The regular excluded minors for signed-graphic matroids

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Dedicated to Thomas Dowling on the occasion of his retirement.

Abstract

We show that the complete list of regular excluded minors for the class of signed-graphic matroids is $M^*(G_1), \ldots, M^*(G_{29}), R_{15}, R_{16}$. Here $G_1, \ldots, G_{29}$ are the vertically 2-connected excluded minors for the class of projective-planar graphs.

1 Introduction

We assume the reader is familiar with matroid theory as in [7]. If the reader is not familiar with signed graphs and their matroids as in [19], then we review all of the relevant material in Section 2. Signed-graphic matroids are exactly the minors of Dowling geometries [3] for the group of order two. Our main result is Theorem 1.1. Here $G_1, \ldots, G_{29}$ are the vertically 2-connected excluded minors for the class of projective-planar graphs[1]. The matroids R_{15} and R_{16} are introduced in Section 4.

Theorem 1.1. *A regular matroid M is a signed-graphic matroid if and only if M contains none of the following as a minor: $M^*(G_1), \ldots, M^*(G_{29}), R_{15}$, and R_{16}.*

Whittle conjectures in [18] that there is a theorem for near-regular matroids similar to Theorem 1.2 that uses signed-graphic matroids and co-signed-graphic matroids as the basic terms in the decomposition. Since the proof of Theorem 1.2 uses the list of excluded minors for the class of graphic matroids, it is possible that a result for near-regular matroids would use the list of excluded minors for the class of signed-graphic matroids.

Theorem 1.2 (Seymour [10]). *Every regular matroid M is constructed by a sequence of k-sums ($k \in \{1, 2, 3\}$) of graphic matroids, cographic matroids, and copies of the matroid R_{10}.*

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[1]There are 35 excluded minors for the class of projective-planar graphs and 29 of these are vertically 2-connected. See Archdeacon [1] and Glover, Huneke, and Wang [4] or see [6, Theorem 6.5.1].

Since we are working within the class of regular matroids, it is no surprise that a proof of Theorem 1.1 starts with Theorem 1.2. Given Theorem 1.2, we know that if M is a regular excluded minor for the class of signed-graphic matroids and M is internally 4-connected, then M is either graphic, cographic, or R_{10}. Since a graphic matroid is signed graphic and since R_{10} is the matroid of the signed graph $-K_5$ (by $-G$ we mean the signed graph with G as its underlying graph and all edges signed negatively) we must have that M is cographic. By Theorem 1.3(1), we then get that $M \in \{M^*(G_1), \ldots, M^*(G_{29})\}$. Theorem 1.3(2) tells us that each of $M^*(G_1), \ldots, M^*(G_{29})$ is indeed an excluded minor for the class of signed-graphic matroids.

Theorem 1.3. *Let M be a cographic matroid.*

(1) If M is 3-connected and an excluded minor for the class of signed-graphic matroids, then $M \in \{M^*(G_1), \ldots, M^*(G_{29})\}$.

(2) If $M \in \{M^*(G_1), \ldots, M^*(G_{29})\}$, then M is an excluded minor for the class of signed-graphic matroids.

Proof. Write $M = M^*(G)$ where G has no isolated vertices. (1) Since $M/e = M^*(G\backslash e)$ and $M\backslash e = M^*(G/e)$ are both signed graphic and connected Theorem 1.4 implies G/e and $G\backslash e$ are both projective planar while G is not projective planar and so $G \in \{G_1, \ldots, G_{29}\}$. (2) Since $M(G)$ is connected, G/e and $G\backslash e$ are both connected and so Theorem 1.4 implies $M^*(G)/e = M^*(G\backslash e)$ and $M^*(G)\backslash e = M^*(G/e)$ are both signed graphic while $M(G)$ is not. □

Theorem 1.4. *Let G be a connected graph.*

(1) If G is projective planar, then $M^*(G)$ is signed graphic ([11]).

(2) If $M^*(G)$ is connected and signed graphic, then G is projective planar ([8] or [12]).

Therefore, our work in this paper is to show that if M has connectivity $k \in \{2, 3\}$, then $M \in \{M^*(G_1), \ldots, M^*(G_{29}), R_{15}, R_{16}\}$ and that R_{15} and R_{16} are excluded minors. The case for $k = 2$ is done in Section 3 and the case for $k = 3$ is done in Section 6. In Section 2 we have some preliminaries, in Section 4 we introduce the matroids R_{15} and R_{16}, and in Section 5 we present some lemmas that we will use in Section 6.

2 Preliminaries

Graphs A graph G consists of a collection of *vertices* (i.e., topological 0-cells), denoted by $V(G)$, and a set of *edges* (i.e., topological 1-cells), denoted by $E(G)$, where an edge has two ends each of which is attached to a vertex. A *link* is an edge that has its ends incident to distinct vertices and a *loop* is an edge that has both of its ends incident to the same vertex.

A *circle* is a connected, 2-regular graph (i.e., a simple closed path). In graph theory a circle is often called a cycle, circuit, polygon, etc. We denote the cycle matroid of the graph G by $M(G)$. If $X \subseteq E(G)$, then we denote the subgraph of G consisting of the edges in X and all vertices incident to an edge in X by $G{:}X$. The collection of vertices in $G{:}X$ is denoted by $V(X)$, the number of vertices in $G{:}X$ is denoted by v_X, and the number of connected components in $G{:}X$ is denoted by c_X.

For $k \geq 1$, a *k-separation* of a graph is a bipartition (A, B) of the edges of G such that $|A| \geq k$, $|B| \geq k$, and $|V(A) \cap V(B)| = k$. A *vertical* k-separation (A, B) of G is a k-separation where

$V(A) \setminus V(B) \neq \emptyset$ and $V(B) \setminus V(A) \neq \emptyset$. A separation or vertical separation (A, B) is said to have *connected parts* when $G{:}A$ and $G{:}B$ are both connected. A connected graph on at least $k+1$ vertices is said to be *vertically k-connected* when there is no vertical r-separation for $r < k$. Vertical k-connectivity is usually called k-connectivity, but here we wish to distinguish between this kind of graph connectivity and the second type used in Tutte's book on graph theory [17].

Given a subgraph H of G, an *H-bridge* is either an edge not in H whose endpoints are both in H or a connected component C of $G \setminus V(H)$ along with the links between C and H. Given an H-bridge B of G: a *foot* of B is an edge of B with an endpoint in H, a *vertex of attachment* of B is a vertex in H that is an endpoint of a foot of B, and $\overline{B}$ denotes the bridge B minus the vertices of attachment of B (i.e., either a connected component of $G \setminus V(H)$ or $\emptyset$ when B is a single edge). An H-bridge of G with n vertices of attachment is called an *n-bridge*.

If G' is a subdivision of a graph G where G has minimum degree at least three, then a *branch vertex* of G' is a vertex of degree at least three in G' and a *branch* is a path in G' corresponding to an edge in G. A G'-bridge B is called *local* if all attachments of B are on the same branch of G'. A useful fact about local bridges that we will need later is Proposition 2.1.

Proposition 2.1 (see [6, Lemma 6.2.1])**.** *Let G be a vertically 3-connected graph. If $H \subseteq G$ is a subdivision of a graph Γ, then there is a subdivision H' of Γ in G such that H' has the same branch vertices as H, if e is a branch in H then the corresponding branch e' in H' connects the same branch vertices, and H' has no local bridges.*

Signed graphs A *signed graph* is a pair (G, σ) in which $\sigma : E(G) \to \{+1, -1\}$. A circle or path in a signed graph Σ is called *positive* if the product of signs on its edges is positive, otherwise the circle or path is called *negative*. If H is a subgraph of Σ, then H is called *balanced* when all circles in H are positive. A *balancing vertex* of an unbalanced signed graph is a vertex whose removal leaves a balanced subgraph.

A *switching function* on a signed graph $\Sigma = (G, \sigma)$ is a function $\eta : V(\Sigma) \to \{+1, -1\}$. The signed graph $\Sigma^\eta = (G, \sigma^\eta)$ has sign function σ^η defined on all edges of G by $\sigma^\eta(e) = \eta(v)\sigma(e)\eta(w)$ where v and w are the endpoints of e. When two signed graphs Σ_1 and Σ_2 satisfy $\Sigma_1^\eta = \Sigma_2$ for some switching function η, Σ_1 and Σ_2 are said to be *switching equivalent*. Two signed graphs with the same underlying graph are switching equivalent iff they have the same list of positive circles (see [19, Proposition 3.2]). Switching equivalent signed graphs are considered to be isomorphic.

In a signed graph $\Sigma = (G, \sigma)$, the deletion of e from Σ is defined as $\Sigma \backslash e = (G \backslash e, \sigma)$ where σ is restricted to the domain $E(G \backslash e)$. The contraction of an edge e is defined for three distinct cases. If e is a link, then $\Sigma / e = (G/e, \sigma^\eta)$ where η is a switching function on Σ satisfying $\sigma^\eta(e) = +1$. Of course then σ^η is restricted to the edges of G/e in Σ/e. Note that Σ/e is well defined up to switching. If e is a positive loop, then $\Sigma/e = \Sigma \backslash e$. If e is a negative loop incident with vertex v, then Σ/e is the signed graph obtained from Σ as follows: links incident to v become negative loops incident to their other endpoint, negative loops incident to v other than e become positive loops incident to v, and edges not incident to v remain unchanged. The reason for this definition of contraction in signed graphs is so that contractions in signed graphs will correspond to contractions in their signed-graphic matroids.

A *minor* of Σ is a signed graph obtained from Σ by a sequence of contractions and deletions of edges, deletions of isolated vertices, and switchings. A *link minor* is a minor that is obtained without contracting any negative loops.

A signed graph is called *tangled* if it is unbalanced, has no balancing vertex, and no two vertex-disjoint negative circles. The proof of Proposition 2.2 is straightforward and is left to the reader.

Proposition 2.2. *If Σ is tangled, then Σ has has exactly one unbalanced block, in particular, Σ has no negative loops.*

Proposition 2.3. *If Σ_1 and Σ_2 are tangled, Σ_1 is a minor of Υ, and Υ is a minor of Σ_2, then Υ is tangled and is a link minor of Σ_2.*

Proof. Let $\mathcal{B}$ be the class of balanced signed graphs, $\mathcal{J}$ be the class of signed graphs that are balanced after removing negative loops, $\mathcal{V}$ be the class of signed graphs with balancing vertices, and $\mathcal{T}$ be the class of tangled signed graphs. By the definitions of these types of signed graphs and the definition of contractions in signed graphs we get the following three facts: since tangled signed graphs do not have negative loops (Proposition 2.2) any one-edge deletion or contraction of a member of $\mathcal{T}$ is in $\mathcal{T}$ or $\mathcal{V}$; any one-edge deletion or contraction of a member of $\mathcal{V}$ is in $\mathcal{V}$, $\mathcal{J}$, or $\mathcal{B}$; and any one-edge deletion or contraction of a member of $\mathcal{J}$ or $\mathcal{B}$ is in $\mathcal{J}$ or $\mathcal{B}$. Hence when obtaining a tangled minor of a tangled signed graph we contract only links and never leave the class of tangled signed graphs. □

When drawing signed graphs, positive edges are drawn as solid curves and negative edges as dashed curves. A signed graph is said to be vertically k-connected when its underlying graph is vertically k-connected.

Signed-graphic matroids The *frame matroid* (often called the *bias matroid*) of Σ is denoted by $M(\Sigma)$. In this paper such a matroid is simply called a *signed-graphic matroid.* The element set of $M(\Sigma)$ is $E(\Sigma)$ and a circuit of $M(\Sigma)$ is either the edge set of a positive circle or the edge set of a subdivision of a subgraph in Figure 1 with no positive circles.

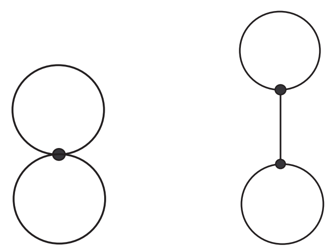


Figure 1.

For any $e \in E(\Sigma)$, we have that $M(\Sigma\backslash e) = M(\Sigma)\backslash e$ and $M(\Sigma/e) = M(\Sigma)/e$ (see [19, Theorem 5.2]). Note that if $\Sigma = (G, \sigma)$ is balanced, then $M(\Sigma) = M(G)$. Hence, the class of signed-graphic matroids contains the class of graphic matroids. Given two signed graphs Σ_1 and Σ_2 with the same underlying graph, $M(\Sigma_1) = M(\Sigma_2)$ iff Σ_1 and Σ_2 have the same positive and negative circles iff Σ_1 and Σ_2 are switching equivalent.

Given $X \subseteq E(\Sigma)$ we denote the number of balanced connected components of $\Sigma{:}X$ by b_X. If $X \subseteq E(\Sigma)$, then $r(X) = v_X - b_X$ (see [19, Theorem 5.1(j)]). For brevity we write $r(\Sigma)$ to mean $r(M(\Sigma))$. The rank function tells us that if Σ is not connected after removing isolated vertices, then $M(\Sigma)$ is not connected. It also tells us that a cocircuit of $M(\Sigma)$ is a minimal set of edges whose removal increases the number of balanced components by one.

Theorem 2.4 is from [14, Theorems 1.3 and 1.4]. It tells us that regularity of $M(\Sigma)$ is almost synonymous with Σ being tangled. Theorem 2.5 is an important fact relating matroid connectivity and graph connectivity.

Theorem 2.4 (Slilaty, Qin [14])**.** *If Σ is connected, then the following are true.*

(1) If Σ is tangled, then $M(\Sigma)$ is regular.

(2) If $M(\Sigma)$ is regular and not graphic, then Σ is tangled.

Theorem 2.5 (Slilaty, Qin [15, Theorem 1.6])**.** *If Σ is tangled and has no isolated vertices and $M(\Sigma)$ is k-connected for any $k \in \{2,3\}$, then Σ is vertically k-connected.*

If Σ is a signed graph with balancing vertex v, then by switching we may assume that all negative edges of Σ are incident to v. Let G_v be the graph obtained from Σ by splitting v into two vertices v_+ and v_- where: positive links incident to v become links incident to v_+, negative links incident to v become links incident to v_-, negative loops incident to v become v_+v_--links, and positive loops incident to v are positive loops anywhere in G_v. Proposition 2.6 is easy to verify.

Proposition 2.6. *If Σ has a balancing vertex v and G_v is the graph obtained from Σ as in the previous paragraph, then $M(\Sigma) = M(G_v)$.*

1-sums Let Σ and Υ be signed graphs with nonempty edge sets such that Υ is balanced. The *1-sum* of Σ and Υ is the identification of Σ and Υ along some vertex and is denoted by $\Sigma \oplus_1 \Upsilon$. Proposition 2.7 is immediate from our definition of a signed-graphic 1-sum and the definition of a matroid 1-sum.

Proposition 2.7. *If Σ and Υ are signed graphs, then $M(\Sigma \oplus_1 \Upsilon) = M(\Sigma) \oplus_1 M(\Upsilon)$.*

2-sums Given two signed graphs Σ and Υ we will define two methods of taking their 2-sum. By $\Sigma \oplus_2 \Upsilon$ we mean a 2-sum that is one of these two types. If both of Σ and Υ are unbalanced, then the *1-vertex 2-sum* is obtained by identifying the signed graphs along a negative loop and then deleting the negative loop. If exactly one of Σ and Υ is unbalanced, then the *2-vertex 2-sum* of the signed graphs is obtained by choosing a link in each signed graph, switching so that the links have the same sign in each, identifying the two signed graphs along the links, and then deleting that link. In both cases it is required that the edge along which the 2-sum is taken is not a coloop in the signed-graphic matroid. The verification of Proposition 2.8 is routine.

Proposition 2.8. *If Σ and Υ are signed graphs, then $M(\Sigma \oplus_2 \Upsilon) = M(\Sigma) \oplus_2 M(\Upsilon)$.*

Proposition 2.9. *If M_1 is a signed-graphic matroid and M_2 is a graphic matroid, then $M_1 \oplus_2 M_2$ is signed graphic.*

Proof. Say that $M_1 = M(\Sigma)$, $M_2 = M(G)$, and e is the edge in each of Σ and G along which the 2-sum is taken. Since e is not a matroid loop, e is a link in G (call its endpoints v and w). If e is a link in Σ, then let Υ be the signed graph with underlying graph G and all edges signed positively. Note that $M(\Upsilon) = M(G)$. If e is a negative loop in Σ, then let Υ be the signed graph with a balancing vertex obtained from G by the reverse of the operation described in Proposition 2.6 performed on the endpoints of e in G. Note that e is then a negative loop in Υ. In either case Proposition 2.8 implies that $M_1 \oplus_2 M_2 = M(\Sigma) \oplus_2 M(\Upsilon) = M(\Sigma \oplus_2 \Upsilon)$, as required. □

3-sums Given a signed graph Σ and a balanced signed graph Υ (or a graph G), their *3-vertex 3-sum* is obtained by selecting a positive triangle in each term, switching so that the edges of the triangle have the same sign pattern in each term, identifying the signed graphs along the triangles, and then deleting the edges.

We also make use of the operation of symmetric differences of binary matroids. If M_1 and M_2 are binary matroids on edge sets E_1 and E_2 with $E_1 \cap E_2 \neq \emptyset$, then there is a binary matroid $M_1 \triangle M_2$ on edge set $E_1 \triangle E_2$ whose circuits are the minimal nonempty elements of $\{C_1 \triangle C_2 : C_i \text{ is a (possibly empty) disjoint union of circuits in } M_i\}$ that are contained in $E_1 \triangle E_2$. An important property of this symmetric difference operation is that $(M_1 \triangle M_2)^* = M_1^* \triangle M_2^*$ [10, p.319]. In [10], the 3-sum $M_1 \oplus_3 M_2$ for binary matroids is defined as $M_1 \triangle M_2$ where $E_1 \cap E_2$ is a triangle in each M_i that is coindependent in each M_i and each $|E_i| \geq 7$. Also, if M_1 and M_2 satisfy these conditions, then $M_1 \oplus_3 M_2$ is the modular sum operation from [2]. See [14, Proposition 3.4] for a verification of Proposition 2.10.

Proposition 2.10. *If Σ is a signed graph such that $M(\Sigma)$ is regular and Υ is a balanced signed graph, then $M(\Sigma \oplus_3 \Upsilon) = M(\Sigma) \oplus_3 M(\Upsilon)$ (or $M(\Sigma) \triangle M(\Upsilon)$ when the 3-sum is not defined).*

Proposition 2.11. *If M_1 is a regular signed-graphic matroid and M_2 is a graphic matroid, then $M_1 \oplus_3 M_2$ is signed graphic.*

Proof. It is known that the class of graphic matroids is closed under 3-summing, so assume that M_1 is not graphic. Now say that $M_1 = M(\Sigma)$, $M_2 = M(G)$, and T is the 3-point line along which the 3-sum is taken. Since $M(\Sigma)$ is not graphic, Σ is tangled (Theorem 2.4) and so T is a positive triangle in Σ because Σ is loopless (Proposition 2.2). Now let Υ be the signed graph with underlying graph G and all edges signed positively. Note that T is a positive triangle in G and so now by Proposition 2.10, $M_1 \oplus_3 M_2 = M(\Sigma) \oplus_3 M(\Upsilon) = M(\Sigma \oplus_3 \Upsilon)$, as required. □

Proposition 2.12. *If G_1 is a vertically 2-connected graph, G_2 is either a vertically 2-connected graph or tangled signed graph, and both are as shown in Figure 2, then the parallel connection of $M^*(G_1)$ with $M^*(G_2)$ along the triad T is $M^*(H)$ where $M^*(H)$ is the cographic matroid of H when H is a graph and is the dual of the signed-graphic matroid of H when H is a signed graph.*

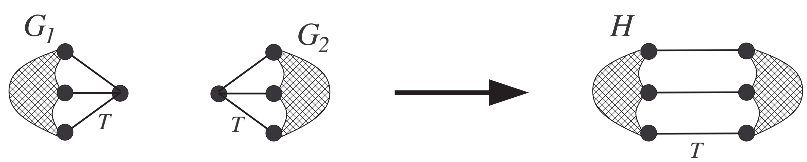


Figure 2.

Proof. Consider G_1 to be an all-positive signed graph when G_2 is a signed graph. A circuit C in H is either a positive circle or a one-vertex join of two negative circles. In the former case, C is either a positive circle in some $G_i \setminus T$ or $C = C_1 \cup C_2$ where C_i is a positive circle in G_i and $C_1 \cap C_2$ consists of two edges of T. In the latter case, C is either the one-vertex join of two negative circles in $G_2 \setminus T$ or $C = C_1 \cup C_2 \cup D$ where D is a negative circle in $G_2 \setminus T$, C_1 is a positive circle in G_1, C_2 is a negative circle in G_2, and $C_1 \cap C_2$ consists of two edges of T. So if $\mathcal{C} = C_1 \cup \cdots \cup C_n$ is a union of circuits in $M(H)$, then there is a corresponding union of circuits $\mathcal{C}_i$ in $M(G_i)$. Thus $E(H) \setminus \mathcal{C} = (E(G_1) \setminus \mathcal{C}_1) \cup (E(G_2) \setminus \mathcal{C}_2)$. Thus a flat in $M^*(H)$ is a flat in the parallel connection $P(M^*(G), M^*(H))$. Conversely any flat in the parallel connection is a flat of one of the terms or the union of a flat from each term with a common intersection in T. Similarly we can show that any flat in the parallel connection will correspond in the same way to a flat in $M^*(H)$. □

3 Excluded minors that are 2-connected but not 3-connected

A collection $\mathcal{N}$ of connected matroids is called *1-rounded* when any connected matroid M containing a minor from $\mathcal{N}$ satisfies the following: for every $e \in E(M)$, M has a minor from $\mathcal{N}$ that uses e.

Theorem 3.1 (Seymour [9]). *The collection $\{U_{2,4}, F_7, F_7^*, M^*(K_5), M^*(K_{3,3}), M^*(K'_{3,3})\}$ is 1-rounded. Here $K'_{3,3}$ denotes the graph obtained from $K_{3,3}$ by adding a link joining two non-adjacent vertices.*

Theorem 3.2. *If M is 2-connected, not 3-connected, and a regular excluded minor for the class of signed graphic matroids, then $M \in \{M^*(G_1), \ldots, M^*(G_{29})\}$.*

Proof. Since M is 2-connected but not 3-connected, $M = M_1 \oplus_2 M_2$. Let e be the edge along which the 2-sum is taken. Since M is minor minimal, each M_i is signed graphic and hence not graphic by Proposition 2.9. Now since M_i is regular it does not contain any of $U_{2,4}$, F_7, and F_7^* as a minor; however, being not graphic implies that M_i contains either $M^*(K_5)$ or $M^*(K_{3,3})$ as a minor. Theorem 3.1 now implies that each M_i contains an $M^*(H_i)$ minor where $H_i \in \{K_5, K_{3,3}, K'_{3,3}\}$ and $e \in H_i$. Thus M contains $M^*(H_1) \oplus_2 M^*(H_2) = M^*(H_1 \oplus_2 H_2)$ as a minor and one can check that the 6 possibilities for $H_1 \oplus_2 H_2$ are all in $\{G_1, \ldots, G_{29}\}$. So since M is minor minimal, $M = M^*(H_1 \oplus_2 H_2) \in \{M^*(G_1), \ldots, M^*(G_{29})\}$. $\square$

4 The matroids R_{15} and R_{16}

In this section, we introduce two regular matroids R_{15} and R_{16} and prove Propositions 4.1–4.4 which show that they are both excluded minors for the class of signed-graphic matroids.

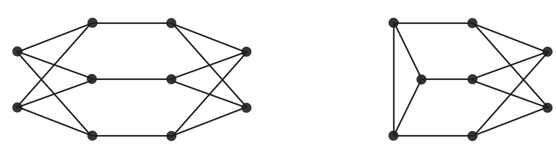
Proposition 4.1. *R_{15} is not signed graphic.*

Proposition 4.2. *Any proper minor of R_{15} is signed graphic.*

Proposition 4.3. *R_{16} is not signed graphic.*

Proposition 4.4. *Any proper minor of R_{16} is signed graphic.*

Figure 3.



The graphs G_1 and H_1, respectively.

Consider the graphs G_1 and H_1 in Figure 3. (Note that G_1 is one of the 29 vertically 2-connected excluded minors for the class of projective-planar graphs and, by Proposition 2.12, $M^*(H_1)$ is the parallel connection of $M^*(K_4)$ and $M^*(K_{3,3})$ along a triad in each graph.) Let T be the 3-edge bond separating the two copies of $K_{2,3}$ in G_1. The matroid R_{15} is obtained from $M^*(G_1)$ by a ΔY exchange along T (i.e., $R_{15} = M^*(G_1) \triangle M(K_4)$). Also $R_{15} = M^*(K_{3,3}) \oplus_3 M^*(H_1)$ where the 3-sum is along a triad in $K_{3,3}$ and the 3-edge bond T' in H_1 separating the triangle and the copy of $K_{2,3}$. R_{15} has 15 elements and rank 7.

The matroid R_{16} is obtained by taking two edge-disjoint triangles of $M(K_5)$ and 3-summing a copy of $M^*(K_{3,3})$ along each of the two triangles. The matroid R_{16} has 16 elements and rank 8.

Proposition 4.5. *The matroids R_{15} and R_{16} are both 3-connected, not graphic, and not cographic.*

Proof. R_{15} and R_{16} are both 3-connected as each is a 3-sum of two 3-connected matroids. Both R_{15} and R_{16} are not graphic because each contains an $M^*(K_{3,3})$ minor. R_{16} is not cographic because it contains an $M(K_5)$ minor. Lastly, we show that R_{15} is not cographic by displaying an $M(K_{3,3})$ minor. By deleting one and contracting two edges in the $K_{2,3}$ subgraph of H_1 we obtain the triangular prism P (i.e., two vertex-disjoint triangles connected by three links) without disturbing the 3-edge bond T' in H_1. Note that $M^*(P) = M(K_5\backslash e)$ and so $R_{15} = M^*(K_{3,3}) \oplus_3 M^*(H_1)$ contains $M^*(K_{3,3}) \oplus_3 M(K_5\backslash e)$ as a minor. Now $M^*(K_{3,3})$ contains an $M(K_4)$ minor using any of the triads of $K_{3,3}$. So now R_{15} contains $M(K_4)\triangle M(K_5\backslash e) = M(K_4 \oplus_3 K_5\backslash e) = M(K_{3,3})$ as a minor. □

The matroid R_{12} is defined as $M^*(K_{3,3}) \oplus_3 M(K_5\backslash e)$ where the 3-sum is along a triad in $K_{3,3}$ and the separating triangle in $K_5\backslash e$. Let $\Sigma_{3,3}$ and Σ_{12}, respectively, be the signed graphs in Figure 4.

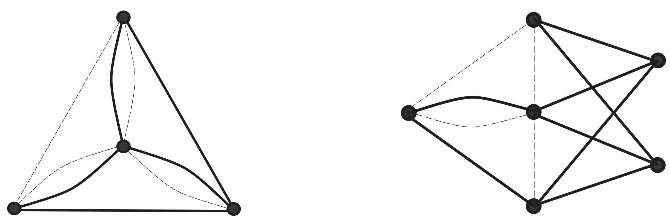


Figure 4.

Proposition 4.6 (Zaslavsky [20, Proposition 4A])**.** *If Σ is a signed graph without isolated vertices, then $M(\Sigma) \cong M^*(K_{3,3})$ iff $\Sigma \cong \Sigma_{3,3}$.*

Proposition 4.7. *If Σ is a signed graph without isolated vertices, then $M(\Sigma) \cong R_{12}$ iff $\Sigma \cong \Sigma_{12}$.*

In order to prove Proposition 4.7 we need Propositions 4.8–4.10. Proposition 4.8 is implied from the main result of [13] and Theorem 1.4.

Proposition 4.8 (Slilaty [13])**.** *If Σ is a tangled signed graph without isolated vertices such that $M(\Sigma)$ is 3-connected, not graphic, not cographic, and not R_{10}, then $\Sigma = \Upsilon \oplus_3 G$ where Υ is tangled, G is all positive, and G has at least 5 vertices.*

Proposition 4.9. *If M_1 and M_2 are binary matroids and $M_1 \oplus_3 M_2$ is 3-connected and contains an $M(K_5)$ minor, then either M_1 or M_2 contains an $M(K_5)$ minor.*

Proof. Suppose that $N \cong M(K_5)$ is a minor of $M_1 \oplus_3 M_2$ such that each $A_i = E(N) \cap E(M_i) \neq \emptyset$. Because $(E(M_1), E(M_2))$ is a 3-separation of $M_1 \oplus_3 M_2$ and this 3-separation induces a separation in any minor, $r_N(A_1) + r_N(A_2) - r(N) \leq 2$. Since N is internally 4-connected, $|A_1| \leq 3$ or $|A_2| \leq 3$, assume the former. Also, since N is simple and has no triads, $|A_1| \leq 3$ implies A_1 is a triangle of N or an independent set of one or two elements. Now when viewing M_1 and M_2 as binary matrices where the triangle along which the 3-sum $M_1 \oplus_3 M_2$ is taken contains three nonzero rows, we see that $M(K_5)$ is a minor of M_2. □

Proposition 4.10. *If Σ is an unbalanced signed graph without isolated vertices, then $M(\Sigma) \cong M(K_5)$ iff Σ is one of the signed graphs in Figure 5.*

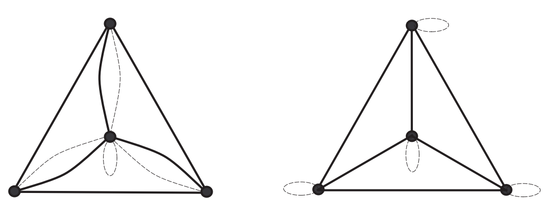


Figure 5.

Proof of Proposition 4.10. Since $M(K_5)$ is 3-connected, it follows from Propositions 2.7 and 2.8 that Σ is vertically 2-connected. It then follows from [14, Theorem 2.6] that Σ either has a balancing vertex, is balanced after removing any negative loops, or is tangled. In the first case, it follows from Proposition 2.6 that Σ is the left-hand signed graph of Figure 5. In the second case, it follows from [14, Proposition 2.2] that Σ is the right-hand signed graph of Figure 5. If Σ is tangled, then by Proposition 2.2 Σ has no negative loops. So since $M(K_5)$ is simple with 10 elements and rank four, Σ has at least four negative digons on four vertices. This will force two vertex-disjoint negative digons, a contradiction. □

Proof of Proposition 4.7. Suppose that $R_{12} = M(\Sigma)$. Since R_{12} is 3-connected and not graphic (because R_{12} has an $M^*(K_{3,3})$ minor), Σ is tangled and vertically 3-connected (Theorems 2.4 and 2.5). Since $R_{12} = M^*(K_{3,3}) \oplus_3 M(K_5 \backslash e)$ contains $M(K_4) \oplus_3 M(K_5\backslash e) = M(K_4 \oplus_3 K_5\backslash e) = M(K_{3,3})$ as a minor, R_{12} is not cographic. Thus $\Sigma = \Upsilon \oplus_3 G$ as in Proposition 4.8 with at least 5 vertices in G. It cannot be that $M(\Upsilon)$ is graphic because otherwise $M(\Sigma) = M(\Upsilon \oplus_3 G) = M(\Upsilon) \oplus_3 M(G)$ would be graphic, a contradiction. Thus $r(\Upsilon) \geq 4$ and so, since $|V(\Sigma)| = 6$ and $|V(G)| \geq 5$, we have that $|V(\Upsilon)| = 4$ and $|V(G)| = 5$.

Since $|V(\Upsilon)| = 4$, it must follow that $M(\Upsilon)$ is cographic unless $M(\Upsilon)$ contains $M(K_5)$ as a submatroid. But then Proposition 4.10 would imply that Υ would contain one of the signed graphs of Figure 5 as a subgraph. But then $\Sigma = \Upsilon \oplus_3 G$ would have a negative loop, a contradiction of tangledness.

Also since $|V(\Upsilon)| = 4$ and $M(\Upsilon)$ is not graphic, it must be that $M(\Upsilon)$ contains $M^*(K_{3,3})$ as a submatroid. Thus $M(\Upsilon) = M^*(H)$ where H is a decontraction of $K_{3,3}$; that is, H is a subdivision of $K_{3,3}$. Note that any 3-edge bond T of H contains three links from three incident branches of H. Using Whitney 2-isomorphisms we can then assume that T is the set of links incident to a 3-valent vertex of H.

Now since $|V(G)| = 5$ it must be that G is planar unless G contains a K_5 subgraph. However G contains no more than 9 edges because $|E(\Upsilon)| \geq 9$ and $|E(\Sigma)| = 12$. Now let a, b, and c be the vertices of G along which the 3-sum with Υ is taken and let x and y be the remaining two vertices of G. Either x and y are adjacent or not. Let these be Cases 1 and 2, respectively.

Case 1: If x and y are adjacent, then since $|E(G)| \leq 9$ we can assume that the simplification of G is contained in $K_5\backslash e$ where without loss of generality e is the xa-link. Now then if T is the edge set of the triangle in $K_5\backslash e$ on vertices a, b, c, then $K_5\backslash e$ is planar and the planar dual graph $(K_5\backslash e)^*$ has T as a vertex bond. So now R_{12} is contained in the parallel connection of $M^*(H)$ and $M^*(K_5\backslash e)$ along a 3-valent vertex in each term. By Proposition 2.12, this parallel connection and then R_{12} are both cographic, a contradiction.

Case 2: If x and y are not adjacent, then since Σ is vertically 3-connected, each of x and y is adjacent to all of a, b, and c. Thus G simplifies to $K_5\backslash e$ where x and y are the 3-valent vertices. So we have that $|E(G)| \geq 9$ and so, since $|E(\Upsilon)| \geq 9$, we get that $|E(G)| = |E(\Upsilon)| = 9$ and so $M(\Upsilon) \cong M^*(K_{3,3})$ and $G = K_5\backslash e$. Thus $\Upsilon \cong \Sigma_{3,3}$ by Proposition 4.6 and so $\Sigma = \Upsilon \oplus_3 G \cong \Sigma_{12}$. □

Proposition 4.11. *R^*_{15} has exactly one triangle A and $R^*_{15}/A \cong M(K_{3,4})$.*

Proof. R^*_{15} is obtained from a $Y\Delta$ switch of $M(G_1)$ along the 3-edge bond T. So now $R^*_{15} = M(G_1)\triangle M(K_4)$. Let P be the prism graph that consists of two vertex-disjoint triangles joined by three links. So now $G_1 = (K_5\backslash e) \oplus_3 P \oplus_3 (K_5\backslash e)$ and so $R^*_{15} = M(K_5\backslash e) \oplus_3 (M(P)\triangle M(K_4)) \oplus_3 M(K_5\backslash e)$. Now $M(P)\triangle M(K_4) = M^*(K_{3,3})$ because $(M(P)\triangle M(K_4))^* = M^*(P)\triangle M^*(K_4) = M(K_5\backslash e)\triangle M^*(K_4) = M(K_{3,3})$ and so $R^*_{15} = M(K_5\backslash e) \oplus_3 M^*(K_{3,3}) \oplus_3 M(K_5\backslash e)$. Now using signed-graphic 3-sums and the signed graph $\Sigma_{3,3}$ in Figure 4 we get that the left-hand signed graph, call it Σ, in Figure 6 has $M(\Sigma) \cong R_{15}^*$.

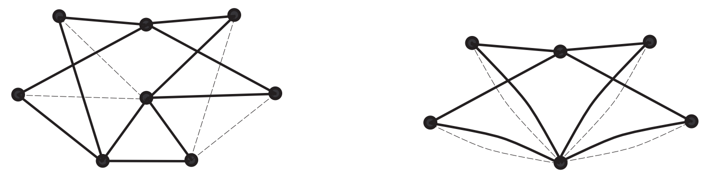


Figure 6.

By inspection we see that there is only one triangle A in $M(\Sigma) \cong R_{15}^*$. Now $R_{15}^*/A = M(\Sigma)/A = M(\Sigma/A)$ and Σ/A is the right-hand signed graph in Figure 6. The graph G_v from Proposition 2.6 that is obtained from Σ/A is $K_{3,4}$ and so $M(\Sigma/A) = M(K_{3,4})$. □

Proof of Proposition 4.1. Suppose that $R_{15} = M(\Sigma)$ where Σ has no isolated vertices. Since R_{15} is 3-connected and is neither graphic nor cographic (Proposition 4.5), R_{15} contains an R_{12}-minor by [10, (7.4) and (14.2)]. Thus Σ contains a Σ_{12} minor by Proposition 4.7. So since R_{15} and R_{12} are both 3-connected, there are three edges c, d, e in Σ such that $\Sigma_{12} \cong \Sigma/c\backslash d\backslash e$.

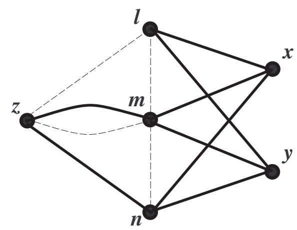


Figure 7.

Consider the labeling of the vertices of $\Sigma/c\backslash d\backslash e \cong \Sigma_{12}$ in Figure 7. Since Σ must be vertically 3-connected (by Theorem 2.5), Σ has minimum degree three.

Claim 1. *Σ has minimum degree four.*

Proof of Claim: Since Σ is vertically 3-connected and has no balancing vertex, the edges incident to any vertex v form a cocircuit of $M(\Sigma)$. So Proposition 4.11 implies that Σ has at most one 3-valent vertex. So by way of contradiction say that v is a 3-valent vertex of Σ. Thus the 3-edge set A in Proposition 4.11 is the collection of edges incident to v and so $M^*(K_{3,4}) \cong R_{15}\backslash A = M(\Sigma\backslash v)$. Since $K_{3,4}$ is nonplanar it follows from [12, Theorem 3] that $\Sigma\backslash v$ is the projective-planar dual signed graph of some imbedding of $K_{3,4}$ in the projective plane. If G is imbedded in the projective plane, then the projective-planar dual signed graph of G is (G^*, σ) where G^* is the topological dual graph of G and σ is a signing such that a circle C in G^* is positive iff C bounds a disk in the projective plane. Up to isomorphism of $K_{3,4}$, the only imbedding $K_{3,4}$ in the projective plane is the one shown in Figure 8 and thus $\Sigma\backslash v$ is isomorphic to the signed graph in Figure 8.

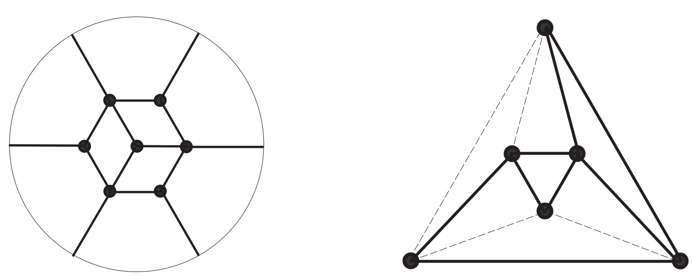


Figure 8.

Note that $\Sigma\backslash v$ is vertically 4-connected. Now since R_{15} is neither graphic, cographic, nor R_{10}, Proposition 4.8 implies that Σ has a 3-separation (A, B) in which $\Sigma{:}B$ is balanced with at least five vertices. Since $\Sigma\backslash v$ is vertically 4-connected, we must then have that all vertices of $\Sigma\backslash v$ are in $\Sigma{:}B$. Given this, it is easily seen that no such 3-separation exists, a contradiction. ♣

Given Claim 1, the 3-valent vertices x and y in $\Sigma/c\backslash d\backslash e$ must have degree at least four in Σ/c. Now since $M(\Sigma)$ is 3-connected, $M(\Sigma/c)$ is 2-connected, regular, and contains an R_{12}-minor. Thus Σ/c is tangled and loopless. Thus d and e are both links in Σ/c and, since $r(\Sigma/c)$=6, Σ/c has the six vertices as in Figure 7. Without loss of generality say that d is incident to x in Σ/c. It cannot be that d is a positive link between x and y because otherwise Σ/c would contain a K_5 minor which would make $M(\Sigma) = R_{15} = M^*(H_1) \oplus_3 M^*(K_{3,3})$ contain an $M(K_5)$ minor and then Proposition 4.9 would imply that one of the cographic matroids $M^*(H_1)$ and $M^*(K_{3,3})$ contains an $M(K_5)$ minor, a contradiction. Furthermore, it cannot be that d is a negative link between x and y because then Σ/c contains two vertex-disjoint negative circles, a contradiction of tangledness. Also, it cannot be that d is a link from x to z of either sign because again we would have two vertex-disjoint negative circles. Finally it cannot be that d is a negative link from x to a vertex in $\{l, m, n\}$ because we would again have two vertex-disjoint negative circles. Thus d is a positive link from x to a vertex in $\{l, m, n\}$. Similarly e must be a positive link from y to a vertex in $\{l, m, n\}$. Since $M(\Sigma)$ has no parallel elements, the endpoints of d and e in $\{l, m, n\}$ are the same. Now in decontracting c to obtain Σ, we cannot leave parallel edges of the same sign and every vertex must have degree at least four (Claim 1) making the only possibilities for Σ the ones given in Figure 9. (Using symmetries including switching at z, flipping horizontally, and permuting the endpoints of c, the reader can verify that these are indeed all of the possibilities for Σ.) In all of these signed graphs, there are two vertex-disjoint negative circles, a contradiction. □

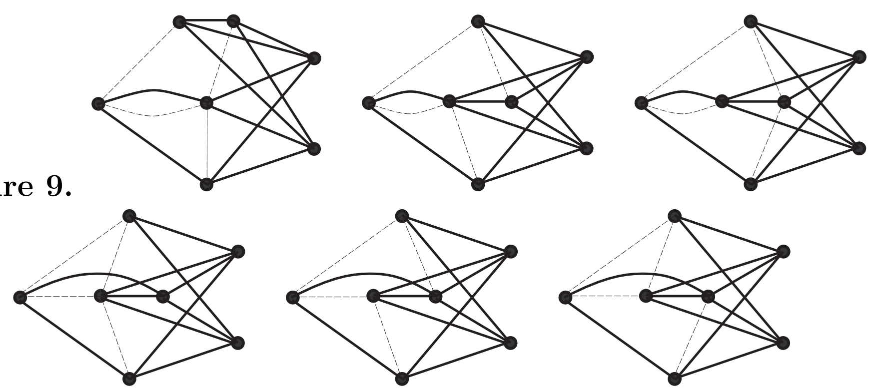


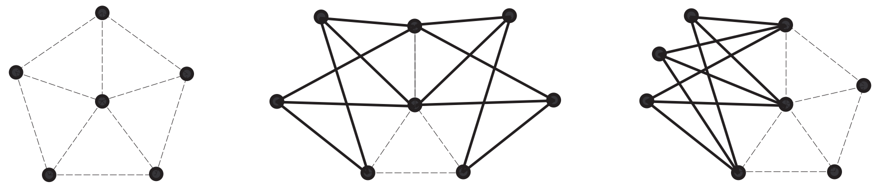
Figure 9.

Proof of Proposition 4.2. We show that $R_{15}\backslash e$ and R_{15}/e are both signed graphic for any $e \in R_{15}$. By symmetry there are two cases to check: when e is in one of the copies of $K_{2,3}$ of G_1 and when e is obtained from the ΔY-exchange on G_1. In the former case, $G_1\backslash e$ and G_1/e are both projective planar. Thus $M^*(G_1\backslash e)$ and $M^*(G_1/e)$ are both signed-graphic by Theorem 1.4 and so $R_{15}/e = M^*(G_1\backslash e)\triangle M(K_4)$ and $R_{15}\backslash e = M^*(G_1/e)\triangle M(K_4)$ are both signed graphic by Proposition 2.11. In the latter case, $R_{15}/e = M^*(G_1)\triangle M(K_4/e)$ is a 1-edge deletion of $M^*(G_1)$ and $R_{15}\backslash e = M^*(G_1)\triangle M(K_4\backslash e)$ is a subdivision of a 2-edge deletion of $M^*(G_1)$ and so both are signed graphic by Theorem 1.4. □

Proposition 4.12. *R^*_{16} has no triangles.*

Proof. Let $-W_5$ be the left-hand signed graph in Figure 10. In [20, Proposition 4A] it is shown that $M(-W_5) = M^*(K_5)$. Since $R_{16} = M^*(K_{3,3}) \oplus_3 M(K_5) \oplus_3 M^*(K_{3,3})$ we can switch edges as necessary and use Proposition 2.12 to get that the middle signed graph, call it Υ, in Figure 10 satisfies $M(\Upsilon) = R_{16}^*$. Evidently $M(\Upsilon)$ has no triangles. $\square$

Figure 10.



Define the matroid $R_{13} = M^*(K_{3,3}) \oplus_3 M(K_5)$ and the signed graph Σ_{13} to be $\Sigma_{12} \cup e$ in which Σ_{12} is as shown in Figure 7 and e is a positive xy-link. Proposition 4.13 can be deduced from Propositions 4.7 and 4.9.

Proposition 4.13. *If Σ is a signed graph without isolated vertices, then $M(\Sigma) \cong R_{13}$ iff $\Sigma \cong \Sigma_{13}$.*

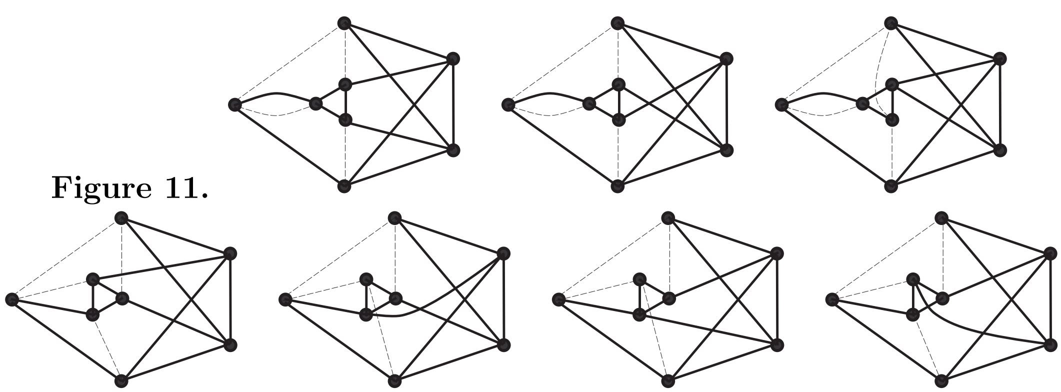
Proof of Proposition 4.3. Say that $R_{16} = M(\Sigma)$ where Σ has no isolated vertices. Since Σ must be vertically 3-connected (Theorem 2.5) and has no balancing vertex (Theorem 2.4), the set of edges incident to any vertex is a cocircuit of $M(\Sigma)$. So since R_{16}^* has no triangles (Proposition 4.12), Σ has minimum degree at least four. Since the number of edges in Σ is 16, Σ must then be 4-regular.

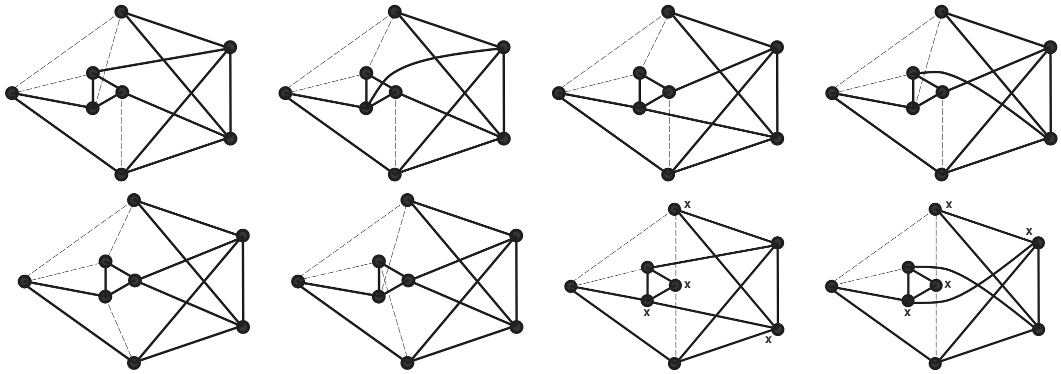
Since $R_{16} = M^*(K_{3,3}) \oplus_3 M(K_5) \oplus_3 M^*(K_{3,3})$ has an $R_{13} = M^*(K_{3,3}) \oplus_3 M(K_5)$ minor, Σ must have a Σ_{13} minor by Proposition 4.13. Thus there are edges c_1, c_2, d such that $\Sigma_{13} \cong \Sigma/c_1/c_2\backslash d$. Since Σ is 4-regular, the degree sequence of Σ/c_1 is 4/4/4/4/4/4/6 and so the degree sequence for $\Sigma/c_1/c_2$ is either 4/4/4/4/6/6 or 4/4/4/4/4/8. Let these be Cases 1 and 2, respectively. In each case use the labeling of $\Sigma/c_1/c_2\backslash d \cong \Sigma_{13}$ in Figure 7. Note that the degree sequence of $\Sigma/c_1/c_2\backslash d$ is 4/4/4/4/4/6.

Case 1: Here d must be a loop on one of the 6-valent vertices of $\Sigma/c_1/c_2$, say vertex v, and that loop must be positive by Proposition 2.2. Without loss of generality, decontracting c_1 splits v and decontracting c_2 splits the other 6-valent vertex of $\Sigma/c_1/c_2$. However, then c_1 and d will be parallel positive links in Σ, a contradiction of the 3-connectedness of $M(\Sigma)$.

Case 2: Here d is a loop on the 8-valent vertex of $\Sigma/c_1/c_2$, call it v, and by Proposition 2.2, d is positive. So now decontracting c_1 and c_2 each split v and so the only possibilities for Σ are as shown in Figure 11.

Figure 11.





All of these signed graphs except the signed graph of the last row second column have two vertex-disjoint negative circles. The vertex-disjoint negative circles are easily seen in all of cases except perhaps the last two where we have marked one negative quadrilateral with x's and the other negative quadrilateral is unmarked. Thus Σ is the signed graph of the last row second column. Note that $\Sigma = \Upsilon \oplus_3 K_5$ where Υ is the signed graph of Figure 8 satisfying $M(\Upsilon) = M^*(K_{3,4})$. Thus $R_{16} \cong M(\Sigma) \cong M^*(K_{3,4}) \oplus_3 M(K_5)$, however, $R_{16} = M^*(K_{3,3}) \oplus_3 M(K_5) \oplus_3 M^*(K_{3,3})$, as defined. As in the proof of Proposition 4.12, the middle signed graph of Figure 10, call it Υ_1, satisfies $M^*(\Upsilon_1) \cong M^*(K_{3,3}) \oplus_3 M(K_5) \oplus_3 M^*(K_{3,3})$. In a similar fashion, the right-hand signed graph of Figure 10, call it Υ_2, satisfies $M^*(\Upsilon_2) \cong M^*(K_{3,4}) \oplus_3 M(K_5)$. Thus $M(\Upsilon_1) \cong M(\Upsilon_2)$.

Claim 1. *If the negative edges of a signed graph Ω form a triangle, then $M(\Omega)$ is graphic.*

Proof of Claim: Let G be the ordinary graph obtained from Ω by replacing the all-negative triangle with a triad on the same vertices. It is easy to check that $M(G) \cong M(\Omega)$. ♣

Claim 2. *Υ_1 has exactly four vertex-deleted subgraphs whose matroids are not graphic and Υ_2 has exactly three vertex-deleted subgraphs whose matroids are not graphic.*

Proof of Claim: For Υ_1, if v is one of its four 3-valent vertices, then $M(\Upsilon_1 \backslash v)$ is not graphic because it contains a $M(-W_5) = M^*(K_5)$ submatroid. If w is any other vertex of Υ_1, then either $\Upsilon_1 \backslash w$ has a balancing vertex or the negative edges of $\Upsilon_1 \backslash w$ form a triangle. In either case $M(\Upsilon_1 \backslash w)$ is graphic by Claim 1 and Proposition 2.6.

For Υ_2, if v is one of the three 3-valent vertices in the upper left, then $M(\Upsilon_2 \backslash v)$ is not graphic because it contains a $M(-W_5) = M^*(K_5)$ submatroid. If w is any other vertex of Υ_2, then $M(\Upsilon_2 \backslash w)$ is graphic because either $\Upsilon_2 \backslash w$ has a balancing vertex or the negative edges of $\Upsilon_2 \backslash w$ form a triangle after switching. ♣

Given the form of the rank function of a signed-graphic matroid $M(\Upsilon)$, a cocircuit is a disjoint union $S \cup B$ where S is a set of edges separating $\Upsilon{:}X$ from $\Upsilon{:}Y$ and B is a minimal balancing set of $\Upsilon{:}X$. Thus a connected hyperplane of $M(\Upsilon)$ is the complement of $S \cup B$ only if either $S = \emptyset$ or $\Upsilon{:}X$ has only one vertex. That is when $S \cup B$ is a minimal balancing set of Υ or $S \cup B$ is the collection of edges incident to a single vertex and that vertex is not a balancing vertex. So by Claim 2, $M(\Upsilon_1)$ has exactly four connected nongraphic hyperplanes and $M(\Upsilon_2)$ has exactly three connected nongraphic hyperplanes. Thus $M(\Upsilon_1) \not\cong M(\Upsilon_2)$, a contradiction. □

Proof of Proposition 4.4. We show that $R_{16} \backslash e$ and R_{16}/e are both signed graphic for any $e \in R_{16}$. By symmetry there are two cases to check: e is an element of one of the two $M^*(K_{3,3})$ terms and e is an element of the $M(K_5)$ term. In the former case $M^*(K_{3,3}\backslash e)$ and $M^*(K_{3,3}/e)$ are both graphic. So since $M(K_5) \oplus_3 M^*(K_{3,3}) \cong M(\Sigma_{13})$, we get that $R_{16}\backslash e = M^*(K_{3,3}/e) \oplus_3 M(\Sigma_{13})$ and $R_{16}/e = M^*(K_{3,3}\backslash e) \oplus_3 M(\Sigma_{13})$ are both signed-graphic by Proposition 2.11.

In the latter case, $K_5\backslash e$ is planar and neither of the two triangles along which the 3-sums are taken is the separating triangle of $K_5\backslash e$. Thus $M^*(K_{3,3}) \oplus_3 M(K_5\backslash e) = M^*(K_{3,3}) \oplus_3 M^*(P)$ where $P = (K_5\backslash e)^*$ is the triangular prism and where the 3-sum is along a triad in each term. Thus $M^*(K_{3,3}) \oplus_3 M^*(P)$ is the cographic matroid as given in Proposition 2.12. So now by Proposition 2.12, $R_{16}\backslash e = M^*(K_{3,3}) \oplus_3 M^*(P) \oplus_3 M^*(K_{3,3})$ is the cographic matroid of the left-hand graph in Figure 12. One can check that this graph is projective planar and so $R_{16}\backslash e$ is signed graphic by Theorem 1.4. By a similar argument R_{16}/e is the cographic matroid of the right-hand graph in Figure 12 and that graph is projective planar. Thus R_{16}/e is signed graphic by Theorem 1.4. □

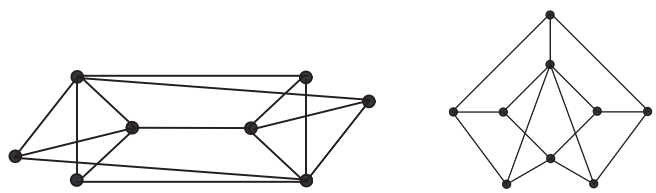


Figure 12.

5 Lemmas for Section 6.

5.1 Lemmas on graphs with $K_{3,3}$ minors.

Lemma 5.1 (Truemper [16, 10.3.9]). *Let G be a graph containing a $K_{3,3}$ minor such that $M(G)$ is 3-connected.*

(1) If G contains a triangle with edge set $\{e_1, e_2, e_3\}$, then G has one of the graphs in Figure 13 as a minor where $\{e_1, e_2, e_3\}$ is shown in bold.

(2) If G has a vertex v of degree 3, then G contains a subdivision of $K_{3,3}$ that uses v as one of its branch vertices.

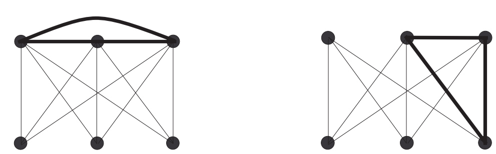


Figure 13.

Lemma 5.2. *Let G be a graph with a $K_{3,3}$ minor such that $M(G)$ is 3-connected. If G contains a 3-edge matching T that is also a bond of G, then G has an H_1 minor (see Figure 3) in which T is the 3-edge bond in H_1 that separates the triangle and the copy of $K_{2,3}$.*

Proof. Since $M(G)$ is 3-connected, we can partition $E(G)$ into A, T, B where A and B are the edge sets of two components of $G \setminus T$. Now if H is a subdivision of $K_{3,3}$ in G, then since H is vertically 2-connected H either uses all the edges of T, two edges of T, or no edges of T. In the first case, without loss of generality, one branch vertex of H is contained in $G{:}A$ and the rest of the branch vertices of H are contained in $G{:}B$. In the second case, without loss of generality, $H \cap (G{:}(A \cup T))$ is a path whose endpoints are in $G{:}B$. In all three cases, G/A contains H/A and H/A is a subdivision of $K_{3,3}$. Also one may show that $M(G/A)$ is 3-connected. Now let v be the vertex of G/A obtained by coalescing the vertices of A. Evidently v has degree 3 in G/A and the edges incident to v are the edges of T. Now by Lemma 5.1, there is a subdivision $\widehat{H}$ of $K_{3,3}$ in G/A that has v as one of its 3-valent vertices. We can now split the vertex v to obtain an H_1 minor containing T as a 3-edge matching. □

Lemma 5.3. *Let G be a graph such that $M(G)$ is 3-connected and G contains a $K_{3,3}$ minor, a triangle T on edges $\{e_1, e_2, e_3\}$, and a 3-valent vertex v not in T. Then G has one of the following:*

(1) a vertical 3-separation (A, B) with $v \in V(A) \setminus V(B)$, $V(T) \subseteq V(A)$, and $|V(B)| \geq 5$ or

(2) one of the graphs in Figure 13 as a minor where $\{e_1, e_2, e_3\}$ is shown in bold and v is a 3-valent vertex not incident to $\{e_1, e_2, e_3\}$.

Proof. By Lemma 5.1, there is a subdivision H of $K_{3,3}$ in G that has v as a branch vertex. By Proposition 2.1 we can rechoose H so that it has no local bridges and still contains v as a branch vertex. In this proof we will use the drawing and labeling of H shown in Figure 14. All edges in Figure 14 represent paths in H except those edges incident to v which are actual edges in H. The crosshatched edges represent paths in H that may have length zero. We use the terms "above" and "below" with respect to Figure 14.

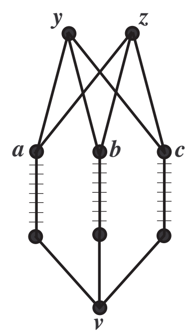


Figure 14.

Now either all vertices of T are in H or all vertices of T are in the same H-bridge, call it B_T. If all vertices of T are in H, then say that $B_T = \emptyset$. Let A_T be the collection of attachments of B_T or if $B_T = \emptyset$, then let $A_T = V(T)$.

Claim 1. *If A_T has a vertex above $\{a, b, c\}$, then $H \cup B_T \cup T$ contains a minor that satisfies Part (2).*

Proof of Claim: Since $M(G)$ is 3-connected and A_T has a vertex above $\{a, b, c\}$, we can choose a 3-element subset A of A_T such that: A contains $V(T) \cap V(H)$, A contains $3 - |V(T) \cap V(H)|$ other vertices chosen from among the attachments of B_T, and A has at least one vertex above $\{a, b, c\}$. Note that if $A = A_T = V(T)$, then not all vertices of T are on the same branch of H because otherwise there is an edge of T that is a local 2-bridge of H, a contradiction. Also, since H has no local bridges, when $|V(T) \cap V(H)| < 3$ we can choose A so that not all of its vertices are on the same branch of H.

Since $M(G)$ is 3-connected, we can use Menger's Theorem to obtain disjoint paths $\gamma_1, \gamma_2, \gamma_3$ in B_T connecting $V(T)$ to A. Now let $\widehat{H}$ be the graph obtained from $H \cup T \cup \gamma_1 \cup \gamma_2 \cup \gamma_3$ by contracting the edges of $\gamma_1 \cup \gamma_2 \cup \gamma_3$. That is $\widehat{H}$ is obtained from H by placing the triangle T on the vertices in A. Now either there are two vertices of A on the same crosshatched path in H or not. If not, then there is $C \subset E(H) \setminus E(T)$ such that $\widehat{H}/C$ is a subdivision H' of $K_{3,3}$ along with the triangle $T' = (\widehat{H}/C){:}\{e_1, e_2, e_3\}$ where all three vertices of T' are branch vertices of H' other than v. We now have a minor satisfying Part (2) in $\widehat{H}/C$. In the former case, since the third vertex of A must then lie above $\{a,b,c\}$, there are C and $D \subseteq E(H)\setminus E(T)$ such that $\widehat{H}/C\setminus D$ is the graph in Figure 15 where e_1, e_2, e_3 are shown in bold.

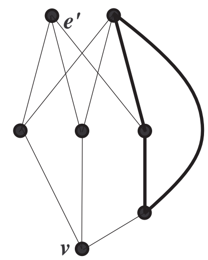


Figure 15.

In $\widehat{H}/C \setminus D$, if we contract the edge e', then we obtain a minor isomorphic to the right-hand graph of Figure 13 which satisfies Part (2) of our conclusion. ♣

Now if A_T has a vertex above $\{a,b,c\}$, then we are done by Claim 1. So suppose that all vertices of A_T are at and below $\{a,b,c\}$. Now rechoose H such that the total of the lengths of the crosshatched paths is a minimum and all vertices of A_T are at and below $\{a,b,c\}$.

Let $H_0 = H \cup B_T \cup T$ and note that an H_0-bridge in G is just an H-bridge that is not B_T and not an edge of T. Let $\mathcal{B}$ be the collection of H_0-bridges with attachments above $\{a,b,c\}$. If $\mathcal{B}=\emptyset$, then we have a 3-separation of G at $\{a,b,c\}$ satisfying Part (1). Otherwise let V be the subgraph of H consisting of v along with the three paths from v to $\{a,b,c\}$. Let $a'=a$ or if there is a bridge in $\mathcal{B}$ with an attachment on the va-path of V below a, then let a' be the lowest such attachment. Since v has degree 3 in G, $a' \neq v$. Define b' and c' similarly. If $\{a',b',c'\}=\{a,b,c\}$, then again there is a 3-separation of G at $\{a,b,c\}$ satisfying Part (1). If not, then we get that V' is a proper subgraph of V where V' is the subgraph of V consisting v along with the three paths from v to $\{a',b',c'\}$. We will now show that there is a subdivision H' of $K_{3,3}$ that contains V' and V and whose branch vertices include v, a', b', c'. After we have obtained H', given that V' is a proper subgraph of V and $A_T \subset V \subset H'$, we will get that the sum of the lengths of the crosshatched paths of H' is less than that sum for H. Hence there is a vertex of A_T above $\{a',b',c'\}$ in H' and so we are done by Claim 1.

Now either there is an H_0-bridge in $\mathcal{B}$ that contains $\{a',b',c'\}$, there is an H_0-bridge in $\mathcal{B}$ that contains two elements from $\{a',b',c'\}$ and no bridge contains all three, or all H_0-bridges in $\mathcal{B}$ contain at most one vertex from $\{a',b',c'\}$. In each case let α be the aa'-path on V, β be the bb'-path on V, χ be the cc'-path on V, Y be the subgraph of H consisting of y along with the three paths from y to $\{a,b,c\}$, and Z be the subgraph of H consisting of z along with the three paths from z to $\{a,b,c\}$. We will call a subdivision of $K_{1,3}$ a *subdivided triad.*

In the first case, let Y' be a subdivided triad in this bridge that has a',b',c' as leaf vertices (which must exist because $M(G)$ is 3-connected) and let $Z' = Z\cup\alpha\cup\beta\cup\chi$ and we have that $H' = V'\cup Y'\cup Z'$ is a subdivision of $K_{3,3}$ that contains V' and V and whose branch vertices include v, a', b', c'.

In the second case, without loss of generality, say that $B\in\mathcal{B}$ contains a' and b' and that either $c'=c$ or $c'\neq c$ and there is $C\in\mathcal{B}$ that contains c'. By the definition of $\mathcal{B}$, B has an attachment on H above $\{a,b,c\}$ and similarly for C. Assume without loss of generality that the attachment for B is on Y. Now if $c'=c$ or the attachment for C is also on Y, then there is a subdivided triad Y' in $Y\cup B\cup C$ which has its 3-valent vertex in the interior of B, has leaf vertices a',b',c', and intersects $\alpha\cup\beta\cup\chi$ at $\{a',b',c'\}$ only. So now let $Z'=Z\cup\alpha\cup\beta\cup\chi$ and we have that $H'=V'\cup Y'\cup Z'$ is a subdivision of $K_{3,3}$ that contains V' and V and whose branch vertices include v, a', b', c'. If $c \neq c'$ and the attachment for C is on Z, then there is a subdivided triad Y' in $Y \cup B \cup \chi$ which has its 3-valent vertex in the interior of B, has leaf vertices a', b', c', contains all of χ, and intersects $\alpha \cup \beta$ at $\{a', b'\}$ only. Also there is a subdivided triad Z' in $Z \cup C \cup \alpha \cup \beta$ which has 3-valent vertex in $C \cup Z$, whose leaf vertices are a', b', c', contains $\alpha \cup \beta$, and intersects χ at c' only. We again have that $H' = V' \cup Y' \cup Z'$ is a subdivision of $K_{3,3}$ that contains V' and V and whose branch vertices include v, a', b', c'.

In the third case, either there is a bridge $B_a \in \mathcal{B}$ that contains a' or $a' = a$. We have the similar property for each of b' and c'. Our desired conclusion follows in much the same fashion as in the previous paragraph. □

5.2 Other lemmas

Theorem 5.4 (Hall [5]). *If $M(G)$ is 3-connected and G contains a K_5 minor, then either $G \cong K_5$ (possibly along with some isolated vertices) or G contains a $K_{3,3}$-subdivision.*

It is almost true that each M_i is 3-connected when $M_1 \oplus_3 M_2$ is 3-connected, the sole exception being some parallel elements along the triangle of summation (Proposition 5.5). Thus we can say that each $si(M_i)$ (i.e, the simplification of M_i) is 3-connected when $M_1 \oplus_3 M_2$ is 3-connected.

Proposition 5.5 (Seymour [10, (4.3)]). *If $M_1 \oplus_3 M_2$ is 3-connected and T is the triangle along which the 3-sum is taken, then each M_i is 3-connected save perhaps for some elements parallel to elements of T.*

Lemma 5.6. *If $M_1 \oplus_3 M_2$ is 3-connected and each M_i is cographic and not graphic, then either $M_1 \oplus_3 M_2$ is cographic or $M_1 \oplus_3 M_2$ has an R_{15}-minor.*

Proof. Let $M_i = M^*(G_i)$ and say T is the triangle along which the 3-sum is taken. Since $si(M_i)$ is 3-connected we can say that G_i is a subdivision of a vertically 3-connected simple graph. Let $\widehat{G}_i$ be obtained from G_i by suppressing vertices of degree 2. Thus any 3-edge bond in $\widehat{G}_i$ is either a vertex bond or a matching. Any series pair of edges in G_i contains at most one edge from T and so in suppressing vertices of degree 2 we need not contract any elements of T and T will still be a 3-edge bond of $\widehat{G}_i$. Thus $\widehat{G}_i{:}T$ is either a vertex bond or a matching. In the case that $\widehat{G}_i{:}T$ is a vertex bond, $G_i{:}T$ is a vertex bond after some possible switching of series pairs of edges and in the case that $\widehat{G}_i{:}T$ is a matching, $G_i{:}T$ is a matching.

Since $M^*(\widehat{G}_i)$ is 3-connected and not graphic Theorem 5.4 implies that $\widehat{G}_i \cong K_5$ or G_i has a $K_{3,3}$-minor. Since $M^*(\widehat{G}_i)$ has a triangle T, we must have the $K_{3,3}$-minor. Now if $G_i{:}T$ is a vertex bond, then Lemma 5.1 yields a $K_{3,3}$-minor of G_i with T as a vertex bond and if $G_i{:}T$ is a matching, then Lemma 5.2 yields a H_1-minor of G_i with T as the matching bond. Now if $G_1{:}T$ and $G_2{:}T$ are both vertex bonds, then $M_1 \oplus_3 M_2$ is cographic by Proposition 2.12 and if say $G_1{:}T$ is a matching, then $M_1 \oplus_3 M_2$ contains $M^*(H_1) \oplus_3 M^*(K_{3,3}) = R_{15}$ as a minor, as required. □

Lemma 5.7. *If $M_1 \oplus_3 M_2$ is 3-connected and signed graphic and each M_i is cographic and not graphic, then $M_1 \oplus_3 M_2$ is cographic.*

Proof. Since $M_1 \oplus_3 M_2$ is signed graphic it cannot contain an R_{15}-minor by Proposition 4.1. So now Lemma 5.6 implies that $M_1 \oplus_3 M_2$ is cographic. □

Lemma 5.8. *If M is a 3-connected regular matroid of rank at least 3 that contains a triangle T, then there is a $M(K_4)$-minor of M that contains T.*

Proof. Bixby's Theorem (see [16, 3.4.36]) says that for every element e of M either the simplification of M/e or the cosimplification of $M\backslash e$ is 3-connected. One can obtain the desired minor by continually applying this fact to elements off of the closure of T until we reach rank 3. Once rank 3 is reached we have an $M(K_4)$-minor. □

The graph $2K_3$ is K_3 with each edge doubled. Lemma 5.9 follows immediately from Menger's Theorem.

Lemma 5.9. *If $M(G)$ is 3-connected save possibly for some parallel edges and T_1 and T_2 are two triangles in G, then there is an $M(2K_3)$-minor of $M(G)$ containing $T_1 \cup T_2$.*

Given a 3-sum $M_1 \oplus_3 (M_2 \oplus_3 M_3)$ we say that M_1 *sums into* M_2 when the triangle of $M_2 \oplus_3 M_3$ that the sum with M_1 is taken along lies in M_2. Given $M_1 \oplus_3 (M_2 \oplus_3 M_3)$ if the triangle $\{e, f, g\}$ of $M_2 \oplus_3 M_3$ along which the 3-sum with M_1 is taken lies neither in M_2 nor M_3, then without loss of generality $M_2 \cap \{e, f, g\} = \{f, g\}$ and $M_3 \cap \{e, f, g\} = \{e\}$. In this case, however, as long as $|M_3| \geq 8$, we get $(M_2 \cup e) \oplus_3 (M_3 \backslash e) = M_2 \oplus_3 M_3$ and so $M_1 \oplus_3 (M_2 \oplus_3 M_3) = M_1 \oplus_3 (M_2' \oplus_3 M_3')$ and M_1 sums into M_2'. In the case that M_1 sums into M_2 we get that $M_1 \oplus_3 (M_2 \oplus_3 M_3) = (M_1 \oplus_3 M_2) \oplus_3 M_3$ where on the right side of the equation M_3 sums into M_2. When we write $M_1 \oplus_3 M_2 \oplus_3 M_3$ we mean $M_1 \oplus_3 (M_2 \oplus_3 M_3)$ where M_1 sums into M_2 and also $(M_1 \oplus_3 M_2) \oplus_3 M_3$ where M_3 sums into M_2. If the triangles T_1 and T_3 of M_2 along which the sums with M_1 and M_3 are taken satisfy $r_{M_2}(T_1 \cup T_3) = 2$, then we say that the three terms are summed *along a common line*.

Lemma 5.10. *If $M_1 \oplus_3 M_2 \oplus_3 M_3$ is 3-connected, M_1 and M_3 are cographic and not graphic, $r(M_2) > 2$, and the three terms are summed along a common line, then $M_1 \oplus_3 M_2 \oplus_3 M_3$ contains an R_{15}-minor.*

Proof. Let T_1 and T_3 be the triangles along which the sums $M_1 \oplus_3 M_2$ and $M_2 \oplus M_3$ are taken. As in the proof of Lemma 5.6, for $i \in \{1, 3\}$, M_i has a $M^*(K_{3,3})$-minor that contains T_i. By Lemma 5.8 we can find a K-minor of M_2 containing $T_1 \cup T_3$ where K is $M^*(K_4)$ with the three edges of one triad subdivided and the resulting 6 edges are $T_1 \cup T_3$. So now $M_1 \oplus_3 M_2 \oplus_3 M_3$ has $M^*(K_{3,3}) \oplus_3 K \oplus_3 M^*(K_{3,3})$ as a minor which by Proposition 2.12 is $M^*(H_1) \oplus_3 M^*(K_{3,3}) = R_{15}$. □

Lemma 5.11. *If $M = M_1 \oplus_3 M_2 \oplus_3 M_3$ is 3-connected, M_1 and M_3 are cographic and not graphic, and M_2 is graphic, then either*

(1) $M_1 \oplus_3 M_2 \oplus_3 M_3$ is cographic,

(2) $M_2 = N_1 \oplus_3 N_2$ where M_1 and M_3 sum into N_1 and N_2 is graphic and of rank at least 4, or

(3) M contains an R_{15} or R_{16}-minor.

Proof. Let T_1 and T_3 be the triangles along which the sums $M_1 \oplus_3 M_2$ and $M_2 \oplus M_3$ are taken. Let $M_i = M^*(G_i)$ for $i \in \{1, 3\}$ and let $M_2 = M(G_2)$. By Proposition 5.5, $M(G_2)$ is 3-connected except perhaps for some doubled edges with $T_1 \cup T_3$. In Case 1 say that G_2 is planar and T_1 and T_3 are both facial triangles up to switching of parallel edges. In Case 2 say that G_2 is nonplanar. In Case 3 say that G_2 is planar and, without loss of generality, T_1 is a separating triangle of G_2.

Case 1: If T_1 and T_3 are facial triangles up to switching of parallel edge pairs, then T_1 and T_3 are vertex bonds in the planar dual graph G_2^*. As in the proof of Lemma 5.6, for each $i \in \{1, 3\}$ either $G_i{:}T_i$ is a vertex bond or a matching. In Case 1.1 say that both $G_1{:}T_1$ and $G_3{:}T_3$ are vertex bonds, in Case 1.2 say that $G_1{:}T_1$ is a vertex bond and $G_3{:}T_3$ is a matching, and in Case 1.3 both $G_1{:}T_1$ and $G_3{:}T_3$ are matchings.

Case 1.1: Here $M_1 \oplus_3 M_2 \oplus_3 M_3 = M^*(G_1) \oplus_3 M^*(G_2^*) \oplus_3 M^*(G_3)$ is cographic by Lemma 2.12.

Case 1.2: Here M_3 has an $M^*(H_1)$-minor as in Lemma 5.2 such that $H_1{:}T_3$ is a matching and M_1 has a $M^*(K_{3,3})$-minor that contains T_1 as a vertex bond. Using Lemma 5.9 on T_1 and T_3 in $M_2 = M(G_2)$ we then obtain $M^*(H_1) \oplus_3 M^*(K_{3,3}) = R_{15}$ as a minor of $M_1 \oplus_3 M_2 \oplus_3 M_3$.

Case 1.3: Here for each $i \in \{1, 3\}$, M_i has an $M^*(H_1)$-minor as in Lemma 5.2 such that $H_1{:}T_i$ is a matching. Thus M_1 has a $M^*(K_{3,3})$-minor in which $H_1{:}T_1$ is a vertex bond and so $M_1 \oplus_3 M_2 \oplus_3 M_3$ has an R_{15}-minor as in Case 1.2.

Case 2: By Theorem 5.4, G_2 is K_5 perhaps with some edges doubled or G has a $K_{3,3}$-subdivision. Let these be Cases 2.1 and 2.2, respectively.

Case 2.1: Let T_1 and T_3 be the two triangles in K_5 that the 3-sums with M_1 and M_3 are taken along. If $|V(T_1) \cap V(T_3)| = 1$, then $M^*(K_{3,3}) \oplus_3 M(K_5) \oplus_3 M^*(K_{3,3}) = R_{16}$ is a minor of M. If $|V(T_1) \cap V(T_3)| = 3$, then $M_1 \oplus_3 M_2 \oplus_3 M_3$ are summed along a common line and so by Lemma 5.10 $M_1 \oplus_3 M_2 \oplus_3 M_3$ has an R_{15}-minor. If $|V(T_1) \cap V(T_3)| = 2$, let e be the link of G_2 that connects the two vertices of $T_1 \cup T_3$ not in $T_1 \cap T_3$. Now T_1 and T_3 share the same vertex set in G_2/e and so $M_1 \oplus_3 (M_2/e) \oplus_3 M_3 = (M_1 \oplus_3 M_2 \oplus_3 M_3)/e$ has its three terms summed along a common line and so by Lemma 5.10, $M_1 \oplus_3 M_2 \oplus_3 M_3$ has an R_{15}-minor.

Case 2.2: Let G' be the graph obtained from G_2 by performing a ΔY switch on T_3. If v is the new 3-valent vertex in G', then v is not on the triangle T_1 and note also that G' still has a $K_{3,3}$ subdivision. Now if Lemma 5.3 Part (1) holds for G', then there is a 3-separation (A, B) of G with T_1 and T_3 in A and $|V(B)| \geq 5$. Thus Part (2) of the current Lemma holds. If Lemma 5.3 Part (2) holds for G', then one can check that G_2 has a $\widehat{K}$ minor that contains $T_1 \cup T_3$ where either $\widehat{K}$ is K_5 with some edges doubled or $\widehat{K}$ is K_4 with the edges of one triangle doubled and this doubled triangle is $T_1 \cup T_3$. As in Case 1, we will get that M has either R_{15} or R_{16} as a minor.

Case 3: Let G' be the graph obtained from G_2 by performing a ΔY switch on T_1. Since T_1 is a separating triangle in G_2 and G_2 is planar, we now have that G' contains a $K_{3,3}$-subdivision with the new 3-valent vertex, call it v, as one branch vertex. In a similar fashion as in Case 2.2 we will get either a 3-separation satisfying part (2) or an R_{15} or R_{16}-minor. □

6 The 3-connected case

Let M be an excluded minor for the class of signed-graphic matroids that is regular and 3-connected. Furthermore, by Theorem 1.3 assume that M is not cographic. Now by Theorem 1.2 and Proposition 5.5, $M = M_1 \oplus_3 M_2$ where each M_i is regular and signed-graphic and each $si(M_i)$ is 3-connected. By Proposition 2.11 and the minimality of M, neither M_1 nor M_2 is graphic. If we assume that both M_1 and M_2 are cographic and not graphic, then Lemma 5.6 and the fact that M is not cographic imply that M has an R_{15}-minor and so since M is minimal, $M \cong R_{15}$. So now assume that M cannot be expressed as a 3-sum of two cographic matroids.

Now if we choose M_1 so that $|M_1|$ is minimal, then it must be that M_1 is cographic. Now among all possible choices for M_1 where M_1 is cographic choose so that $|M_1|$ is maximal. Let T_2 be the triangle along which the sum $M_1 \oplus_3 M_2$ is taken. Now let $k \geq 2$ be the maximum possible integer such that $M = N_1 \oplus_3 N_2 \oplus_3 \cdots \oplus_3 N_k$ where $N_1 = M_1 \cup T_2 \cup \cdots \cup T_k$, the sum of N_i with N_1 is along triangle T_i, each $r(N_i) > 2$, and $r_{N_1}(T_2 \cup \cdots \cup T_k) = 2$. Note that N_1 is still cographic and that by Proposition 2.11 none of $N_1, N_2, \ldots, N_k$ are graphic. In Case 1 say that some N_i for $i \geq 2$ is cographic and in Case 2 say none of $N_2, \ldots, N_k$ are cographic.

Case 1: Without loss of generality say that N_2 is cographic. By assumption M is not a 3-sum of two cographic matroids and so $k \geq 3$. So we have at least three terms summed along a common line with two being cographic and not graphic. So by Lemma 5.10, we have an R_{15}-minor and by the minimality of M we get $M \cong R_{15}$.

Case 2: Since N_2 is neither graphic nor cographic we can write $N_2 = P_1 \oplus_3 P_2$ where $r(P_1) > 2$, P_1 sums into N_1, and P_2 sums into P_1. However by the maximality of k, the triangle along which the 3-sum $P_1 \oplus_3 P_2$ is taken, call it T_2', must satisfy $r_{P_1}(T_2 \cup T_2') > 2$. Therefore, we can now choose P_1 and P_2 so that $|P_1|$ is minimal and we must get that P_1 is either graphic or cographic. In Case 2.1 say that P_1 is cographic and not graphic and in Case 2.2 say that P_1 is graphic.

Case 2.1: In this case we cannot have that $k = 2$ because then $M = (M_1 \oplus_3 P_1) \oplus_3 P_2$ where, by the minimality of M, $M_1 \oplus_3 P_1$ is signed graphic. So since both M_1 and P_1 are cographic and not graphic, Lemma 5.7 implies that $M_1 \oplus_3 P_1$ is cographic which contradicts the maximality of M_1. So now that $k \geq 3$ we have that $M_1 \oplus_3 P_1 \oplus_3 N_3$ is a minor of M and these three terms are summed along a common line and so Lemma 5.10 implies that M has an R_{15}-minor and so $M \cong R_{15}$.

Case 2.2: Among all possible choices for P_1 and P_2 where P_1 is graphic choose so that $|P_1|$ is maximal. Now let $m \geq 2$ be the maximum integer such that $N_2 = Q_1 \oplus_3 Q_2 \oplus_3 \cdots \oplus_3 Q_m$ where $Q_1 = P_1 \cup T_2' \cup \cdots \cup T_m'$, the sum of Q_i with Q_1 is along triangle T_i', each $r(Q_i) > 2$, and $r_{Q_1}(T_2' \cup \cdots \cup T_m') = 2$. Note that Q_1 is still graphic and by the maximality of $|P_1|$ none of $Q_2, \ldots, Q_m$ are graphic. In Case 2.2.1 say that at least one of $Q_2, \ldots, Q_m$ is cographic and $k = 2$, in Case 2.2.2 say that at least one of $Q_2, \ldots, Q_m$ is cographic and $k > 2$, and in Case 2.2.3 say that none of $Q_2, \ldots, Q_m$ are cographic.

Case 2.2.1: Without loss of generality say that Q_2 is cographic. Now consider $M_1 \oplus_3 P_1 \oplus_3 Q_2$. Since M_1 and Q_2 are cographic and not graphic we can apply Lemma 5.11. It cannot be that part (1) holds because then $M = M_1 \oplus_3 P_1 \oplus_3 M'$ where $M_1 \oplus_3 P_1$ is cographic, a contradiction of the maximality of M_1. If part (2) of Lemma 5.11 holds, then $M = M'' \oplus_3 M(G)$ where M'' is signed graphic by the minimality of M and $M(G)$ is graphic. But now Lemma 2.11 implies that M is signed graphic, a contradiction. Thus M has an R_{15} or R_{16}-minor and so $M \cong R_{15}$ or R_{16}.

Case 2.2.2: Without loss of generality say that Q_2 is cographic. Now consider $(N_1 \oplus_3 N_3 \oplus_3 P_1) \oplus_3 Q_2$ minus any parallel edges along triangles. Since the terms in $N_1 \oplus_3 N_3 \oplus_3 P_1$ are summed along a common line and since Q_2 sums into P_1 which is graphic we can use Lemma 5.9 to get that $N_1 \oplus_3 N_3 \oplus_3 Q_2$ is a 3-connected minor of M where the three terms are summed along a common line. So now by Lemma 5.10 we get an R_{15}-minor of $N_1 \oplus_3 N_3 \oplus_3 Q_2$ and so $M \cong R_{15}$.

Case 2.2.3: Since Q_2 is neither graphic nor cographic $Q_2 = R_1 \oplus_3 R_2$ where $r(R_1) > 2$, $T_2' \subset R_1$, and by the maximality of m the triangle along which the 3-sum is taken, call it T_2'', satisfies $r_{Q_2}(T_2'' \cup T_2') > 2$. As before we can choose R_1 so as to minimize $|R_1|$ which will then make R_1 either graphic or cographic. In Case 2.2.3.1 say that R_1 is cographic and not graphic and $k = 2$, in Case 2.2.3.2 say that R_1 is cographic and not graphic and $k > 2$, and Case 2.2.3.3 say that R_1 is graphic.

Case 2.2.3.1: Consider the minor $M_1 \oplus_3 P_1 \oplus_3 R_1$ of M and we are done as in Case 2.2.1.

Case 2.2.3.2: Consider the minor $(N_1 \oplus_3 N_3 \oplus_3 P_1) \oplus_3 R_1$ of M and we are done as in Case 2.2.2.

Case 2.2.3.3: Since R_1 is graphic, we can use Lemma 5.9 on R_1 to obtain $N_2' = Q_1 \oplus_3 Q_3 \oplus_3 \cdots \oplus_3 Q_m \oplus_3 R_2$ as a minor of $N_2 = (Q_1 \oplus_3 Q_3 \oplus_3 \cdots \oplus_3 Q_m) \oplus_3 (R_1 \oplus_3 R_2)$ where all terms in the sum for N_2' are along a common line. So now $M'' = N_1 \oplus_3 N_2' \oplus_3 N_3 \oplus_3 \cdots \oplus_3 N_k$ is a minor of M and is 3-connected.

Let $n \geq 1$ be the largest integer such that $R_2 = S_1 \oplus_3 \cdots \oplus_3 S_n$ where all terms are summed along a common line parallel to T_2'' in R_2 and each $r(S_i) > 2$. It cannot be that any S_i from $S_1, \ldots, S_n$ is graphic because then $M = M' \oplus_3 S_i$ would be signed graphic by Proposition 2.11 and the minimality of M. So then either there is some S_i that is cographic or not. In the case that there is some S_i that is cographic and $k = 2$, we get that M'' has an R_{15} or R_{16}-minor in a similar way as in Case 2.2.1 and hence $M \cong R_{15}$ or R_{16}. In the case that some S_i is cographic and $k > 2$, then we get that M'' has an R_{15}-minor in a similar way as in Case 2.2.2 and hence $M \cong R_{15}$. In the case that none of $S_1, \ldots, S_n$ are cographic, we can repeat this process in Case 2.2.3 on S_1 in M''. Eventually this process must halt with the conclusion that $M \cong R_{15}$ or R_{16}.

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