# CLONES IN 3-CONNECTED FRAME MATROIDS

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Abstract. We determine the structure of clonal classes of 3-connected frame matroids in terms of the structure of biased graphs. Robbins has conjectured that a 3-connected non-uniform matroid with a clonal class of size $q-1$ is not $GF(q)$-representable. We confirm the conjecture for the class of frame matroids.

## 1. Introduction

We will assume that the reader is familiar with matroid theory as presented by Oxley in [4]. The graph-theory terminology used is mostly standard. A *frame matroid* is a matroid $M$ which has an extension $M_J$ satisfying the following.

- $r(M) = r(M_J)$
- $E(M_J) = E(M) \cup J$ and $J$ is a basis of $M_J$.
- For every nonloop $e \in E(M)$ that is not parallel to some $b \in J$, there is a unique $\{b_1, b_2\} \subseteq J$ such that $e \in \mathrm{cl}_{M_J}\{b_1, b_2\}$.

The elements of $J$ are usually referred to as *joints*. A fundamental example of frame matroids is the following. Given a field $\mathbb{F}$, an $\mathbb{F}$-*frame matrix* is an $\mathbb{F}$-matrix for which every column has at most two nonzero components. The vector matroid of a frame matrix is a frame matroid where the joints are the columns of an identity matrix prepended to the frame matrix. An $\mathbb{F}$-representable matroid $M$ that may be represented by a frame matrix over $\mathbb{F}$ we shall refer to as an $\mathbb{F}$-*frame matroid*. Not every frame matroid that is $\mathbb{F}$-representable is an $\mathbb{F}$-frame matroid, however.

Any frame matroid has a graphical structure called a *biased graph* that was first explored by Zaslavsky [13, 14]. *Gain graphs* (sometimes called *voltage graphs*, especially in the area of topological graph theory) are a specific type of biased graph that are especially useful for studying Dowling geometries and their minors.

The relative importance of frame matroids among all matroids was first displayed by Kahn and Kung [3]. They found that there are only two classes of matroid varieties that can contain 3-connected matroids: simple matroids representable over $GF(q)$ and Dowling geometries and their simple minors (which are frame matroids). More recently the matroid-minors project of Geelen, Gerards, and Whittle [2] has found the following far-reaching generalization of Seymour's decomposition theorem for regular matroids [8]. If $\mathcal{M}$ is a proper minor-closed class of the class of $GF(q)$-representable matroids, then any member of $\mathcal{M}$ of sufficiently high vertical connectivity is either a bounded-rank perturbation of a frame

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*Date*: November 20, 2015.

1991 *Mathematics Subject Classification.* 05B35.

*Key words and phrases.* frame matroids, biased graph, clones, 3-connected.

Slilaty is partially supported by a grant from the Simons Foundation #246380.

matroid, the dual of a bounded-rank perturbation of a frame matroid, or is representable over some subfield of $GF(q)$.

Two elements $e$ and $e'$ of a matroid $M$ are said to be *clones* when the bijection from $E(M)$ to $E(M)$ that exchanges $e$ and $e'$ and leaves all other elements fixed is an automorphism of $M$. Hence clone relation on the elements of $M$ is an equivalence relation whose equivalence classes are called *clonal classes*. Clones in matroids are important in studying inequivalent representations of matroids over finite fields [1] and, as mentioned in the previous paragraph, so are frame matroids. In this paper we will determine the structure of clonal classes of 3-connected frame matroids in terms of the structure of biased graphs (Theorems 2.5 and 2.6). Robbins (the first author listed in this paper) has conjectured in [5] that a 3-connected, non-uniform matroid $M$ with a clonal class of size $q-1$ is not $GF(q)$-representable. Robbins' conjecture is known to be true [5, 6] for $q \in \{2, 3, 4, 5, 7, 8\}$ and in this paper we will confirm the conjecture for the class of frame matroids (Theorem 2.7).

## 2. Preliminaries and statements of main results

Given a graph $G$, the set of edges of $G$ is denoted by $E(G)$. An edge with two distinct endpoints is called a *link* and an edges with only one endpoint is called a *loop*. Given $X \subseteq E(G)$, by $V(X)$ we mean the set of vertices in $G$ that are incident to edges in $X$ and by $G{:}X$ we mean the subgraph of $G$ with edges $X$ and vertices $V(X)$. A *k-separation* of $G$ is a bipartition $(X, Y)$ of the edges of $G$ for which $|X| \geq k$, $|Y| \geq k$, and $|V(X) \cap V(Y)| = k$. A connected graph $G$ is said to be *k-connected* when it has no $t$-separation for $t < k$. A connected graph $G$ is said to be *vertically k-connected* when either $G$ has at least $k+2$ vertices and there is no set of $t < k$ vertices whose removal leaves a disconnected subgraph or $G$ has $k+1$ vertices and contains a spanning complete subgraph.

We require the notion of an $H$-bridge in a graph. Let $H$ be a subgraph of $G$. Then an *$H$-bridge* is either an edge not in $H$ whose endpoints are both in $H$; or a connected component of $G \backslash V(H)$ along with the edges that link that component to $H$.

A *$\Theta$-subgraph* of a graph $G$ is a 2-connected subgraph with exactly two vertices of degree 3 and all other vertices having degree 2. Refer to Figure 1 for an illustration. A *biased graph* is pair $(G, B)$ where $G$ is a graph and $B$ is a collection of cycles of $G$ such that every $\Theta$-subgraph of $G$ has 0, 1, or 3 cycles in $B$. Cycles in $B$ are called *balanced cycles*. The biased graph $(G, B)$ is said to be *balanced* when every cycle is balanced; otherwise $(G, B)$ is said to be *unbalanced*. When a biased graph has no balanced cycles it is said to be *contrabalanced*. A *subgraph* of $(G, B)$ is a pair $(H, B')$ for which $H$ is a subgraph of $G$ and $B'$ is $B$ restricted to the cycles of $H$. Sometimes we refer to such a subgraph as the subgraph of $(G, B)$ on $H$. A *balancing set* of $(G, B)$ is a collection of edges of $(G, B)$ whose removal leaves a balanced subgraph.

The *frame matroid* $M(G, B)$ is the matroid with ground set $E(G)$ and a circuit is the edge set of either a balanced cycle, a contrabalanced *handcuff* or a contrabalanced $\Theta$-*subgraph* (see Figure 1). Graphic matroids, signed-graphic matroids, and bicircular matroids are all special cases of frame matroids: for graphic matroids, all cycles are balanced; in a signed-graphic matroid, every $\Theta$-graph contains either 1 or 3 balanced cycles; and in a bicircular matroid, every $\Theta$-graph contains 0 balanced cycles. Given a subset $X \subseteq E(G)$, the rank of $X$ in $M(G, B)$, denoted $r(X)$, is $|V(X)| - b_X$ in which $b_X$ is the number of connected components of $G{:}X$ that are balanced.

Given a biased $(G,B)$, if we add an unbalanced loop at each vertex of $G$ and denote the new biased graph by $(\bar{G},B)$. Let $L=\{l_v|v\in V(\bar{G})$ and $l_v$ is an unbalanced loop at v$\}$. Then the set $L$ can be regarded as the set of joints in the frame matroid $M(\bar{G},B)$; as every link (i.e., an edge that is not a loop) is spanned by the two unbalanced loops at its two end vertices.

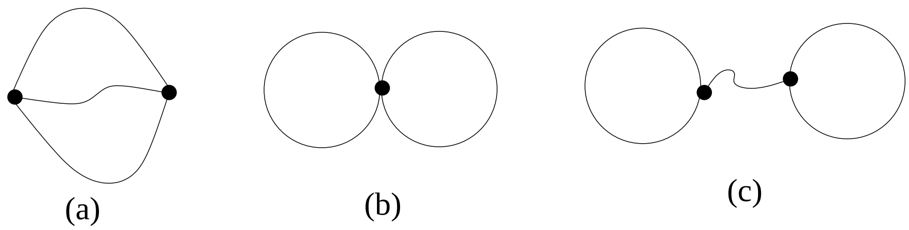


FIGURE 1. A $\Theta$-graph is a subdivision of the graph (a), a *tight handcuff* is a subdivision of the graph (b), and a *loose handcuff* is a subdivision of the graph (c).

We will characterize clonal classes in 3-connected frame matroids. We use $A\triangle B$ to denote the symmetric difference of two sets $A$ and $B$. Proposition 2.1 follows immediately from the definition of clones.

**Proposition 2.1.** *Two elements $e$ and $f$ in a matroid $M$ form a clonal pair if and only if for each circuit $C$ of $M$ with $|C\cap\{e,f\}|=1$ we have that $C\triangle\{e,f\}$ is also a circuit.*

Using the form of the rank function for $M(G,B)$ we immediately get Proposition 2.2.

**Proposition 2.2.** *If $(G,B)$ is connected and unbalanced, then an edge $e$ in $(G,B)$ is a coloop of $M(G,B)$ iff either*

- *$e$ is an isthmus and the subgraph on $G\backslash e$ consists of one balanced connected component and one unbalanced connected component or*
- *$e$ is a balancing edge of $(G,B)$.*

Our analysis of biased graphs requires characterizing 3-connectivity of frame matroids in terms of the structure of the biased graphs. Given a biased graph $(G,B)$ and a $k$-separation $(X,Y)$ of $G$, we call the subgraphs of $(G,B)$ on $G{:}X$ and $G{:}Y$, the *sides* of the separation.

**Proposition 2.3** (Qin, Slilaty [10, Theorem 1.4])**.** *If $(G,B)$ is a connected and unbalanced biased graph with at least three vertices, then $M(G,B)$ is 3-connected iff $(G,B)$ is vertically 2-connected, every 2-separation has both sides unbalanced, every 3-separation has at least one side unbalanced, there are no balanced cycles of length 1 or 2, there are no two loops incident to the same vertex, and there is no balancing set of 1 or 2 edges.*

A vertex $u$ in a biased graph $(G,B)$ is called a *balancing vertex* if the subgraph on $G\backslash u$ is balanced.

**Proposition 2.4** (Zaslavsky [12])**.** *Let $(G,B)$ be an unbalanced biased graph where $G$ is connected and let $X\subseteq V(G)$ be the collection of balancing vertices of $(G,B)$.*

- *If $|X|\geq 3$, then $(G,B)$ has the structure shown on the right in Figure 2 where the vertices shown are the vertices of $X$ and the shaded regions are balanced subgraphs.*

*Furthermore, a cycle of $(G, B)$ is unbalanced iff it uses edges from more than one shaded region.*

- *If $|X| = 2$, then $(G, B)$ has the structure shown on the left in Figure 2 where the vertices shown are the vertices of $X$ and the shaded regions are balanced subgraphs. Furthermore, a cycle of $(G, B)$ is unbalanced iff it uses edges from more than one shaded region.*

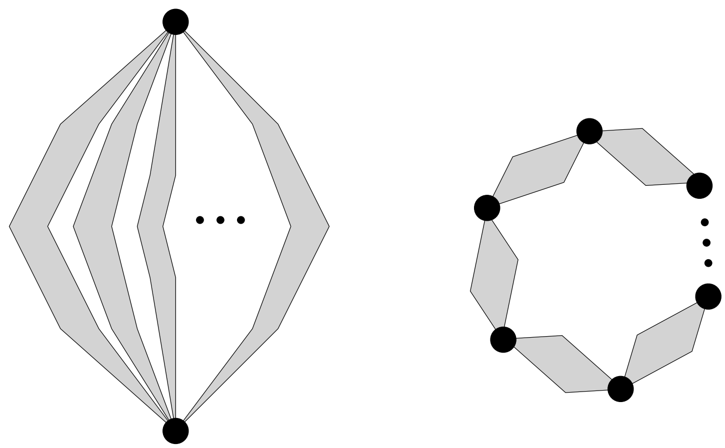

FIGURE 2. Structure of biased graphs with more than one balancing vertex.

Theorem 2.5 is the first result of this paper.

**Theorem 2.5.** *Let $(G, B)$ be a biased graph such that $M(G, B)$ is 3-connected. If $e$ and $f$ are clones in $M(G, B)$, then one of the following holds.*

- *$e$ and $f$ are parallel links in $G$ and every cycle $C$ of $G$ with $|C \cap \{e, f\}| = 1$ is unbalanced;*
- *$e$ and $f$ are loops at two distinct vertices in $G$ and $M(G, B)$ is isomorphic to $U_{2,n}$;*
- *$e$ is a loop and $f$ is a link in $G$ and either $M(G, B)$ is isomorphic to $U_{3,5}$, $M(G, B)$ is isomorphic to $U_{2,n}$, or $(G, B)$ has the first structure shown in Figure 3.*
- *$e$ and $f$ are both links in $G$ and either $M(G, B)$ is isomorphic to $U_{4,6}$, $(G, B)$ has the second or third structure shown in Figure 3, or $(G, B)$ has 3 or 4 vertices and is one of the biased graphs from Figure 4.*

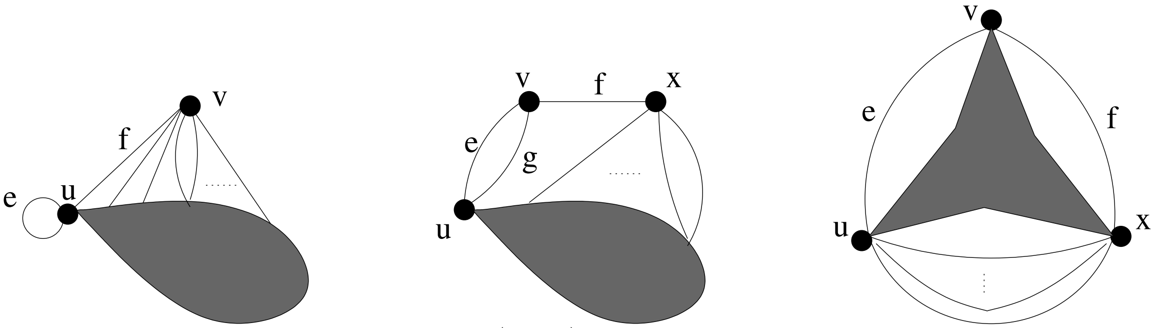


FIGURE 3. Three structures for $(G, B)$ supporting a clonal pair $e, f$ where, in each, the shaded region represents a balanced subgraph. In the first structure, there may be links parallel to $f$ and every cycle containing $f$ is unbalanced. In the second structure, the degree of $v$ is exactly three, every cycle containing $f$ but not containing $e$ is unbalanced, and $(G\backslash v, B')$ is unbalanced; moreover, if there is no balanced cycle containing both $e$ and $f$, then $e$, $f$, and $g$ will form a clone triple. In the third structure, any cycle not contained completely within the shaded region is unbalanced.

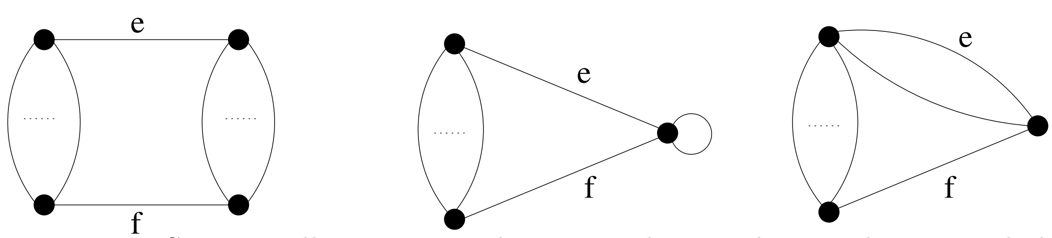


FIGURE 4. Some small structures where every loop and every digon is unbalanced; and in the third structure, every triangle containing $f$ but not containing $e$ is unbalanced.

As corollaries to Theorem 2.5 we have Theorems 2.6 and 2.7. The latter is conjectured by Robbins in [5] to hold for all 3-connected matroids and not just the class of 3-connected frame matroids. A clonal class $X$ in $M(G, B)$ is said to be a *canonical clonal class* when either $X$ is a collection of parallel edges in $(G, B)$ for which no edge is in a balanced cycle or $X\backslash e$ is such a collection of parallel edges with $e$ being an unbalanced loop incident to the edges in $X\backslash e$. The next theorem follows immediately from Theorem 2.5.

**Theorem 2.6.** *Let $(G, B)$ be a biased graph with at least three vertices such that $M(G, B)$ is 3-connected and $M(G, B) \ncong U_{3,5}$, $U_{3,6}$, or $U_{4,6}$ and let $X$ be a clonal class of elements in $M(G, B)$ with $|X| \geq 3$. One of the following holds.*

- *$X$ is a canonical clonal class.*
- *$|X| = 4$, $G$ has four vertices, and $(G, B)$ has the structure shown in Figure 5.*
- *$|X| = 3$ and $(G, B)$ has one of the structures shown in Figure 6.*

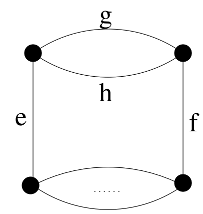


FIGURE 5. A frame matroid with a clonal class of size 4, where $\{g, h\}$ is a parallel class of size exactly two and the graph is contrabalanced.

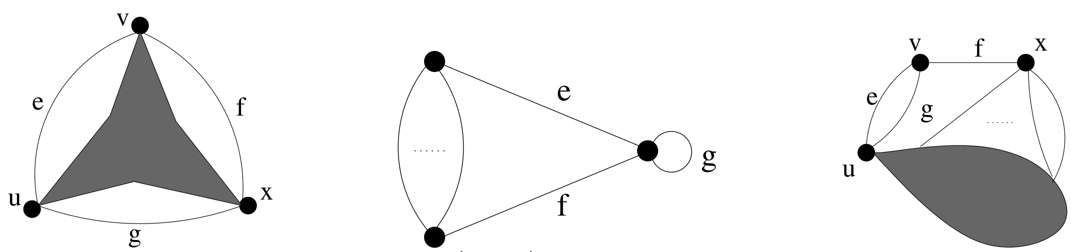


FIGURE 6. Three structures for $(G, B)$ supporting a clonal triple, where, each shaded region represents a balanced subgraph. In the first structure, any cycle not contained completely within the shaded region is unbalanced. In the second structure, the biased graph is contrabalanced. In the third structure, the degree of $v$ is exactly three, $(G\backslash v, B')$ is unbalanced, and there is no balanced cycle containing the edge $f$.

**Theorem 2.7.** *If $M(G, B)$ is 3-connected, not a uniform matroid, and contains a clonal class of size $q - 1 \geq 2$, then $M(G, B)$ is not $GF(q)$-representable.*

*Proof.* The result is known for $q \in \{3, 4, 5\}$ for general matroids and not just frame matroids [5]. So assuming that $q \geq 7$, we get that any clonal class in $M(G, B)$ of size at least $q - 1$ is a canonical clonal class by Theorem 2.6. If $X$ is a canonical clonal class in $M(G, B)$ is size at least $q - 1$, then the fact that $M(G, B)$ is 3-connected and non-uniform forces $G$ to have at least three vertices. One can check that $(G, B)$ now contains a minor from Figure 7 in which each biased graph shown is contrabalanced.

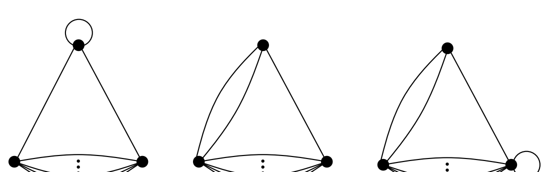

FIGURE 7. The clonal class $X$ is the collection of edges on the bottom two vertices of each configuration.

These three biased graphs have frame matroids that are obtained by applying a $\triangle Y$ operation to $U_{2,l}$ for $l \geq q + 2$ and this matroid is not $GF(q)$-representable. $\square$

## 3. Proof of Theorem 2.5

First suppose $e$ and $f$ are parallel links. If $\{e, f\}$ is not a balanced cycle, then $e$ and $f$ are clones iff every cycle $C$ with $|C \cap \{e, f\}| = 1$ is unbalanced. For the rest of the section, we will assume that $e$ and $f$ are not parallel links.

Suppose that $e$ and $f$ are both loops not at the same vertex and neither is balanced. Assume $e$ is incident with $u$ and $f$ is incident with $v$. Now if $e$ and $f$ are clones, then both $G\backslash u\backslash f$ and $G\backslash v\backslash e$ are balanced. Therefore, both $u$ and $v$ are balancing vertices of $G\backslash\{e, f\}$. So $(G\backslash\{e, f\}, B)$ has one of the structures of Figure 2. Since $M(G, B)$ is 3-connected, each shaded region must be a single link, and hence, $M(G, B)$ must be isomorphic to $U_{2,n}$.

Next assume that $e$ is a loop and $f$ is a link. If $e$ and $f$ are clones, $e$ must be unbalanced and, furthermore, no balanced cycle of $(G, B)$ contains $f$. Suppose that $e$ is incident to $u$ and $f$ is incident to $v$ and $w$. We may assume that $u \neq v$. Since $M(G, B)$ is 3-connected, $G\backslash v$ is a connected graph, $G\backslash v$ must be balanced after removing the loop $e$ as otherwise there would be a handcuff in $G\backslash v$ containing the loop $e$, contrary to the fact that $e$ and $f$ are clones.

If $G\backslash v$ has exactly one vertex, then $M(G, B)$ is isomorphic to $U_{2,n}$. Assume that $|V(G\backslash v)| \geq 2$. Consider the following two cases.

*Case 1:* $u \neq w$.

Note that $G\backslash e$ is vertically 2-connected. If $G\backslash e$ has an unbalanced cycle not using $v$ or $w$, then by the 2-connectivity and the fact that every cycle containing $f$ is unbalanced, we can construct a $\Theta$-subgraph containing $f$ in $G\backslash e$, say $\Theta_1$. Since $u \notin \{v, w\}$ by the assumption, $\Theta_1\backslash\{f\} \cup \{e\}$ is not a circuit in $M(G, B)$, contrary to the fact that $e$ and $f$ are clones. Therefore, both $v$ and $w$ are balancing vertices of $G\backslash e$. So $G\backslash e$ has one of the structures of Figure 2. Moreover, it follows from the 3-connectivity of $M(G, B)$ that each $\{v, w\}$-bridge not containing the vertex $u$ must be a single link. There are at most two such links including $f$; as otherwise, we can find a $\Theta$-graph, say $\Theta_2$, containing $f$ and two other links parallel to $f$. Clearly $\Theta_2\backslash\{f\} \cup \{e\}$ is not a circuit of $M(G, B)$, a contradiction. Since $M(G, B)$ is 3-connected, $\{e, f\}$ is not a series pair, and hence, $G\backslash\{e, f\}$ is unbalanced. So there must be exactly two links between $v$ and $w$ including $f$. Now in the $\{v, w\}$-bridge that contains $u$, $u$ must be a cut vertex. It follows from the 3-connectivity that $G$ has exactly three vertices $u$, $v$ and $w$ and $G$ consists of a pair of parallel links between $v$ and $w$, a loop at $u$, a link between $u$ and $v$, and a link between $u$ and $w$. Since $M(G, B)$ is 3-connected, $(G, B)$ contains no balanced cycles, and hence, $M(G, B)$ is isomorphic to $U_{3,5}$.

*Case 2:* $u = w$.

Now $G\backslash v\backslash e$ must be balanced and $f$ is not in any balanced cycle and $G$ has the first structure shown in Figure 3.

Finally assume both $e$ and $f$ are links. Suppose that $e$ is incident with $u$ and $v$ , and $f$ is incident with $w$ and $x$. Since $e$ and $f$ are not parallel to each other, we may assume that $u \notin \{w, x\}$. Since $M(G, B)$ is 3-connected, $G$ is vertically 2-connected, and hence, $G\backslash u$ is connected. Since $e$ and $f$ are clones, $f$ is a coloop in $M(G\backslash u, B')$. Then either $f$ is an isthmus or $f$ is a balancing edge of $(G\backslash u, B')$. (See Proposition 2.2.)

*Case 1:* $f$ is an isthmus of $G\backslash u$.

Let $G_1$ and $G_2$ be the two components of $G\backslash u\backslash f$. Assume that $w \in G_1$ and $x \in G_2$. Furthermore and without loss of generality we may assume that $G_1$ is balanced. Consider the following two subcases:

*Subcase 1.1:* $G_1$ has more than one vertex.

In this case first assume $v \in G_1$. Since there is a 2-separation of $G$ at $\{u, w\}$, the 3-connectivity of $M(G, B)$ implies that the $\{u, w\}$-bridge $G_1'$ that contains $G_1$ is unbalanced. Hence $u$ is a balancing vertex of $G_1'$ and so there is an unbalanced cycle $C_e$ in $G_1'$ that contains $e$.

Similarly the $\{u, x\}$-bridge $G_2'$ that contains $G_2$ is unbalanced and there is an unbalanced cycle $C_2$ in $G_2'$. Therefore there exists a handcuff $H_e$ in $G$ that contains $C_e \cup C_2$ and does not use the edge $f$. It follows that $v = w$ since $v$ has degree-1 in $H_e\backslash\{e\}$ and has degree at least 2 in $H_e \triangle \{e, f\}$.

Now since $M(G, B)$ is 3-connected, there exists an unbalanced cycle $C_u$ in $G_1'$ not containing $e$. Hence we can find a $\Theta$-subgraph $\Theta_1$ containing $e$ and $C_u$. Note that since $e$ and $f$ are clones and they are not parallel, every cycle containing $e$ and not containing $f$ is unbalanced. So $\Theta_1$ is a circuit of $M(G, B)$. Now clearly $\Theta_1 \triangle \{e, f\}$ is not a circuit of $M(G, B)$, contrary to the fact that $e$ and $f$ are clones.

Therefore we must have that $v \in G_2$. Again in the $\{u, x\}$-bridge $G_2'$ that contains $G_2$, there exists an unbalanced cycle $C_1$; and in the $\{u, w\}$-bridge $G_1'$ that contains $G_1$, there exists an unbalanced cycle $C_u$ containing the vertex $u$. Clearly there exists a handcuff $H_e$ containing $C_1 \cup C_u \cup \{e\}$. Since $e$ and $f$ are clones, the cycle $C_u$ must use the vertex $w$. Therefore, both $u$ and $w$ are balancing vertices for $G_1'$. By Proposition 2.4, $G_1'$ has one of the structures shown in Figure 2. So since $G_1'$ has at least three vertices, $(G, B)$ has a 2-separation with one side balanced, contradicting the 3-connectivity of $M(G, B)$.

*Subcase 1.2:* $G_1$ has one vertex.

In this case, $w$ is the only vertex of $G_1$. If $v = w$, then by the 3-connectivity of $M(G, B)$ and the fact that $e$ and $f$ are clones, there exists a unique link $g$ parallel to $e$ and the digon $\{e, g\}$ is unbalanced. Since $e$ and $f$ are clones, every cycle containing $f$ and $g$ must be unbalanced. Furthermore, in the $\{u, x\}$-bridge $G_2'$ that contains $G_2$, $x$ must be a balancing vertex: for otherwise there exists a unbalanced cycle $C_4$ in $G_2'$ not using $x$ and we can construct a handcuff $H_e$ using $C_4$ and the digon $\{e, g\}$, and not using $x$. Clearly $H_e \triangle \{e, f\}$ is not a circuit of $M(G, B)$, a contradiction. Therefore $(G, B)$ has the second structure shown in Figure 3.

If $v \in G_2$, then there exists a pair of links $g$ and $h$ between $u$ and $w$ and the digon $\{g, h\}$ must be unbalanced. Now consider the $\{u, x\}$-bridge $G_2'$ that contains $G_2$. First assume there exists an unbalanced cycle $C$ not using $v$. If $u \notin V(C)$, then we can find a handcuff $H_f$ containing $C$ and $\{f, g, h\}$ and not using $v$. Clearly $H_f \triangle \{e, f\}$ is not a circuit of $M(G, B)$, a contradiction. If $u \in V(C)$, then we can find a $\Theta$-graph $\Theta_e$ containing $C$ and the link $e$. Again $\Theta_e \triangle \{e, f\}$ is not a circuit of $M(G, B)$; a contradiction. Therefore $v$ is a balancing vertex of $G_2'$. Similarly we can show $x$ is a balancing vertex of $G_2'$. Again by 3-connectivity and Proposition 2.4 that $G_2'$ has either two or three vertices. Therefore $G$ has three or four vertices. A routine case analysis shows $G$ must be one of graphs shown in Figure 4. Note that in the third graph, every cycle containing $f$ but not containing $e$ must be unbalanced.

*Case 2:* $f$ is a balancing edge of $(G\backslash u, B')$.

Note that we may further assume that $e$ is a balancing edge of $(G\backslash w, B')$. Therefore, every unbalanced cycle of $G$ either uses the edge $f$ or uses the vertex $u$; similarly, every unbalanced cycle of $G$ either uses the edge $e$ or the vertex $w$. So in the graph $G\backslash\{e, f\}$, both $u$ and $w$ are balancing vertices. So $G\backslash\{e, f\}$ has one of the structures from Figure 2.

If $e$ and $f$ lie in two different $\{u, w\}$-bridges, then $G\backslash\{e, f\}$ must have exactly two $\{u, w\}$-bridges; one contains $e$ and the other contains $f$. (As otherwise, we can find a contrabalanced $\Theta$-subgraph containing $e$ only; and removing $e$ and adding $f$ does not produce a $\Theta$-subgraph or a handcuff, contrary to the fact $e$ and $f$ are clones.) Similarly we can find a contrabalanced $\Theta$-subgraph $C_e$ containing $e$ such that $C_e \triangle \{e, f\}$ is not a circuit if $x$ is not a cut vertex of the $\{u, w\}$-bridge containing $f$. Thus $x$ is a cut vertex of $\{u, w\}$-bridge containing $f$ and also $v$ is a cut vertex of $\{u, w\}$-bridge containing $e$. It follows from the 3-connectivity of $M(B, G)$ that $G$ has exactly four vertices and six edges and $M(G, B)$ is isomorphic to $U_{4,6}$.

If $e$ and $f$ lie in the same $\{u, w\}$-bridge, say $G_1$, then by the 3-connectivity, all other $\{u, w\}$-bridges contain a single link. Note that if $v = x$, then $(G, B)$ has the third structure shown in Figure 3. So now assume that $v \neq x$. By Case 1 and symmetry, we may assume that $e$ is a balancing edge of $G\backslash x$ and $f$ is a balancing edge of $G\backslash v$. It follows that $G\backslash\{e, f\}$ has exactly two $\{u, w\}$-bridges: one that contains the vertices $v$ and $x$, and the other consists of a single link, say $g$, connecting $u$ and $w$. Moreover, $\{e, f, g\}$ is a balancing set for $(G, B)$.

Since $M(G, B)$ is 3-connected, $G\backslash\{e, f\}$ is not balanced, and hence contains an unbalanced cycle $C_g$. Clearly $C_g$ contains the link $g$ and a path from $u$ to $w$. If $v$ is not on the cycle $C_g$, then we can find a path from $v$ to the cycle $C_g$, thus construct a contrabalanced $\Theta$-subgraph $\Theta_e$ containing $e$. Since $v \neq x$, $v$ has degree 1 in $\Theta_e \triangle \{e, f\}$, contrary to the fact that $e$ and $f$ are clones. By symmetry, $C_g$ must also contain the vertex $x$. Therefore, in $G\backslash g$, every path from $u$ to $w$ must contain both $v$ and $x$.

Consider the $\{u, w\}$-path $C_g\backslash g$ and let $u$ be the starting vertex. Then either the vertex $v$ comes before $x$ or it comes after the vertex $x$. In the former case, since $M(G, B)$ is 3-connected, the path $C_g\backslash g$ consists of exactly three links, from $u$ to $v$, from $v$ to $x$, and from $x$ to $w$, and $G$ has exactly four vertices and six edges, and $G$ is the graph obtained from a pair of disjoint digons connected by a matching. It is routine to check that $M(G, B)$ is isomorphic to $U_{4,6}$. In the latter case, it follows from the 3-connectivity of $M(G, B)$ again that $G$ is isomorphic to $K_4$ and $M(G, B)$ is isomorphic to $U_{4,6}$.

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