# Unavoidable minors of large 4-connected bicircular matroids

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**Abstract**

It is known that any 3-connected matroid that is large enough is certain to contain a minor of a given size belonging one of a few special classes of matroids. This paper proves a similar unavoidable minor result for large 4-connected bicircular matroids. The main result follows from establishing the list of unavoidable minors of large 4-biconnected graphs, which are the graphs representing the 4-connected bicircular matroids. This paper also gives similar results for internally 4-connected and vertically 4-connected bicircular matroids.

*Key words:* bicircular matroid, 4-connected, internally 4-connected, vertically 4-connected, unavoidable minor

Dedicated to Dr. James G. Oxley on the occasion of his 60th birthday

## 1 Introduction

Our notation and terminology will generally follow [5]. The following result of Ding, Oporowski, Oxley, and Vertigan, from [2], shows that each sufficiently large 3-connected matroid is guaranteed to contain a large minor isomorphic to one of a few types of 3-connected matroids.

**Theorem 1.1** *For every integer $n$ exceeding two, there is an integer $N(n)$ such that every 3-connected matroid with at least $N(n)$ elements has a minor*

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*isomorphic to one of $U_{n,n+2}$, $U_{2,n+2}$, $M(K_{3,n})$, $M^*(K_{3,n})$, $M(\mathcal{W}_n)$, $\mathcal{W}^n$, or a uniform $n$-spike.*

Evidently, corollaries for various minor-closed classes of matroids follow by filtering out the members of the list in Theorem 1.1 that are not in the class of interest. For instance, we may choose to restrict to graphic matroids.

**Corollary 1.2** *For every integer $n$ exceeding two, there is an integer $N(n)$ such that every simple, 3-connected graph having at least $N(n)$ edges has a minor isomorphic to one of $K_{3,n}$ or $\mathcal{W}_n$.*

The following result of Oporowski, Oxley, and Thomas, from [4], is a stronger version of Corollary 1.2. Refer to Figure 1 for an illustration of $V_k$, which can be formed by contracting a pair of consecutive rungs of the circular $k$-ladder and simplifying the resulting graph.

**Theorem 1.3** *For every integer $k \geq 3$, there is an integer $N$ such that every 3-connected graph with at least $N$ vertices contains a subgraph isomorphic to a subdivision of one of $\mathcal{W}_k$, $V_k$, and $K_{3,k}$.*

The focus of this paper is an unavoidable minor result for bicircular matroids. As noted above, a result of this type for 3-connected bicircular matroids is merely a corollary of Theorem 1.1. However, a 4-connected analog of Theorem 1.1 is not known. The following theorem is the main result of this paper. Here, $\mathcal{W}_n^2$ can be constructed from the $n$-spoked wheel by adding an edge in parallel to each spoke. The graph $K_{3,n}^+$ is formed by adding a loop at each of the $n$ degree-3 vertices of $K_{3,n}$. Finally, $K_{3,n}^2$ is constructed from $K_{3,n}$ by adding an edge in parallel to each of the edges incident with a single degree-$n$ vertex.

**Theorem 1.4** *For every integer $n$ exceeding four, there is an integer $N(n)$ such that every 4-connected bicircular matroid with at least $N(n)$ elements has a minor isomorphic to one of $B(\mathcal{W}_n^2)$, $B(K_{3,n}^+)$, or $B(K_{3,n}^2)$.*

The proof of this result makes use of a type of graph connectivity called *biconnectivity*. Section 2 provides an equivalent characterization of $n$-biconnectivity that is used in Section 4 to prove Theorem 1.4.

In Section 3 we analyze the graphic structure of size-$n$ cocircuits in $n$-connected bicircular matroids. This is used in Section 5 to prove the following internally 4-connected analog of Theorem 1.4.

**Theorem 1.5** *For every integer $n$ exceeding four, there is an integer $N'(n)$ such that every internally 4-connected bicircular matroid with at least $N'(n)$ elements has a minor isomorphic to $B(\mathcal{W}_n)$ or $B(K_{3,n})$.*

Finally, we prove a vertically 4-connected version of the main result in Section 6. Recall that, by definition, a vertically 4-connected may not be 3-connected. For simplicity, we assume in the next result the matroids under consideration are 3-connected.

**Theorem 1.6** *For each integer $n$ exceeding four, there is an integer $N''(n)$ such that every vertically 4-connected and 3-connected bicircular matroid on at least $N''(n)$ elements has a restriction isomorphic to $U_{2,n}$, or a minor isomorphic to one of $B(\mathcal{W}_n^2)$, $B(K_{3,n}^+)$, or $B(K_{3,n}^2)$.*

## 2 Preliminaries

Let $G$ be a graph. The bicircular matroid of $G$, denoted by $B(G)$, is the matroid with ground set $E(G)$, and a subset of $E(G)$ is a circuit if it is the edge set of a minimal connected subgraph of $G$ that contains at least two cycles. A subgraph of $G$ is called a $\Theta$-*graph* if it consists of two distinct vertices and three internally disjoint paths connecting them; a subgraph is called a *tight handcuff* if it consists of two cycles having just one vertex in common; and a subgraph is called a *loose handcuff* if it consists of two disjoint cycles and a minimal connecting path. It is easy to see that a circuit of $B(G)$ is either a $\Theta$-graph, a tight handcuff, or a loose handcuff, shown in Figure 3. A subgraph of $G$ is called a *bicycle* if it is a $\Theta$-graph, a tight handcuff, or a loose handcuff.

Wagner defines $n$-biconnectivity in [7] with respect to $k$-biseparations as follows.

Let $(E_1, E_2)$ partition the edge set $E$ of a connected graph $G = (V, E)$. For $i \in \{1, 2\}$, let $G_i$ denote the subgraph of $G$ induced by $E_i$. We say $(E_1, E_2)$ is a $k$-*biseparation* of $G$, for $k \geq 1$, if each of $|E_1|$ and $|E_2|$ is at least $k$, and

$$|V(G_1) \cap V(G_2)| = \begin{cases} k-1 & \text{if neither } G_1 \text{ nor } G_2 \text{ is acyclic} \\ k & \text{if exactly one or all three of } G_1,\, G_2, \text{ and } G \text{ are acyclic} \\ k+1 & \text{if both } G_1 \text{ and } G_2 \text{ are acyclic, but } G \text{ is not acyclic} \end{cases}$$

For $n$ a positive integer, a graph is $n$-*biconnected* if it has no $k$-biseparation for $k < n$.

The next theorem of Wagner from [7] shows that biconnectivity is the version of graphic connectivity corresponding to matroid connectivity in bicircular matroids.

**Theorem 2.1** *Let $G$ be a connected graph. Then $B(G)$ is $n$-connected if and only if $G$ is $n$-biconnected.*

Here we give an equivalent characterization for $n$-biconnectivity.

**Lemma 2.2** *For $n \geq 3$, a graph $G$ on at least $n$ vertices and at least $2n-2$ edges is $n$-biconnected if and only if each of the following holds:*

*(1) $G$ has no vertex cut of size at most $n-2$.*
*(2) $\delta(G)$, the minimum degree of $G$, is at least $n$*
*(3) $G$ has no bicycle of size at most $n-1$*

*Proof.*

Equivalence holds for $n = 3$ by Wagner in [7]. Suppose that $G = (V, E)$ is $n$-biconnected for a fixed $n > 3$ and that the lemma holds for smaller values of $n$. Since $G$ is $(n-1)$-biconnected, $\delta(G) \geq n-1$, and $G$ has no vertex cut of size less than $n-2$. Suppose $G$ has a vertex cut $W$ of size $n-2$. Let $H$ be a component of $G - W$. Let $E_1$ denote the edges of $G$ having at least one end in $V(H)$. Let $E_W$ denote the edges of $G$ having both ends in $W$. Let $E_2 = E - E_1 \cup E_W$. By $\delta(G) \geq n-1$ and the minimality of the vertex cut $W$, we have that each of $|E_1|$ and $|E_2|$ is at least $n-1$. Since $G$ is $n$-biconnected, we have that $(E_i, E_j \cup E_W)$ is not an $(n-1)$-biseparation for $(i,j) \in \{(1,2),(2,1)\}$. Up to relabeling, we have that the subgraph $G_1$ of $G$ induced by $E_1$ is acyclic. Since $\delta(G) \geq n-1$ we have that a leaf vertex in $G_1 - W$ must be adjacent to all $n-2$ vertices of $W$. By acyclicity, there can be no such vertex. This contradicts that $W$ is a vertex cut.

Suppose $G$ has a vertex $v$ of degree $n-1$. Since $G$ has no vertex cut of size at most $n-2$, the subgraph induced by the edges incident with $v$ is acyclic. Thus $G - v$ is acyclic since $G$ has no $(n-1)$-biseparation. Each leaf vertex of $G - v - N(v)$ is adjacent to at least $\delta(G) - 1 \geq n-2$ members of $N(v)$, where $N(v)$ denotes the set of neighbors of $v$. Since $G - v - N(v)$ is acyclic, each connected component of $G - v - N(v)$ consists of exactly one vertex. Since $\delta(G) \geq n-1$, every such vertex must be adjacent to all vertices of $N(v)$. Therefore, $G - v - N(v)$ consists of exactly one vertex of degree $n-1$, so $G$ is isomorphic to $K_{2,n-1}$, a contradiction to $\delta(G) > 2$.

By the inductive assumption, $G$ has no bicycle of size less than $n-1$. Suppose $G$ has a bicycle of size $n-1$ with edge set $E_1$. Let $E_2 = E - E_1$. Then $|E_2| \geq 2n-2-(n-1) = n-1$, and $|V(G_1) \cap V(G_2)| = |V(G_1)| = n-2$. Since $G$ has no $(n-1)$-biseparation, $G_2$ must be acyclic. However, $G_2$ has at least $n-(n-2) = 2$ vertices and therefore at least two leaf vertices; every such leaf vertex is adjacent to all members of $V(G_1)$, a contradiction to acyclicity.

Now suppose $G = (V, E)$ is a graph satisfying the three conditions in the statement of the lemma for some $n > 3$ and that the equivalence holds for smaller values of $n$. By assumption, $G$ has no $k$-biseparation for $k < n-1$. Suppose $G_1$ and $G_2$ are induced by an $(n-1)$-biseparation $(E_1, E_2)$. First, suppose that $|V(G_1) \cap V(G_2)| = n-2$. Since $G$ has no size-$(n-2)$ cutset, at least one of $V(G_1)-V(G_2)$ and $V(G_2)-V(G_1)$ is empty – assume the former. Then $|E_1| \geq n-1$ and $|V_1| = n-2$, so $G_1$ contains a bicycle of size at most $n-1$, a contradiction. Hence $|V(G_1) \cap V(G_2)| \geq n-1$.

Next suppose that $|V(G_1) \cap V(G_2)| = n-1$. The graph $G$ is not acyclic by assumption, so we may assume $G_1$ is acyclic. Since $|E_1| \geq n-1$ and $|V(G_1) \cap V(G_2)| = n-1$, it follows that $V(G_1) - V(G_2) \neq \emptyset$. Since $\delta(G) \geq n$, a leaf vertex of $V(G_1) - V(G_2)$ is adjacent to all vertices of $V(G_1) \cap V(G_2)$. As $G_1$ is acyclic, there is only one such vertex. This contradicts the fact that $\delta(G) \geq n$.

Therefore, we may assume that $|V(G_1) \cap V(G_2)| = n$, so both $G_1$ and $G_2$ are acyclic. First we show that one of $V(G_1) - V(G_2)$ and $V(G_2) - V(G_1)$ is empty. Suppose that neither $V(G_1) - V(G_2)$ nor $V(G_2) - V(G_1)$ is empty. Since each of $G_1$ and $G_2$ is acyclic and $\delta(G) \geq n$, each of $V(G_1) - V(G_2)$ and $V(G_2) - V(G_1)$ must have only one vertex by the pigeonhole principle. So $G$ is isomorphic to $K_{2,n}$, a contradiction.

Therefore we may assume that $V(G_1) - V(G_2) = \emptyset$. Then $|E_1| = n-1$. Thus $|V(G_2) - V(G_1)| \in \{0, 1\}$. If $V(G_2) - V(G_1) \neq \emptyset$ then a leaf of $G_1$ has degree 2 in $G$, a contradiction. Therefore $V(G_2) - V(G_1) = \emptyset$. Hence $G$ is a graph on $2n-2$ edges and $n$ vertices. The sum of the degrees of vertices of $G$ is at least $n\delta(G) \geq 4n$. However, $2|E| = 4n-4$, a contradiction. Thus, $G$ has no $(n-1)$-biseparation, so $G$ is $n$-biconnected. □

## 3 The graphic structure of small cocircuits in $n$-connected bicircular matroids

The following from [3] is Matthews's description of a hyperplane of $B(G)$ in the underlying graph $G$, which we assume to be connected and containing a bicycle. A hyperplane $H$ is a collection of edges of $G$ such that the subgraph with vertex set $V(G)$ and edge set $H$ consists of

(1) exactly one acyclic component $H_0$, which may be an isolated vertex; and
(2) a collection of other components, each of which is cyclic;

such that all edges of $E(G) \backslash H$ have at least one endpoint in $H_0$.

Evidently, a cocircuit of $B(G)$ is a minimal set of edges $X$ such that $G - X$ has exactly one acyclic component. In general, the edges of a cocircuit need not form a bond in $G$ as they would in the case of graphic matroids. The results below describe small cocircuits in the underlying graphs of $n$-connected bicircular matroids. Before exploring this graphic structure, we consider the following trivial consequence of the minimum degree condition in Lemma 2.2 that will be used frequently in our description of these small cocircuits.

**Lemma 3.1** *Let $G$ be a connected graph. Suppose $B(G)$ is $n$-connected, for some $n \geq 3$. Let $X$ be a cocircuit of $B(G)$. Let $H_0$ denote the unique acyclic component of $G - X$. Then*

$$2|X| \geq \sum_{\substack{v \in V(H_0); \\ d_{G-X}(v) < n}} n - d_{G-X}(v)$$

Recall that a triangle is a 3-element circuit and a triad is a 3-element cocircuit. We now consider triads in 3-connected bicircular matroids.

**Lemma 3.2** *Let $G$ be a connected graph having at least seven edges. Suppose $B(G)$ is 3-connected. If $X \subseteq E(G)$ is a triad of $B(G)$, then the edges of $X$ are all incident with a common vertex; or $G|X$ is isomorphic to $P_4$, and the set of edges incident to either of the two internal vertices of this path consists of the edges of $X$ along with a single edge in parallel to the middle edge of the path.*

*Proof.* We have that $G - X$ contains exactly one acyclic component $H_0$. Evidently $G - X$ has at most one cyclic component $H_1$ since $G$ is 2-connected by Lemma 2.2. If $H_0$ has exactly one vertex, we are done. Assume $H_0$ is a tree containing at least two vertices. Thus, $H_0$ has at least two leaf vertices. By Lemma 3.1, $H_0$ has at most three leaf vertices.

If all edges of $X$ have both ends in $H_0$, then $H_0$ is a tree and $|E(H_0)| = |E(G)| - 3 \geq 7 - 3 = 4$. Since $H_0$ has at most three leaves, it is easy to see that either $H_0$ is a path of length at least 4, or $H_0$ has exactly three leaves and at least one degree-2 vertex. However, each of these contradicts Lemma 3.1.

So we may assume that an edge of $X$ has one end in $H_0$ and one end in a cyclic component $H_1$ of $G - X$. Since $G$ is 2-connected, there is at least one other $H_0$-$H_1$ edge of $X$. Therefore, $H_0$ has exactly two leaf vertices, say $u$ and $v$, and these are the only vertices in $H_0$. Each is incident with an $H_0$-$H_1$ edge of $X$. Since $\delta(G) \geq 3$, the third edge of $X$ must be incident to both $u$ and $v$.

□

A similar proof technique establishes the graphic structure of $n$-cocircuits in $n$-connected bicircular matroids for $n \geq 4$.

**Lemma 3.3** *Suppose $G$ is a connected graph having at least seven edges, and $B(G)$ is $n$-connected for some $n \geq 4$. If $X \subseteq E(G)$ is a size-$n$ cocircuit of*

*$B(G)$, then the edges of $X$ are all incident with a common vertex.*

*Proof.* As in the proof of Lemma 3.2, we may assume that $H_0$ has at least two vertices. Since $2n < 3(n-1)$ , $H_0$ has exactly two leaf vertices by Lemma 3.1, so $H_0$ is a path. Furthermore, $2n < 2(n-1) + 2(n-2)$ so $H_0$ is $P_2$ or $P_3$.

First suppose that all edges of $X$ have both ends in $H_0$. So $|V(G)| = 2$ or 3, and $|E(G)| \geq 7$. It is easy to see that $G$ must contain a bicycle of size at most 3, contradicting the $n$-biconnectivity.

Thus there is an edge in $X$ that has an end in a cyclic component $H_1$ of $G-X$. By the $(n-1)$-connectivity of $G$, there are least 2 such edges. Then there are at most $2n-2$ ends of the edges of $X$ in $H_0$. Thus $H_0$ is $P_2$. Since bicycles of $G$ must have at least four edges, at most one edge of $X$ has both ends in $H_0$. Then there are at most $n-1+2 = n+1$ ends of the edges of $X$ in $H_0$. Since $n+1 < 2n-2$, this is a contradiction. □

## 4 Unavoidable minors of 4-connected bicircular matroids

Before proving the main result of the paper, we recall that if a graph $H$ is a minor of a graph $G$, then the bicircular matroid $B(H)$ is a minor of $B(G)$ [8]. The next result can be found in Biedl [1]; one may proved it by a simple counting argument.

**Lemma 4.1** *A maximal matching in a max-deg-$k$ graph with $m$ edges has size at least $\frac{m}{2k-1}$.*

The next lemma is the main result of this section.

**Lemma 4.2** *For each $n$ there is an $R(n)$ such that every 3-connected graph on at least $R(n)$ vertices having minimum degree at least four has a minor isomorphic to one of $W_n^2$, $K_{3,n}^+$, or $K_{3,n}^2$.*

*Proof.* By Theorem 1.3, there is an $R$ such that each 3-connected graph on at least $R$ vertices has a subgraph isomorphic to a subdivision of $W_k$, $K_{3,k}$, or $V_k$ for $k = 4n^2 - 2n - 4$. Suppose $G$ is a 3-connected graph on at least $R$ vertices. Since $k = 4n^2 - 2n - 4 > 4n$, if $G$ has a $W_k$- or $V_k$-subdivision as a subgraph, then $G$ has a $W_n^2$-minor, and we are done. Assume then that $G$ has a $K_{3,k}$-subdivision as a subgraph. That is, $G$ has vertices $u_1$, $u_2$, $u_3$, $v_1$, $v_2$, $\ldots$, $v_k$ such that there for each $i \in \{1, 2, \ldots, k\}$ there are paths $P_{i,1}$, $P_{i,2}$, and $P_{i,3}$ from $v_i$ to $u_1$, $u_2$, and $u_3$, respectively, such that $P_{i_1,j_1}$ and $P_{i_2,j_2}$ are internally vertex-disjoint whenever $(i_1, j_1) \neq (i_2, j_2)$.

Let $e \in E(G)$. Note that if $e$ satisfies either of the following conditions, then $G/e$ contains a $K_{3,k}$ subdivision having small and large sides $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, \ldots, v_k\}$, respectively, such that $d_{G/e}(v_j) \geq 4$ for each $j \in \{1, 2, \ldots, k\}$.

(1) For some $a \in \{1, 2, 3\}$ and $b \in \{1, 2, \ldots, k\}$, $e$ is an edge on the path $P_{a,b}$ that is incident with $u_a$ but has its other end in $V(G) - \{v_1, v_2, \ldots, v_k\}$.
(2) Each path $P_{i,j}$ has length one and $e$ is an edge of $G$ with one end in $\{v_1, v_2, \ldots, v_k\}$ and the other end in $V(G) - \{u_1, u_2, u_3, v_1, v_2, \ldots, v_k\}$.

Obtain a minor $H$ of $G$ by consecutively contracting edges of the types given above until no such edges remain, followed by deleting all edges not incident with some $\{v_1, v_2, \ldots, v_k\}$.

Now, $H$ consists of a $K_{3,k}$-subgraph with some extra edges added incident with the vertices on the large side of the bipartition. By construction, no step of the algorithm above decreases the degree of a vertex in $\{v_1, v_2, \ldots, v_k\}$. Hence, each of the $k = 4n^2 - 2n - 4$ vertices is incident with at least one such extra edge. If at least $n$ of these vertices have adjacent loops, then $H$ has a $K^+_{3,n}$-minor. If at least $3n - 2$ of these vertices are adjacent to a vertex in $\{u_1, u_2, u_3\}$ by an edge not in the $K_{3,k}$-graph, then at least $n$ are adjacent to the same vertex by the pigeonhole principle, so $H$ has a $K^2_{3,n}$ minor. Assume neither of these cases occurs. Let $E_1$ be the set of non-loop edges of $H$ that have both ends in $\{v_1, v_2, \cdots, v_k\}$, let $H_1 = span_H(E_1)$ and let $Z = V(H_1)$. Then $|Z| \geq (4n^2 - 2n - 4) - (n - 1) - (3n - 3) = 4n^2 - 6n$ and every vertex in $Z$ is adjacent to some other vertex in $Z$. We have that $H_1$ has at least $\frac{|Z|}{2} \geq 2n^2 - 3n$ edges. If some vertex $v_i \in Z$ has degree greater than $n - 1$ in $H_1$, then $H$ has $K^2_{3,n}$-minor by contraction of the edge $v_i u_1$. Assume then that the maximum degree in $H_1$ is at most $n - 1$. Then by Lemma 4.1, $H_1$ has a matching of size at least $\frac{2n^2-3n}{2n-3} = n$. Thus $H$ has a $K^2_{3,n}$-minor by contraction of each edge in this matching. □

**Corollary 4.3** *For each $n$ there is an $N(n)$ such that every 4-biconnected graph on at least $N(n)$ edges has a minor isomorphic to one of $W^2_n$, $K^+_{3,n}$, or $K^2_{3,n}$.*

*Proof.* Note that a 4-biconnected graph $G$ contains at most one loop at each vertex, and each parallel class of edges has size at most two. Therefore, $|E(G)| \leq |V(G)| + 2\binom{|V(G)|}{2} = |V(G)|^2$. Hence $|V(G)| \geq \sqrt{|E(G)|}$. Fix $n$. Let $R(n)$ be given as in Lemma 4.2. If $|E(G)| \geq R(n)^2$ then $G$ is a 3-connected graph with $\delta(G) \geq 4$ on at least $R(n)$ vertices, so $G$ has one of the given minors. □

It is a trivial matter to prove Theorem 1.4 from the above corollary.

*Proof.* [Proof of Theorem 1.4]The theorem follows from Corollary 4.3 since a sufficiently large 4-connected bicircular matroid can be represented by a large 4-biconnected graph, which in turn must have one of the given large minors. □

## 5 Unavoidable minors of internally 4-connected bicircular matroids

Recall that a matroid $M$ is internally 4-connected if $M$ is 3-connected and for every 3-separation $(X, Y)$ of $M$, either $|X| = 3$ or $|Y| = 3$. It is clear that a triangle in a bicircular matroid $B(G)$ is a set of three parallel edges, a set of two parallel edges and a loop at one end, or two loops at two distinct vertices and an edge between them in the associated graph $G$. Lemma 3.2 describes what a triad looks like in a 3-connected bicircular matroid. Note that the exceptional case in Lemma 3.2 gives rise to a 3-separating set of size 4, thus does not occur in an internally 4-connected bicircular matroid $B(G)$ when $|E(G)| \geq 8$. Therefore, every triad in an internally 4-connected bicircular matroid corresponds to either a degree-3 vertex, or a degree-4 vertex incident to exactly one loop in the underlying graph.

By Lemma 2.2, the graph underlying an internally 4-connected bicircular matroid is 2-connected and has a minimum degree of at least three. However, using Wagner's original definition of biconnectivity, we see that the 2-separations in such a graph are highly restricted.

**Lemma 5.1** *Let $G$ be a connected graph having at least six edges. If $B(G)$ is internally 4-connected and $G$ has a 2-vertex cut, then one side of the separation consists of a single vertex having exactly three incident edges.*

*Proof.* Since $\delta(G) \geq 3$, each side of the 2-separation is cyclic. Therefore, the 2-vertex cut in $G$ naturally induces a "small" 3-biseparation $(E_1, E_2)$ in $G$. Assume $|E_1| = 3$ since $G$ is internally 4-connected. Thus $|V(G_1) - V(G_2)| = 1$. □

Each 2-separation in the graph underlying an internally 4-connected bicircular matroid must have one of the configurations given in Figure 4.

Now it is easy to see that we have the following graphic characterization for a bicircular matroid to be internally 4-connected.

**Lemma 5.2** *Let $G$ be a connected graph having at least eight edges. Then $B(G)$ is internally 4-connected if and only if each of the following holds.*

*(1) $G$ is 2-connected.*
*(2) There exists at most one loop at each vertex.*

*(3) $\delta(G)$, the minimum degree of $G$, is at least* 3

*(4) Every vertex cut of size 2 must have one of the forms shown in Figure 4; moreover, there exists no edges between and no loops at the two cut vertices.*

*(5) Every parallel class of edges has size at most 3.*

*(6) For each parallel class of size 3, there exists no loop at either end.*

*(7) For each parallel class of size 2, there exists at most one loop at the two ends.*

We now prove our result on the unavoidable minors of large internally 4-connected bicircular matroids.

*Proof.*[Proof of Theorem 1.5] First note that the matroids $B(\mathcal{W}_n)$ and $B(K_{3,n})$ are internally 4-connected by Lemma 5.2.

Suppose $G$ is a connected graph for which $B(G)$ is internally 4-connected. A parallel class of edges in $G$ has size at most three, and there is at most one loop at each vertex. Therefore, $|E(G)| \leq |V(G)| + 3\binom{|V(G)|}{2} \leq \frac{3}{2}|V(G)|^2$. Thus $|V(G)| \geq \sqrt{\frac{2}{3}|E(G)|}$.

Now suppose $G$ is a connected graph underlying an internally 4-connected bicircular matroid $B(G)$ having at least $\frac{3}{2}R^4$ elements in its ground set, where $R$ is an integer for which any 3-connected graph on at least $R$ vertices has a minor isomorphic to $\mathcal{W}_n$ or $K_{3,n}$ as given by Corollary 1.2.

If $G$ has a 2-separation, we have by Lemma 5.1 that one side of the separation consists of a single degree-3 vertex that is adjacent to exactly two vertices, namely the two cut vertices. Call such a degree-3 vertex a *tick*. A vertex that is not a tick is a *non-tick*. There is a natural injection between the set of ticks and the set of pairs of non-ticks given by matching a tick with its associated pair of 2-separating non-tick vertices. Let $\tau$ denote the number of ticks in $G$, and let $\eta$ denote the number of non-tick vertices. We have that $\tau \leq \binom{\eta}{2}$ and $\eta + \tau = |V(G)|$. By $\eta \geq 1$ we have $\frac{\eta-1}{2} + 1 \leq \eta$, so

$$\eta^2 \geq \eta\left(\frac{\eta-1}{2} + 1\right) = \binom{\eta}{2} + \eta \geq \tau + \eta = |V(G)|$$

Note that the graph resulting from the contraction of a link edge incident with a tick is still 2-connected. Furthermore, any 2-separations of the resultant graph are also (up to identification of vertices via contraction) 2-separations of $G$. Thus, we can consecutively contract link edges incident with ticks to obtain a 3-connected graph $H$ having $\eta \geq \sqrt{|V(G)|}$ vertices.

Recall that $G$ has at least $\frac{3}{2}R^4$ edges, so $G$ has at least $R^2$ vertices. Hence,

$G$ has a 3-connected minor having at least $R$ vertices. Thus, $G$ has a minor isomorphic to one of $\mathcal{W}_n$ or $K_{3,n}$, so $B(G)$ has a minor isomorphic to one of $B(\mathcal{W}_n)$ or $B(K_{3,n})$. □

## 6 Bicircular matroid that are vertically 4-connected and 3-connected

In this section we study bicircular matroids that are both vertically 4-connected and 3-connected. Since a rank-2 flat in a 3-connected bicircular matroid is a class of parallel non-loop edges plus the set of loops at the two end vertices, the next result follows easily from Lemma 2.2.

**Lemma 6.1** *If $G$ is a connected graph on at least four vertices such that $B(G)$ is vertically 4-connected and 3-connected, then $G$ is 3-connected and $\delta(G) \geq 4$.*

Now we are ready to prove Theorem 1.6.

*Proof.*[Proof of Theorem 1.6] Suppose $G$ is a connected graph such that $B(G)$ is 3-connected and vertically 4-connected and $|E(G)| \geq N'' = \frac{n-1}{2}R(n)^2$, where $R(n)$ is given as in Lemma 4.2.

If $G$ has a parallel class of edges of size at least $n$, then $B(G)$ has a $U_{2,n}$-restriction. So we may assume that each parallel class of edges has size at most $n-1$. Since $B(G)$ is 3-connected, $G$ has at most one loop at each vertex. Therefore we have $|E(G)| \leq |V(G)| + (n-1)\binom{|V(G)|}{2} = \frac{n-1}{2}|V(G)|^2 - \frac{n-3}{2}|V(G)|$. Since $n \geq 4$, $|E(G)| \leq \frac{n-1}{2}|V(G)|^2$. Therefore, $|V(G)| \geq \sqrt{\frac{2}{n-1}|E(G)|} \geq \sqrt{\frac{2}{n-1} \cdot \frac{n-1}{2}R(n)^2} = R(n)$. By Lemma 6.1, $G$ is a 3-connected graph having minimum degree at least four. By Lemma 4.2, $G$ has a minor isomorphic to one of $\mathcal{W}_n^2$, $K_{3,n}^+$, or $K_{3,n}^2$. Thus, $G$ has one of these minors, so $B(G)$ has a minor isomorphic to the bicircular matroids of one of these graphs.

□

## 7 Conclusion

The class of 4-connected bicircular matroids is admittedly restrictive. However, the techniques in this paper center around the biconnectivity property and do not readily extend to more general classes of bias matroids. Slilaty and Qin offer a version of Wagner's biconnectivity that is generalized to bias matroids in [6]. Evidently, the extra attention that must be paid to balanced cycles is the inherent complication in obtaining an analog of Lemma 2.2, which we have relied upon in our proof. An extension to 4-connected signed graphic matroids might be much more easily obtained and would still have the benefit of providing the list of unavoidable minors of large 4-connected graphs.

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## List of Figures


1 Illustration of $V_k$ 14

2 Unavoidable minors for 4-biconnectivity 15

3 Three types of bicycles 16

4 The 2-separations in graphs underlying internally 4-connected bicircular matroids 17

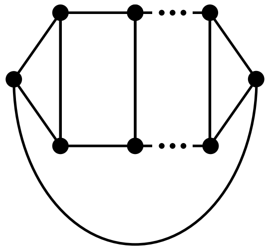
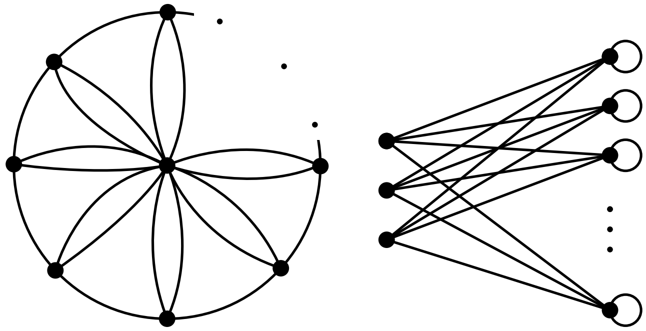


Fig. 1. Illustration of $V_k$



(a) $\mathcal{W}_n^2$ (b) $K_{3,n}^+$ (c) $K_{3,n}^2$

Fig. 2. Unavoidable minors for 4-biconnectivity

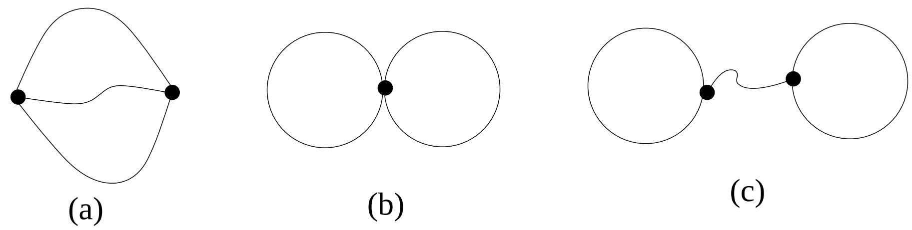



Fig. 3. Three types of bicycles

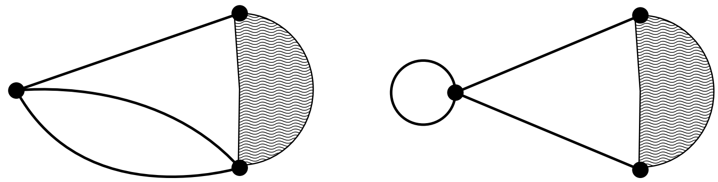


Fig. 4. The 2-separations in graphs underlying internally 4-connected bicircular matroids