The signed-graphic representations of wheels and whirls

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Abstract

We characterize all of the ways to represent the wheel matroids and whirl matroids using frame matroids of signed graphs. The characterization of wheels is in terms of topological duality in the projective plane and the characterization of whirls is in terms of topological duality in the annulus.

1 Introduction

Throughout this paper we assume that the reader is familiar with matroid theory as in [3]. Let $\mathcal{W}_n$ and $\mathcal{W}^n$ denote, respectively, the *wheel* and *whirl* matroids of rank n. Tutte's "Wheels and Whirls Theorem" from [9] and Seymour's "Splitter Theorem" from [4] tell us that wheels and whirls play a special role within the class of 3-connected matroids. That is, for induction proofs within the class of 3-connected matroids, wheels and whirls can always be used as a base case. Thus more knowledge of the structure of $\mathcal{W}_n$ and $\mathcal{W}^n$ is desirable. In this paper we will find all signed graphs whose frame matroids are $\mathcal{W}_n$ and all signed graphs whose frame matroids are $\mathcal{W}^n$. The classification for $\mathcal{W}_n$ is in terms of topological duality in the projective plane and the classification for $\mathcal{W}^n$ is in terms of topological duality in the annulus.

In Section 2 we will state definitions and background results. In Section 3 we will define and discuss our notion of imbedding signed graphs in the annulus. In Section 4 we will state and prove our main results for $\mathcal{W}_n$ and $\mathcal{W}^n$.

2 Definitions and background

Matroids We assume that the reader is familiar with matroid theory as in [3]. We use the terminology and notation for matroids found in [3]. Proposition 2.1 is a characterization of matroid duality from [5, §2]

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that we will use in this paper.

Proposition 2.1 (Slilaty [5]). *If M and N are matroids on E, then $M^* = N$ iff*

(1) $r(M) + r(N) = |E|$ and

(2) for each circuit C of M and each circuit D of N, $|C \cap D| \neq 1$.

Graphs We denote the vertex set of a graph G by $V(G)$ and its edge set by $E(G)$. A graph has four types of edges: *links*, *loops*, *half edges*, and *loose edges*. Links have their ends attached to distinct vertices, loops have both ends attached to the same vertex, half edges have one end attached to a vertex and the other unattached, and loose edges have both ends unattached. A graph containing neither half edges nor loose edges is called an *ordinary* graph. The notion of half edges and loose edges is used in [10]. It is most always the case that this notion is unnecessary in the study of signed graphs and their matroids, but in this paper we will use of half and loose edges in a way that cannot be avoided.

A graph is *connected* if it has no loose edges and has a path connecting any two vertices. A graph on $n \geq k+1$ vertices is called *vertically k-connected* if there are no $r < k$ vertices whose removal leaves a disconnected subgraph.

If $X \subseteq E(G)$, then we denote the subgraph of G consisting of the edges in X and all vertices incident to an edge in X by $G{:}X$. A graph is called *separable* if it has an isolated vertex, or there is a partition (A, B) with nonempty parts of the edges of G such that $|V(G{:}A) \cap V(G{:}B)| \leq 1$. A nonseparable graph on at least three vertices is vertically 2-connected but a vertically 2-connected graph is nonseparable iff it does not contain any loops or half edges. A *block* is a maximal subgraph that is either an isolated vertex or nonseparable.

A *circle* is a vertically 2-connected ordinary graph (i.e., a simple closed path). We denote the binary *cycle space* of an ordinary graph G by $Z(G)$. It is the vector subspace of $\mathbb{Z}_2^{E(G)}$ whose elements are edge sets of subgraphs in which each vertex has even degree.

Graphic Matroids Given an ordinary graph G, the *graphic matroid* $M(G)$ is the matroid whose element set is $E(G)$ and whose circuits are the edge sets of circles in G. If $X \subseteq E(G)$, then $r(X) = |V(G{:}X)| - c(G{:}X)$ where $c(G{:}X)$ denotes the number of components of $G{:}X$. The graphic matroid $M(G)$ is connected iff G is nonseparable save any isolated vertices.

Signed Graphs Given a graph G, let $E'(G)$ denote the collection of links and loops of G. A *signed graph* is a pair $\Sigma = (G, \sigma)$ in which $\sigma : E'(G) \to \{+1, -1\}$. A circle in a signed graph Σ is called *positive* if the product of signs on its edges is positive, otherwise the circle is called *negative*. If H is a subgraph of Σ, then H is called *balanced* if it has no half edges and all circles in H are positive. A *balancing vertex* is a vertex of an unbalanced signed graph whose removal leaves a balanced subgraph. Not all unbalanced signed graphs have balancing vertices. When drawing signed graphs, positive edges are represented by solid curves and negative edges by dashed curves. We write $\|\Sigma\|$ to denote the underlying graph of Σ.

A *switching function* on a signed graph $\Sigma = (G, \sigma)$ is a function $\eta : V(\Sigma) \to \{+1, -1\}$. The signed graph $\Sigma^\eta = (G, \sigma^\eta)$ has sign function σ^η on $E'(G)$ defined by $\sigma^\eta(e) = \eta(v)\sigma(e)\eta(w)$ where v and w are the end vertices (or end vertex) of the link or loop e. The signed graphs Σ and Σ^η have the same list of positive circles. When two signed graphs Σ and Υ satisfy $\Sigma^\eta = \Upsilon$ for some switching function η, the two signed graphs are said to be *switching equivalent*. An important notion in the study of signed graphs is that two signed graphs with the same underlying graph are switching equivalent iff they have the same list of positive circles (see [10, Prop. 3.2]). Given $\sigma : E'(G) \to \{+1, -1\}$, there is an induced linear transformation $\hat{\sigma} : Z(G) \to \mathbb{Z}_2$ in which for $H \subseteq E(G)$, $\hat{\sigma}(H) = \sum_{e \in H} \sigma'(e)$ where $\sigma'(e) = 0$ iff $\sigma(e) = +1$. Evidently (G, σ_1) and (G, σ_2) are switching equivalent iff $\hat{\sigma}_1 = \hat{\sigma}_2$. Thus we can define a signed graph up to sign switching by the pair $(G, \hat{\sigma})$. Conversely, if $\phi : Z(G) \to \mathbb{Z}_2$, then there is $\sigma : E'(G) \to \{+1, -1\}$ such that $\hat{\sigma} = \phi$. The signing σ is constructed by taking a maximal forest F of G and defining $\sigma(e) = +1$ iff $e \in F$ or the unique circle C_e in $F \cup e$ has $\phi(C_e) = 0$.

Signed-graphic matroids The matroid of a signed graph introduced in [10] is often called the *frame matroid* or *bias matroid* (see [11]). Within this paper we simply call this matroid of a signed graph a *signed-graphic matroid*. Signed-graphic matroids are precisely the minors of Dowling geometries for the group of order two.

We denote the signed-graphic matroid of Σ by $M(\Sigma)$. The element set of $M(\Sigma)$ is $E(\Sigma)$ and a circuit is either a loose edge, the edge set of a positive circle, or the edge set of a subgraph in which all circles are negative and is a subdivision of one of the two graphs shown in Figure 2.2 where a negative loop may be replaced by a half edge. The latter type of circuit is called a *handcuff*.

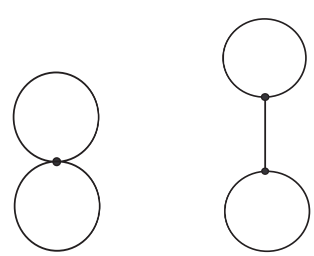

Figure 2.2.

Given this definition of circuits, half edges and negative loops are indistinguishable in $M(\Sigma)$ and positive loops and loose edges are indistinguishable in $M(\Sigma)$. That is, if Σ' is obtained from Σ by exchanging a half edge for a negative loop or a loose edge for a positive loop, then $M(\Sigma) = M(\Sigma')$. We use that term *joint* to mean an edge that is either a half edge or negative loop.

Since switching a signed graph does not change the list of positive circles, $M(\Sigma) = M(\Sigma^\eta)$ for any switching function η. Conversely, if $\|\Sigma\| = \|\Upsilon\|$ and $M(\Sigma) = M(\Upsilon)$, then Σ and Υ must have the same list of positive circles which is true iff Σ and Υ are switching equivalent.

If $X \subseteq E(\Sigma)$, then the rank of X is $r(X) = |V(\Sigma{:}X)| - b(\Sigma{:}X)$ where $b(\Sigma{:}X)$ denotes the number of balanced components of $\Sigma{:}X$ (see [10, Theorem 5.1(j)]). By convention, a loose edge is not considered to contribute to the number of balanced components of $\Sigma{:}X$. Two situations in which a signed-graphic matroid $M(\Sigma)$ is not connected are when Σ is disconnected after removing isolated vertices and when Σ is the one-vertex join of Υ_1 and Υ_2 with Υ_1 balanced.

If Σ is a balanced signed graph, then the ordinary graph G obtained by removing any loose edges from Σ satisfies $M(G) = M(\Sigma)$ up to addition of matroid loops. Two other classes of signed graphs that have graphic matroids are joint-unbalanced signed graphs and signed graphs with balancing vertices. A signed graph is called *joint unbalanced* when it is balanced aside from the existence of joints. If Σ is in one of these classes, then Σ is obtained canonically as described below from an ordinary graph G such that $M(G) = M(\Sigma)$ up to addition of matroid loops.

Let Σ be a joint-unbalanced signed graph. Let G be the ordinary graph obtained from Σ by removing loose edges, adding a new vertex v, and replacing each joint of Σ with a link from the joint endpoint to v. Proposition 2.3 is easy to verify.

Proposition 2.3. *If Σ is joint-unbalanced, then the ordinary graph G obtained as above satisfies $M(\Sigma) = M(G)$ up to addition of matroid loops.*

Let Σ have a balancing vertex v. By sign switching we may assume that all negative links of Σ are incident to v. Let G be the ordinary graph obtained from Σ by removing loose edges and then splitting v into two vertices v_+ and v_- where positive links incident to v are incident to v_+, negative links incident to v are incident to v_-, and joints incident to v are links between v_+ and v_-. Proposition 2.4 is easy to verify.

Proposition 2.4. *If Σ has a balancing vertex, then the ordinary graph G obtained as above satisfies $M(\Sigma) = M(G)$ up to addition of matroid loops.*

Since graphic matroids are binary matroids, we see from Propositions 2.3 and 2.4 that signed graphs with balancing vertices and joint-unbalanced signed graphs have binary matroids. Outside of these two classes of signed graphs, Theorem 2.5 from [8, Thm. 3.6] characterizes the vertically 2-connected signed graphs with binary matroids.

Theorem 2.5. *If Σ is vertically 2-connected, is unbalanced, has no balancing vertex, and is not joint unbalanced, then $M(\Sigma)$ is binary iff Σ is jointless and has no two vertex-disjoint negative circles.*

Since $\mathcal{W}_n$ and $\mathcal{W}^n$ are both 3-connected matroids, we present some useful facts about signed graphs whose matroids are 3-connected in Proposition 2.6.

Proposition 2.6. *Let Σ be an unbalanced signed graph without isolated vertices such that $M(\Sigma)$ is 3-connected.*

(1) Σ is vertically 2-connected.

(2) If Σ has no balancing vertex, then for every $v \in V(\Sigma)$, the edges incident to v form a cocircuit of $M(\Sigma)$.

(3) If Σ has no balancing vertex, is jointless, and has no two vertex-disjoint negative circles, then Σ is vertically 3-connected.

Proof. (1) It must be that Σ is connected, because a signed graph with edges in two or more components will have a disconnected matroid. By way of contradiction assume that Σ has a vertical 1-separation (X, Y). Thus $r(X) + r(Y) - r(\Sigma) = v_X - b_X + v_Y - b_Y - v_\Sigma = 1 - (b_X + b_Y) \leq 1$, a contradiction of $M(\Sigma)$ being 3-connected. Thus Σ is vertically 2-connected.

(2) Since Σ is unbalanced, vertically 2-connected, and does not have a balancing vertex, $r(\Sigma \backslash v) = r(\Sigma) - 1$ for any $v \in V(\Sigma)$ and if e is an edge incident to v, then e is a link or joint and so $r((\Sigma \backslash v) \cup e) = r(\Sigma)$. Thus the collection of edges incident to v is a cocircuit.

(3) This is [7, Thm. 4.1]. □

Imbeddings An imbedding of a graph G in a closed surface is called an *open 2-cell* imbedding if the interior of each face of G in the surface is homeomorphic to an open 2-cell. The topological dual graph of G imbedded in S is denoted by G^*. Theorem 2.7 and Corollary 2.8 are results of J. Edmonds from [1].

Theorem 2.7 (Edmonds)**.** *A one-to-one correspondence between the edges of two connected graphs is a duality with respect to some 2-cell surface imbedding iff for each vertex v of each graph, the edges which meet v correspond in the other graph to the edges of a subgraph G_v which is connected and which has an even number of edge ends to each of its vertices (where the image in G_v of a loop at v is counted twice).*

Corollary 2.8 (Edmonds)**.** *A necessary and sufficient condition for a graph G to have a 2-cell imbedding in a surface of Euler characteristic χ is that it have an edge correspondence with another graph G^* for which*

(1) the conditions of Theorem 2.7 are satisfied and

(2) $|V(G)| - |E(G)| + |V(G^*)| = \chi$.

If G has a 2-cell imbedding in S, then let $B(G)$ denote the subspace of $Z(G)$ generated by the facial boundary cycles of the imbedding. By invariance of homology (see, for example, [2, Ch. 5]) if G and H have 2-cell imbeddings S, then $Z(G)/B(G) \cong Z(H)/B(H)$.

Consider a 2-cell imbedding of a graph G in the projective plane with projective-planar dual graph G^*. It is well known that $Z(G^*)/B(G^*) \cong \mathbb{Z}_2$. So let $\natural : Z(G^*) \to \mathbb{Z}_2$ be the natural map defined by this quotient. We call $\Sigma = (G^*, \natural)$ the *projective-planar dual signed graph* of the imbedded graph G. A circle in Σ is negative iff it is a nonseparating closed curve in the projective plane. Theorem 2.9 below is found in [5, §2].

Theorem 2.9. *If G is a connected graph that is 2-cell imbedded in the projective plane, then $M^*(G) = M(G^*, \natural)$.*

3 Imbeddings and duality in the annulus

Imbeddings of graphs in surfaces is a popular and well-studied topic in graph theory. Usually the surfaces that are considered are closed surfaces. In this section we will develop a notion of imbedding signed graphs in the annulus, one of the simplest connected surfaces with boundary. This notion is particularly appealing in that it makes distinct uses of half edges and negative loops.

Given a signed graph Σ without half and loose edges, Σ is called *cylindrical* if it is connected and Σ imbeds in the plane with exactly two negative faces. If we remove two disks from the interior of these negative faces, then we have Σ imbedded in the interior of an annulus (or cylinder) in which its positive circles are contractible and its negative circles wind once around the annulus. Thus Σ subdivides the annulus into two annuli and $n \geq 0$ 2-cells.

So now given a signed graph Σ with half edges H_Σ we say that Σ is *annular* if Σ is connected, $\Sigma \setminus H_\Sigma$ is cylindrical, and we can draw in the half edges H_Σ without crossings as curves from their endpoints to the boundary of the annulus. Now suppose that Σ is imbedded in the annulus so that it touches $i \in \{0, 1, 2\}$ of the two circular boundaries of the annulus. Thus $\Sigma \setminus H_\Sigma$ subdivides the annulus into 2 annuli and $n \geq 0$ 2-cells and Σ subdivides the annulus into $2 - i$ annuli and $n + |H_\Sigma|$ 2-cells. On the left in Figure 3.1 is an example of a signed graph with three half edges imbedded in the annulus touching only one circular boundary.

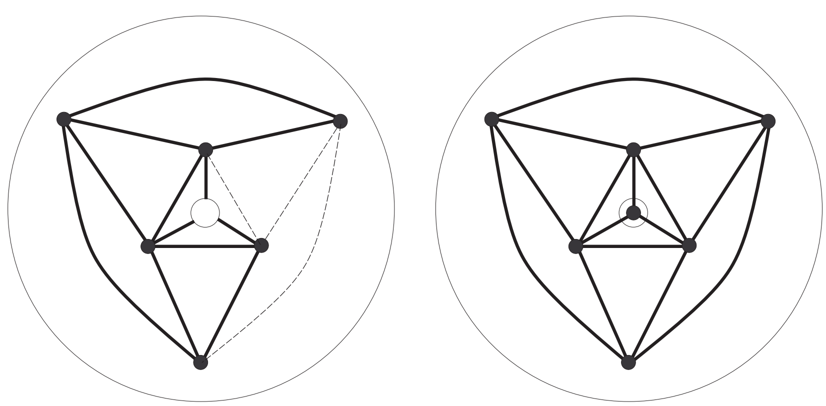

Figure 3.1.

Given Σ imbedded in the annulus define the *faces* of the imbedding as the 2-cells into which Σ subdivides the annulus. Let $F(\Sigma)$ be the collection of faces of the imbedding of Σ. We will now construct an ordinary graph Γ imbedded in the sphere that we will associate with Σ. We will call Γ the *planar graph associated with* Σ. Let γ_1 and γ_2 be the boundary circles of the annulus. If there are no half edges touching γ_j, then attach a disk D_j to γ_j. If there are half edges touching γ_j, then attach a disk D_j to γ_j and extend these half edges (without crossing) to a common endpoint in the interior of D_j. On the right of Figure 3.1 is the planar graph associated with the imbedding on the left of Figure 3.1. Let $i \in \{0, 1, 2\}$ be the number of circular boundaries of the annulus that Σ touches. We have now constructed a graph Γ with $|V(\Gamma)| = |V(\Sigma)| + i$, $|E(\Gamma)| = |E(\Sigma)|$, and $|F(\Gamma)| = |F(\Sigma)| + (2 - i)$. We therefore now have Proposition 3.2 which is a variation of Euler's formula for the annulus.

Proposition 3.2. *If Σ is imbedded in the annulus, then $|V(\Sigma)| - |E(\Sigma)| + |F(\Sigma)| = 0$.*

Proof. Given the planar graph Γ associated with Σ, Euler's formula yields $|V(\Gamma)| - |E(\Gamma)| + |F(\Gamma)| = 2$. Therefore $|V(\Sigma)| + i - |E(\Sigma)| + |F(\Sigma)| + (2 - i) = 2$ and so $|V(\Sigma)| - |E(\Sigma)| + |F(\Sigma)| = 0$. □

Let G be a graph imbedded in the annulus with at least one circle orbiting the annulus and let $\overline{G}$ be the maximal ordinary subgraph of G. Let $B(\overline{G})$ be the subspace of $Z(\overline{G})$ generated by the boundary cycles of $F(\overline{G})$. Thus $Z(\overline{G})/B(\overline{G}) \cong \mathbb{Z}_2$. Thus the natural map $\natural : Z(\overline{G}) \to \mathbb{Z}_2$ defines a cylindrical signed graph $(G, \natural)$.

Now let Γ be the associated planar graph of G and let γ_1 and γ_2 be the two boundary circles of the annulus. Let G^* be the graph whose associated planar graph is Γ^* and that has half edges touching γ_i iff G does not have half edges touching γ_i. (This latter property is important in the proof of Theorem 3.4.) We call G^* the *annular dual graph* of G. In Figure 3.3 is an example of a graph imbedded along with its annular dual graph. So now given a signed graph $\Sigma = (G, \sigma)$ imbedded in the annulus, the *annular dual signed graph* is defined as $\Sigma^* = (G^*, \natural)$.

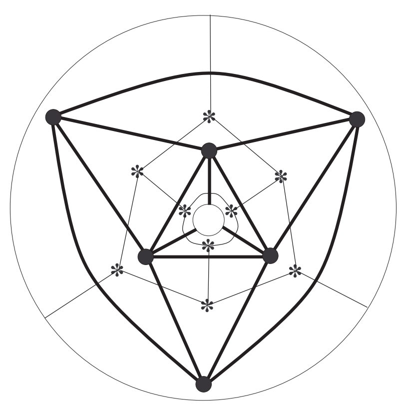

Figure 3.3.

Theorem 3.4. *If Σ is imbedded in the annulus, then $M^*(\Sigma) = M(\Sigma^*)$.*

Proof. Our proof will use Proposition 2.1. Of course Σ and Σ^* are not really on the same edge set but on corresponding edge sets. Furthermore, if $C \subseteq E(\Sigma)$ and $D \subseteq E(\Sigma^*)$, then topologically the intersection of

C with D is a collection of points with that exact same order as the set-theoretic intersection of C with the dual edges of D.

By Proposition 3.2, $|V(\Sigma)| - |E(\Sigma)| + |F(\Sigma)| = 0$. Thus $|V(\Sigma)| + |V(\Sigma^*)| = |E(\Sigma)|$. Since Σ must be connected and unbalanced, $r(\Sigma) = |V(\Sigma)|$ and the associated planar graphs Γ and Γ^* are connected. Thus every component of Σ^* that has a vertex is unbalanced. Thus $r(\Sigma^*) = |V(\Sigma)|$ and so $r(\Sigma)+r(\Sigma^*) = |E(\Sigma)|$. So now by Proposition 2.1 we can complete the proof by showing that for every circuit C in $M(\Sigma)$ and circuit D in $M(\Sigma^*)$, $|C \cap D| \neq 1$.

First, since Σ is connected, C is not a loose edge. If D is a loose edge in Σ^*, then one can check that the dual of D in Σ is a coloop of $M(\Sigma)$, thus $C \cap D = \emptyset$. So for the remainder of the proof each of C and D is either a positive circle or a handcuff. In the first case, suppose both are positive circles. In the second, that one is a handcuff, and in the third that both are handcuffs. In the cases where C is a handcuff write $C = C_1 \cup \gamma_C \cup C_2$ where C_i is a minimal unbalanced subgraph of C (i.e., a negative circle or half edge) and γ_C is the minimal connecting path between C_1 and C_2. Note that γ_C may consist of just a single vertex. Similarly we will write $D = D_1 \cup \gamma_D \cup D_2$ when D is a handcuff.

Case 1: Here $|C \cap D|$ is even (in particular $|C \cap D| \neq 1$) because circles on the annulus or plane separate the surface into two regions and because C and D only intersect at transverse crossings.

Case 2: Without loss of generality say that C is a handcuff and D is a positive circle. Thus D encloses a disk on the annulus. Say that u and v are the endpoints (or endpoint) of γ_C. If both of u and v are contained on the inside disk of D, then since C_i is a negative circle or half edge, $|C_i \cap D| \geq 1$ and so $|C \cap D| \geq 2$. If one of u and v is contained in the inside of disk of D, then, without loss of generality, $|C_1 \cap D| \geq 1$ and $|\gamma_C \cap D| \geq 1$. Thus $|C \cap D| \geq 2$. If neither u nor v is contained in the interior of D, then $|D \cap \gamma_C|$ is even. If $|D \cap \gamma_C| \geq 2$, we are done. So suppose that $|D \cap \gamma_C| = 0$. When C_i is a circle, $|C_i \cap D|$ is even and when C_i is a half edge $|C_i \cap D| = 0$ because the endpoint of C_i is not in the interior of D. In both cases $|C \cap D|$ will be even when $|D \cap \gamma_C| = 0$.

Case 3: First assume that neither C nor D contains half edges. Note that when C_i and D_j are both circles that $|C_i \cap D_j|$ is even. So we may assume that $|C_i \cap D_j| = 0$ when they are both circles. Thus C_1 and C_2 separate the annulus into three annuli and each D_i is contained entirely in one of these annuli. If D_1 and D_2 are contained in different annuli, then $|\gamma_D \cap C| \geq 1$ and $|\gamma_C \cap D| \geq 1$. Thus $|C \cap D| \geq 2$. If D_1 and D_2 are contained in the same annulus, then $|(C_1 \cup C_2) \cap \gamma_D| = 0$, $|(D_1 \cup D_2) \cap \gamma_C|$ is even, and $|\gamma_C \cap \gamma_D| = 0$ unless γ_C and γ_D are contained in the same annulus; however, if they are contained in the same annulus, then $|(D_1 \cup D_2) \cap \gamma_C| \geq 2$. In all cases, $|C \cap D| \neq 1$.

Second assume that C or D contains a half edge. By the definition of duality in the annulus, each boundary circle of the annulus is touched by half edges of exactly one of Σ and Σ^*. Without loss of generality, the five cases to consider here are: C contains two half edges and D contains two half edges, C

contains two half edges and D contains one, C contains two half edges and D contains none, C contains one half edge and D contains one, and C contains one half edge and D contains none.

Case 3.1: Since C and D both have two half edges, C is a circle in the planar graph associated with Σ and D is a circle in the planar graph associated with Σ^*. Thus $|C \cap D|$ is even.

Case 3.2: Since C has two joints and D has one, C is a circle in the planar graph associated with Σ. Let v_1 and v_2 be the endpoints (or endpoint) of γ_D where v_i is contained in D_i. Say that D_1 is the half edge of D. Either v_1 is contained in the interior of the associated circle of C or not. If so, then since the half edge D_1 must touch the boundary of the annulus, $|C \cap D_2| \geq 2$ and so $|C \cap D| \neq 1$. If not, then $|D_2 \cap C| \geq 2$ or $|D_2 \cap C| = 0$. In the former case we are done and in the latter case, D_2 separates the annulus into two annuli with C in one and $D_1 \cup \gamma_D$ in the other. Thus $C \cap D = \emptyset$.

Case 3.3: There are two subcases here. In the first, both of the half edges of C touch the same boundary circle and in the second case, the half edges touch different boundary circles.

Case 3.3.1: This case is very similar to Case 3.2.

Case 3.3.2: Here C is imbedded as a simple path connecting the two boundary circles of the annulus. Thus each $|C \cap D_i| \geq 1$ and so $|C \cap D| \geq 2$.

Case 3.4: Without loss of generality say that C_1 and D_1 are negative circles. So $|C \cap D| \geq 2$ when $|C_1 \cap D_1| \neq 0$. So assume that $|C_1 \cap D_1| = 0$. So now C_1 separates the annulus into two annuli, A_1 and A_2. Say that γ_C is in A_1. If D_1 is in A_1, then $|C_1 \cap \gamma_D| \geq 1$ and $|D_1 \cap \gamma_C| \geq 1$. Thus $|C \cap D| \geq 2$. If D_1 is in A_2, then we must have $|C \cap D| = 0$.

Case 3.5: Without loss of generality say that C_1 is a negative circle and C_2 is a half edge. So $|C \cap D| \geq 2$ when some $|C_1 \cap D_i| \neq 0$. So assume that each $|C_1 \cap D_i| \neq 0$. Thus C_1 separates the annulus into two annuli, A_1 and A_2. Say that γ_C is in A_1. If both D_1 and D_2 are contained in A_1, then each $|D_i \cap \gamma_C| \geq 1$ and so $|C \cap D| \geq 2$. If one D_i is contained in A_1, then as in Case 3.4, $|C \cap D| \geq 2$. If both D_1 and D_2 are contained in A_2, then $|C \cap D| = 0$. □

4 Main results

4.1 Wheels

The graph W_n shown in Figure 4.1 is the *wheel graph*. Since W_n is vertically 3-connected, W_n is the only ordinary graph that represents $\mathcal{W}_n = M(W_n)$ (see, for example, [3, Lemma 5.3.2].)

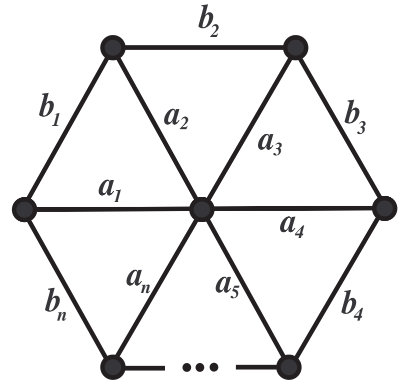

Figure 4.1.

Now let Σ be a signed graph without isolated vertices such that $M(\Sigma) \cong \mathcal{W}_n$: if Σ is balanced then $\Sigma \cong W_n$, if Σ is joint-unbalanced then Σ is obtained from W_n as in Proposition 2.3, and if Σ has a balancing vertex then Σ is obtained from W_n as in Proposition 2.4. So it only remains to classify Σ when it is unbalanced, not joint unbalanced, and has no balancing vertex. This is done in Theorem 4.2. The proof technique utilized for Theorem 4.2 is essentially the proof technique of [6, Thm. 3].

Theorem 4.2. *Let Σ be an unbalanced signed graph that is not joint unbalanced, does not have a balancing vertex, and has no isolated vertices. If $M(\Sigma) = \mathcal{W}_n$, then Σ is vertically 3-connected, jointless and is a projective-planar dual signed graph of some imbedding of W_n. Furthermore, if W_n is imbedded in the projective plane, then $M(W_n^*, \natural) \cong \mathcal{W}_n$.*

Proof. The furthermore statement follows by Theorem 2.9 and the fact that $\mathcal{W}_n^* \cong \mathcal{W}_n$.

Since $\mathcal{W}_n$ is regular, Theorem 2.5 says that Σ is jointless and has no two vertex-disjoint negative circles. That Σ is vertically 3-connected follows from Proposition 2.6(3). So now every circuit of Σ is either a balanced circle or a pair of negative circles that intersect in a single vertex.

Since $\mathcal{W}_n^* \cong \mathcal{W}_n$ we can write $M^*(W_n) \cong M(\Sigma)$. So now let C_v be the edges meeting $v \in V(W_n)$. Since W_n is 3-connected, C_v is a bond and consists only of links. Thus C_v is a cocircuit of $M(W_n)$. Thus C_v is a circuit of $M(\Sigma)$ and so $\Sigma{:}C_v$ is a positive circle or a union of two negative circles meeting in a single vertex. Thus C_v satisfies the conditions of Theorem 2.7.

Now consider $w \in V(\Sigma)$ and let T_w be the set of edges in Σ incident to w. Since Σ is jointless and vertically 3-connected, T_w contains only links. Proposition 2.6 implies that T_w is a cocircuit of $M(\Sigma)$ and so T_w is a circuit of $M(W_n)$. Thus $W_n{:}T_w$ is a circle and so T_w satisfies the conditions of Theorem 2.7.

In the previous two paragraphs we have shown that $\|\Sigma\|$ and W_n satisfy the conditions of Theorem 2.7. Thus $\|\Sigma\|$ and W_n are topological dual graphs in some 2-cell imbedding in a closed surface S. By Corollary 2.8, the Euler characteristic of S is $|V(W_n)| - |E(W_n)| + |V(\Sigma)| = n + 1 - 2n + n = 1$. Thus S is the projective plane.

Theorem 2.9 implies that $M(\Sigma) = M^*(W_n) = M(W_n^*, \natural)$ and so since $\|\Sigma\| = W_n^*$, Σ and $(W_n^*, \natural)$ are switching equivalent. Since projective-planar dual signed graphs are only well defined up to switching,

$\Sigma = (W_n^*, \natural)$. □

4.2 Whirls

The whirl of rank n, denoted $\mathcal{W}^n$, is obtained from the wheel $\mathcal{W}_n$ by relaxing the unique circuit hyperplane $\{b_1, \ldots, b_n\}$. One can check that both signed graphs in Figure 4.3 represent $\mathcal{W}^n$. Denote the left-hand signed graph by W^n and the right-hand signed graph by S^n. Evidently $W^n \cong S^n$ iff $n = 2$. By $\widetilde{W}^n$ and $\widetilde{S}^n$ we mean the signed graphs obtained from W^n and S^n by replacing all negative loops with half edges. One can check that there is only one imbedding of S^n in the annulus and its annular dual signed graph is $\widetilde{S}^n$.

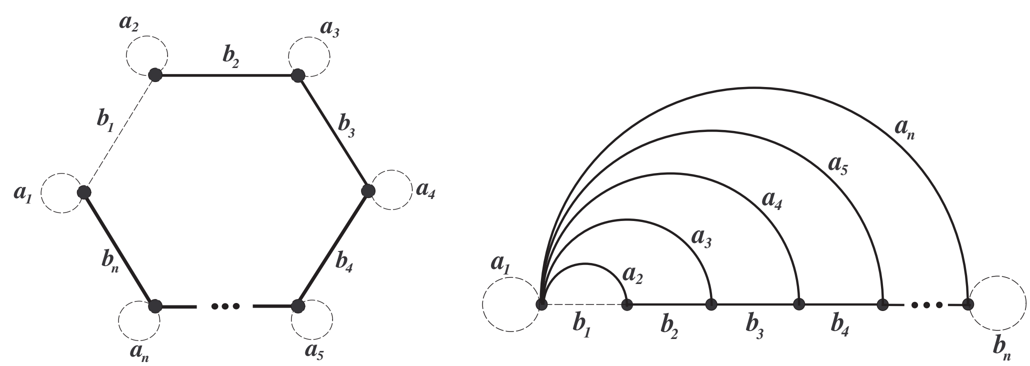

Figure 4.3.

Now let Σ be a signed graph without isolated vertices such that $M(\Sigma) = \mathcal{W}^n$. Of course one can always switch half edges with negative loops and not change the matroid, so we will assume that all joints in Σ are negative loops. Theorem 4.4 characterizes the structure of Σ.

Theorem 4.4. *If $M(\Sigma) = \mathcal{W}^n$, then after removing any isolated vertices, Σ is vertically 2-connected and either*

(1) $\Sigma \cong W^n$,

(2) $\Sigma \cong S^n$, *or*

(3) Σ *is an annular dual signed graph of some imbedding of* $\widetilde{W}^n$.

Furthermore, if Σ is the annular dual signed graph of some imbedding of $\widetilde{W}^n$, then $M(\Sigma) \cong \mathcal{W}^n$.

Lemma 4.5. *If $M(\Sigma) = \mathcal{W}^n$ for $n \geq 3$, $\Sigma \not\cong W^n$, and $\Sigma \not\cong S^n$, then $\Sigma{:}\{b_1, \ldots, b_n\}$ is a vertex-disjoint union of two negative circles, each a_i is a link in Σ with one endpoint in each of these circles, and there is at most one joint in Σ.*

Proof. Since $\{a_i, b_1, \ldots, b_n\}$ is a circuit for each i, $\{b_1, \ldots, b_n\}$ is a circuit in Σ with one edge removed that may be completed by the inclusion of any a_i. Up to subdivision of edges, there are eight possible topological types for a circuit in a signed graph Σ with one edge removed. They are shown in the Figure 4.6. (Recall that Σ has no half edges.)

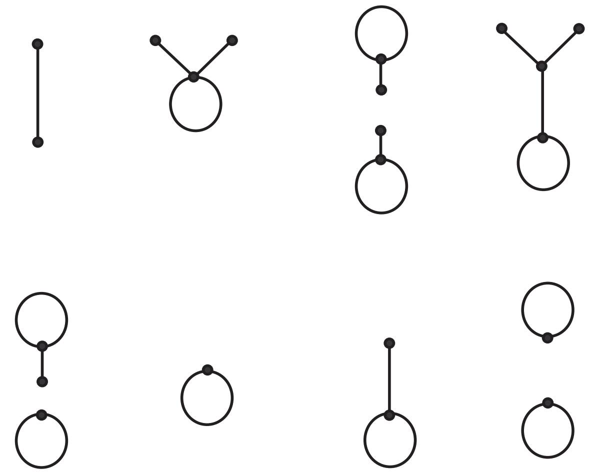

Figure 4.6.

In the first four cases, a_i may be added in only one way to complete a circuit and a_i must be a link. However, we cannot place more than two links with the same endpoints without creating parallel elements in $M(\Sigma)$. Thus the first four cases are not possible.

In the fifth case, each a_i must be a link from the leaf vertex to the other negative circle. So after adding the edges $a_1, \ldots, a_n$, we will have that Σ has a vertical 1-separation. But this contradicts Proposition 2.6 which says that Σ is vertically 2-connected aside from isolated vertices.

In the sixth case, the b_n's form a negative circle and so each a_i must then be a negative loop attached to the circle. Thus $\Sigma \cong W^n$.

In the seventh case, the only way to add a_i to complete a circuit is if we place a_i as a negative loop at the one end of the path or if a_i is a link with one endpoint at the end of the path and the other somewhere in the middle and the sign on the edge is such that the circle formed with a_i is negative. In order so that we do not have parallel elements in $M(\Sigma)$, we cannot have that more than one a_i as a negative loop. Furthermore, we cannot have more than two a_i's that are links sharing the same two endpoints. Thus we must have that the path has n vertices and so $\Sigma \cong S^n$.

In the eighth case we have that each a_i is a link connecting the two vertex-disjoint negative circles formed by $b_1, \ldots, b_n$. Our result follows. □

Proof of Theorem 4.4. The furthermore statement follows by Theorem 3.4 and the fact that $(\mathcal{W}^n)^* \cong \mathcal{W}^n$.

So now suppose that $\Sigma \ncong W^n$ and $\Sigma \ncong S^n$. Since the proof is trivial for $n = 2$, assume that $n \geq 3$. We want to show that Σ is an annular dual signed graph of $\widetilde{W}^n$. To do this we will actually show that $\|\Sigma\|$ is the planar dual graph of an associated planar graph of some annular imbedding of $\widetilde{W}^n$. Once we do this, we must have that $M(\Sigma) = M^*(\widetilde{W}^n) = M(\|\widetilde{W}^n\|^*, \natural)$ where $\|\Sigma\| = \|\widetilde{W}^n\|^*$. Since signed graphs with equal underlying graphs have the same matroid iff they are switching equivalent, we get that $\Sigma = (\|\widetilde{W}^n\|^*, \natural)$.

First $M^*(\Sigma) = (\mathcal{W}^n)^* \cong \mathcal{W}^n$ where the isomorphism is given by $a_i \leftrightarrow b_i$. By $\overline{W}^n$ we will mean $\widetilde{W}^n$ with this switch in labels. In Lemma 4.5, $\Sigma{:}\{b_1, \ldots, b_n\}$ is a disjoint union of two negative circles. Thus we have a natural bipartition (B_1, B_2) of $\{b_1, \ldots, b_n\}$ with nonempty parts. Let G be the graph obtained from $\overline{W}^n$ by adding two new vertices, v_1 and v_2 and extending the half edges B_i to links incident to v_i. We can imbed G in the plane with v_1 in the outside region of the circle $G{:}\{a_1, \ldots, a_n\}$ and v_2 inside the circle. We will now show that $\|\Sigma\|$ is the planar dual graph of this imbedding of G.

Let S be the collection of edges incident to $v \in V(G)$. Note that $G{:}S$ contains only links. If $v \in \{v_1, v_2\}$, then $\Sigma{:}S$ is a circle, which satisfies the condition of Theorem 2.7. If $v \notin \{v_1, v_2\}$, then S is a 3-element cocircuit of $M(\overline{W}^n)$ and so S is a 3-element circuit of $M(\Sigma)$. Since Σ has at most one joint and that joint is a loop, $\Sigma{:}S$ is either a positive triangle or a negative loop along with a negative digon. Either case satisfies the conditions of Theorem 2.7.

Now let S be the collection of edges incident to $v \in V(\Sigma)$. By Lemma 4.5, all vertices of Σ are in $\Sigma{:}(B_1 \cup B_2)$. Say without loss of generality that $v \in \Sigma{:}B_1$. Now either $|B_1| = 1$ or $|B_1| \geq 2$. In the first case, by Lemma 4.5, $S = \{b_i, a_1, \ldots, a_n\}$ and $\Sigma{:}b_i$ is a negative loop. Now $G{:}\{a_1, \ldots a_n\}$ is a circle and $G{:}b_i$ is a link with exactly one endpoint in the circle $G{:}\{a_1, \ldots a_n\}$. This satisfies the conditions of Theorem 2.7. In the latter case S consists of two elements of B_1, say b_i and b_j, and at least one element of $\{a_1, \ldots, a_n\}$ and all elements of S are links in Σ. Now $\overline{W}^n{:}S$ is a circuit containing two members of B_1, which are all half edges. Thus $\overline{W}^n{:}S$ is a handcuff consisting of two half edges and a connecting path with edges from $\{a_1, \ldots, a_n\}$. Thus $G{:}S$ is a circle. This also satisfies the conditions of Theorem 2.7.

So by the previous two paragraphs and Corollary 2.8, G and $\|\Sigma\|$ are dual graphs in some surface S whose Euler characteristic is $|V(G)| - |E(G)| + |V(\Sigma)| = n + 2 - 2n + n = 2$. Thus S is the sphere. Thus $\|\Sigma\|$ and $\overline{W}^n$ are topological duals in the annulus, as required. □

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