did have a $\mathbb{Z}_4$-antivoltage, then G° would have a $\mathbb{Z}_4$-antivoltage and so $B(G^\circ)$ would be $GF(5)$-representable, a contradiction of Proposition 24.

Because K_4 has a $\mathbb{Z}_4$-antivoltage, $B(K_4^\circ)$ is a $GF(5)$-frame matroid. One can check that $B(K_4^\|) \cong B(2C_4\backslash e)$ and so $B(K_4^\|)$ has a $GF(5)$-frame-matrix representation as given by the $\mathbb{Z}_4$-antivoltage in Figure 14. Any $GF(5)$-frame matrix representing $B(W_4)$ must come from a $\mathbb{Z}_4$-antivoltage on some connected graph G with $B(G) \cong B(W_4)$ (Proposition 15). Again $B(G) \cong B(W_4)$ iff $G \cong W_4$ and we proved in the previous paragraph that W_4 does not have a $\mathbb{Z}_4$-antivoltage. $\qquad\square$

Proposition 26 is the specific realization of Proposition 3 for $GF(5)$.

Proposition 26. *If G is a vertically 3-connected graph, then $B(G)$ is $GF(5)$-representable iff G is a minor of K_4° or W_4 (see Figure 23).*

Proof of Proposition 26. By Tutte's Wheel Theorem, there is a sequence of single-edge deletions and contractions taking G to a wheel graph W_n such that each intermediate graph is vertically 3-connected. It cannot be that $n \geq 5$ because W_5 contains a $G_{7,2}$-minor. If $n = 4$, then $G \cong W_4$ by Proposition 24. If $n = 3$, then G is a minor of K_4° or $K_4^{\|}$ by Proposition 24. $\qquad\square$

Proof of Theorem 4. The converse part of the theorem follows from Propositions 16 and 20 and the fact that $GF(5)$-representability is closed under 2-summing and parallel connections.

Now for the initial part of the theorem. Since $B(G)$ is connected, G is connected. If G has exactly two vertices, then G satisfies one of Parts (1)–(3) by Proposition 21 and the fact that $U_{2,7}$ is not $GF(5)$-representable. If G is vertically 3-connected, then by Proposition 26, G satisfies Part (1) or (3). If G has at least three vertices, is connected, and is not vertically 2-connected, then G satisfies Part (4). So for the remainder of the proof assume that G has at least three vertices, is vertically 2-connected, and not vertically 3-connected. Furthermore, we may smooth out vertices of degree 2 without affecting $GF(5)$-representability (Proposition 12(3)) and so $B(G)$ is 3-connected by Proposition 12(6).

Let G' be G with its loops removed. So now G' is vertically 2-connected, possibly with vertices of degree 2 which must have loops attached to them. Let T be the canonical tree decomposition of G' with $V(T) = \{G_1, \ldots, G_n\}$ and $E(T) = \{e_1, \ldots, e_{n-1}\}$. (See one of [3], [13, pp.308–315], or [15] for a description of the canonical tree decomposition of a vertically 2-connected graph.) For each i there is $t_i \geq 3$ such that G_i is one of the following: a t_i-cycle, a t_i-multilink, or a vertically 3-connected simple graph. Furthermore, there are no two adjacent vertices in T that are both cycles or both multilinks. In Case 1, say that more than one of the graphs $G_1, \ldots, G_n$ are vertically 3-connected. In Case 2, say that exactly one of $G_1, \ldots, G_n$ is vertically 3-connected. In Case 3, no G_i is vertically 3-connected.

Case 1 Here G contains a $(K_4 \oplus_2 K_4)$-minor and $K_4 \oplus_2 K_4$ contains a $\overline{G}_{7,3}$-minor and we know that $B(\overline{G}_{7,3})$ is not $GF(5)$-representable, a contradiction.

Case 2 Say that G_1 is vertically 3-connected. Each other G_i is either C_{t_i} or $t_i K_2$ for some $t_i \geq 3$. By Proposition 26(1) and the fact that G_1 is simple, $G_1 \cong K_4$ or W_4. In the latter case, 3-connectivity and Proposition 24 implies that $G \cong W_4$. Now assume that $G_1 \cong K_4$. If there are no multilink vertices in T (i.e., T consists of G_1 with adjacent vertices which are all cycles) then G satisfies Part (3). If T contains a multilink vertex, then G has a $K_4^{\|}$-minor and so $G \cong K_4^{\|}$ or W_4 by Proposition 24.

Case 3 Here each G_i is either C_{t_i} or $t_i K_2$ and no G_i that is of one type is adjacent in T to a G_j that is another type. (In particular, notice that G is series parallel.) Furthermore, for $G_i \cong t_i K_2$ we must have that $t_i \leq 6$. So if we root T at a vertex G_i, then the vertices of T at level $2k+1$ are of the opposite type as G_i and the vertices at level $2k$ are of the same type as G_i.

First suppose that there is some $G_i \cong 6K_2$. Root T at G_i. If T has height at least 2, then G' will contain a 7-multilink minor, a contradiction of $GF(5)$-representability. If T has height 1, then G will contain a $6K_2$-with-a-loop-minor, a contradiction of $GF(5)$-representability. Thus $G = 6K_2$, which satisfies Part (1).

Second suppose that there is some $G_i \cong 5K_2$. Root T at G_i. If T has height at least 2, then G' will contain a $G'_{7,1}$-minor, a contradiction of $GF(5)$-representability. If T has height 1, then G will contain a $G''_{7,1}$-minor, a contradiction of $GF(5)$-representability. Thus G is a subgraph of $5K_2 \cup \ell$ where ℓ is a loop, and thus G satisfies Part (2).

Third suppose that there is some $G_i \cong 4K_2$. Root T at G_i. If T has height at most 1, then G is a subgraph of H° where H is a subdivision of $4K_2$ and so satisfies Part (3). If T has height 2, then no child of G_i can have more than two children itself without creating a $G_{7,2}$-minor in G, a contradiction of $GF(5)$-representability. If G_i has two children which each have their own child, then G has a $G'_{7,1}$-minor, a contradiction of $GF(5)$-representability. Thus T has exactly one grandchild G_j. Say the parent of G_j in T is G_k. It must be that $G_k \cong C_3$ because otherwise, the facts that G has no vertices of degree 2 and that G_k has

only one child would imply that G contains a $\overline{G}_{7,1}$-, or $G'_{7,3}$-minor, a contradiction. Additionally G_j must be a 3-multilink because otherwise G would have a $G_{7,3}$-minor, a contradiction. Finally G_i has only one child in T or else G contains a $G_{7,1}$- or $G'_{7,1}$-minor. Thus G is a minor of the graph in Part (2). If T has height at least 3, then any path of length three in T down from G_i to a vertex at level 3 would yield a $G'_{7,3}$-minor in G, a contradiction. If T has height at least 4, then any path of length 4 in T down from G_i would yield a $G_{7,3}$-minor in G, a contradiction.

Lastly, each G_i that is a multilink is a 3-multilink. If no G_i is a multilink, then $G' \cong C_n^\circ$ and so G satisfies Part (3). So say T is rooted at G_i which is a 3-multilink. If T has height at most 1, then G satisfies Part (3). If T has height 2, then we will consider the cases where G_i has grandchildren from one, two, or three different children. If G_i has grandchildren from three different children, then G has a $G'_{7,1}$-minor, a contradiction. If G_i has grandchildren from two different children, then because G does not have a $\overline{G}_{7,3}$-minor, G must be a minor of the graph in Part (2). If G has grandchildren from only one child, then G is a subgraph of H° where H is a subdivision of $2C_n$, thus satisfying Part (3). If G has height 3, then from the case where G has height 2, it must be that G_i has grandchildren from at most two distinct children. If G_i has grandchildren from two distinct children, then G has both $G_{7,1}$- and $G'_{7,3}$-minors, a contradiction. Thus G_i has grandchildren from only one of its children. Since T has height 3, G is a subgraph of H° where H is a subdivision of $2C_n$, thus satisfying Part (3). If T has height 4, then, because G does not have an $\overline{G}_{7,3}$-minor, any path P of length 4 in T down from G_i produces a minor H in G that is a subgraph of the graph in Part (2). If $T = P$, then G satisfies Part (2). If $T \supset P$ then: an additional child of G_i will produce $G_{7,1}$- or $G'_{7,1}$-minor (a contradiction), an additional child of the child of G_i on P will produce $G_{7,2}$-minor (a contradiction), an additional child of the grandchild of G_i on P will produce a $G_{7,1}$- or $G'_{7,1}$-minor (a contradiction), an additional child of the great grandchild of G_i on P will produce a $G_{7,2}$-minor (a contradiction). If T has height at least 5, then a path of length 5 or 6 in T down from G_i in T will produce a $G_{7,1}$- or $G'_{7,1}$-minor (a contradiction). $\qquad\square$

Proof of Theorem 7. That the matroids listed are excluded minors for $GF(5)$ is from [1]. Now say that $B(G)$ has none of these matroids as a minor. Thus G has none of the graphs of Proposition 21 as a minor. Now in the proof of Theorem 4, we actually prove that if G has none of the graphs from Proposition 21 as a minor, then G satisfies one of Parts (1)–(4) of Theorem 4. Because each term of a cut-vertex 2-sum or parallel connection is a minor of G, we now get that G either satisfies Part (1) or G is constructed from cut-vertex 2-sums and parallel connections of graphs from Parts (2) and (3). The converse part of Theorem 4 states that such a graph's bicircular matroid is $GF(5)$-representable. $\qquad\square$

4 $GF(4)$-representable bicircular matroids

Proof of Theorem 10. First we prove the converse part of the theorem. Recall that the additive and multiplicative groups for $GF(4)$ are isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_3$, respectively. For the graphs in Part (1), the matroids $B(5K_2) \cong U_{2,5}$ and $M(2C_3) \cong U_{3,6}$ are both $GF(4)$-representable. The graph in Part (2) has a $\mathbb{Z}_2 \times \mathbb{Z}_2$-antivoltage and so its bicircular matroid is $GF(4)$-representable. The graphs $3K_2$ has a $\mathbb{Z}_3$-antivoltage and so $B(H^\circ)$ is $GF(4)$-representable. Now cut-vertex 2-sums and parallel connections of bicircular matroids from Parts (1) – (3) are still $GF(4)$-representable.

Now for the initial part of the theorem. Since $B(G)$ is connected, G is connected. If G has exactly two vertices, then G satisfies Part (1) or Part (3) because of the fact that $U_{2,6}$ is not $GF(5)$-representable. Now say that G has at least three vertices. Denote the graphs of Figure 9, respectively, by $G_{6,1}$, $G_{6,2}$, $G_{6,3}$, and $G_{6,4}$.

If G is vertically 3-connected, then G contains a K_4-minor; however, $B(K_4) \cong U_{4,6}$ which is not $GF(4)$-representable, a contradiction. If G has at least three vertices, is connected, and is not vertically 2-connected, then G satisfies Part (4). So for the remainder of the proof assume that G has at least three vertices, is vertically 2-connected, and not vertically 3-connected. Furthermore, we may smooth out vertices of degree 2 without affecting $GF(4)$-representability (Proposition 12(3)) and so $B(G)$ is 3-connected by Proposition 12(6).

Let G' be G with its loops removed. So now G' is vertically 2-connected, possibly with vertices of degree 2 which must have loops attached to them. Let T be the canonical tree decomposition of G' with $V(T) = \{G_1, \ldots, G_n\}$ and $E(T) = \{e_1, \ldots, e_{n-1}\}$. As mentioned in the previous paragraph, $B(K_4)$ is not $GF(4)$-representable and so for each i, G_i is either a t_i-multilink for some $3 \leq t_i \leq 5$ or a t_i-cycle for some $t_i \geq 3$. If there is no G_i that is a multilink, then $T = G_1 = G'$ and is a cycle and so G satisfies Part (3). So we assume that some G_i is a multilink.

Suppose that there is some $t_i = 5$ and root T at G_i. If T has height greater than zero, then G would contain a $G_{6,3}$- or $G_{6,4}$-minor, a contradiction. Thus $T = G_i$ and $G \cong 5K_2$, as required.

Next suppose that there is some $t_i = 4$ and root T at G_i. If T has height 2 or more, then G would contain a $G_{6,3}$-minor, a contradiction. Thus T has height at most 1; however, if G_i has a child in T, then G contains a $G_{6,4}$-minor, a contradiction. Thus $T = G_i$ and so G is a subgraph of $4K_2^\circ \backslash \ell$ which satisfies Part (2).

Lastly we assume that each multilink is a 3-multilink. Root T and some G_i that is a 3-multilink. The height of T is at most 2 because height 3 or more would imply that G has a $G_{6,2}$- or $G_{6,3}$-minor, a contradiction. If the height of T is zero then G is a subgraph of $3K_2^\circ$, which satisfies Part (3). If the height of T is one, then G is a subgraph of H° where H is a subdivision of $3K_2$, which satisfies part (3). So assume that the height of T is 2. If root G_i has two children in T, then G contains either $G_{6,1}$, $G_{6,2}$, or $G_{6,3}$ as a minor, a contradiction. So G_i has only one child in T. If G_i has only one grandchild, then because G does not contain a $G_{6,1}$- or $G_{6,2}$-minor, we must have that G is a subgraph of the graph in Figure 11, which satisfies Part (2). If G has exactly two grandchildren, then because G does not contains a $G_{6,1}$- or $G_{6,2}$-minor, we must have that G is a subgraph of $2K_3$, which satisfies Part (1). If G has three grandchildren, then G contains a $G_{6,1}$-minor, a contradiction. $\qquad\square$

5 Some comments on $GF(q)$-representability

Given a prime power $q \geq 7$, the same ideas used in the proof of Theorem 4 should be enough to find the analogous result for $GF(q)$ although the jump in complexity for even $q = 7$ may be significant. (Even finding the particular realization of Proposition 3 for $GF(7)$ may be quite challenging.) One particular example which hints at a jump in complexity is the following.

Note that if $B(G)$ is $GF(5)$-representable, then the canonical tree decomposition of G has at most one vertex labeled by a vertically 3-connected graph and if $B(G)$ is $GF(4)$-representable, then the canonical tree decomposition of G has no vertex labeled by a vertically 3-connected graph. It might seem reasonable to think that if $B(G)$ is $GF(q)$-representable, then there would be some finite bound on the number of vertices in the canonical tree decomposition of G that are labeled by vertically 3-connected graphs. This, however, is not the case. Consider the graph $G = C_n \oplus_2 K_4 \oplus_2 \cdots \oplus_2 K_4$ where there are n vertices labeled by K_4 and each such term is 2-summed onto its own edge of C_n. The canonical tree decomposition of G has n vertices labeled by vertically 3-connected graphs for n as large as we like. Now for any odd prime p and $p^e \geq 7$, $B(G)$ has a $GF(p^e)$-frame-matrix representation by Proposition 15 and the $\mathbb{Z}_{p^e-1}$-antivoltage shown in Figure 27.
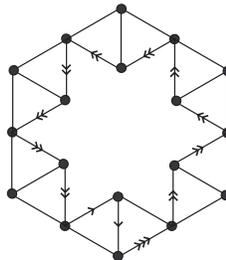

Figure 27.

On the other hand, a fact that would help curb the complexity of characterizing when $B(G)$ is $GF(q)$-representable is the following proposition which places a bound on the diameter of the tree decomposition of G when $B(G)$ is $GF(q)$-representable.

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Proposition 28. *Let G be a vertically 2-connected graph having canonical tree decomposition T. If a path in T has $n \geq 4$ vertices, then $B(G)$ is not $GF(q)$-representable for $q \leq \lfloor n/2 \rfloor$.*

Proof. At most $\lceil n/2 \rceil$ of the vertices in the path are labeled by cycles, so at least $\lfloor n/2 \rfloor$ of the vertices in the path are labeled by bonds of size at least 3 or vertically 3-connected graphs. Each of the vertically 3-connected graphs labeling a vertex in this path has a $3K_2$-minor containing the basepoints used in the 2-sums with the graphs labeling the adjacent vertices in the path in T. Therefore, there is a minor of G isomorphic to $B_1 \oplus_2 B_2 \oplus_2 \ldots \oplus_2 B_{\lfloor n/2 \rfloor}$, where each B_i is a bond of size at least three. Thus G has a minor isomorphic to $(\lfloor n/2 \rfloor + 2)K_2$, so $B(G)$ has a $U_{2,\lfloor n/2 \rfloor+2}$-minor, which is not $GF(q)$-representable. $\qquad \square$

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