Bias matroids with unique graphical representations.

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Abstract

Given a 3-connected biased graph Ω with three node-disjoint unbalanced circles, at most one of which is a loop, we describe how the bias matroid of Ω is uniquely represented by Ω.

1 Introduction

In the study of representations of matroids using matrices, graphs, signed graphs, biased graphs, etcetera, unique representability can be useful. For instance, Theorem 1 is a lemma to the proof of Hassler Whitney's 2-Isomorphism Theorem as presented in [3, §5.3].

Theorem 1 (Whitney)**.** *Let* Γ *and* Γ_0 *be graphs without loops and isolated nodes. If* Γ *is 3-connected, then* $G(\Gamma) \cong G(\Gamma_0)$ *iff* $\Gamma \cong \Gamma_0$.

In this paper we present Theorem 2 which describes sufficient conditions for the bias matroid of a biased graph Ω to be uniquely represented by Ω. For an introduction to biased graphs and their matroids see Section 2.

Theorem 2. *Let* Ω *and* Ω_0 *be biased graphs without balanced loops, loose edges, and isolated nodes. Replace all half edges with unbalanced loops. If* Ω *is 3-connected and contains three node-disjoint unbalanced circles, at most one of which is a loop, then* $G(\Omega) \cong G(\Omega_0)$ *iff* $\Omega \cong \Omega_0$.

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The statement of Theorem 2 is reminiscent of Theorem 1 and its proof is an adaptation of A.K. Kelmans' and J. Edmonds' proof of Theorem 1 in [2, pp. 644–645].

It is worth noting that, for a biased graph Ω, removing or adding isolated nodes has no effect on $G(\Omega)$, replacing half edges with unbalanced loops has no effect on $G(\Omega)$, and loose edges and balanced loops form loops in $G(\Omega)$. Thus the conditions in the first two sentences of Theorem 2 are simply meant to eliminate some of the trivial variations that biased graphs can have without affecting their matroids.

Some results on contrabalanced biased graphs uniquely representing their matroids (which are exactly the bicircular matroids) were obtained by D.K. Wagner in [4]. For instance, Proposition 5 in [4] implies that if Ω is a contrabalanced biased graph whose underlying graph is the wheel graph with $n \geq 4$ spokes, then Ω is the only contrabalanced biased graph representing $G(\Omega)$. Surprisingly, there are no results for unique representability of bicircular matroids in [1].

Finding the correct necessary and sufficient or almost necessary and sufficient conditions to guarantee unique representability of bias matroids by biased graphs seems to be a very difficult problem.

2 Definitions

In this paper we assume the reader is thoroughly familiar with matroid theory as in [3] and somewhat familiar with biased graphs as in [5] and [6]. We will review all of the pertinent information about biased graphs and their matroids here in this section in an effort to make the presentation more self contained. We follow the notation and terminology for biased graphs and their matroids in [5] and [6].

A graph Γ has node set denoted by $N(\Gamma)$ and edge set denoted by $E(\Gamma)$. There are four types of edge in a graph: *links* have ends attached to distinct endpoints, *loops* have both ends attached to the same endpoint, *half edges* have one end attached to a node and the other unattached, *loose edges* have both ends unattached. A *circle* is a simple closed path. A *theta graph* is a graph consisting of two nodes joined by three internally disjoint paths.

A *biased graph* is a pair $(\Gamma, \mathcal{B})$ where Γ is a graph and $\mathcal{B}$ is a collection of circles of Γ, called *balanced*, such that every theta subgraph of Γ contains either 0, 1, or 3 balanced circles. A biased graph is called *balanced* if it contains no half edges and no unbalanced circles. A *bal-*

ancing node of an unbalanced biased graph Ω is a node whose removal (along with its incident edges) leaves a balanced biased graph. Not all unbalanced biased graphs have balancing nodes. A biased graph is called *contrabalanced* if it contains no loose edges and no balanced circles.

Given a biased graph Ω, the *bias matroid* of Ω, denoted by $G(\Omega)$, is the matroid on $E(\Omega)$ in which the rank of $X \subseteq E(\Omega)$ is given by $\mathrm{rk}(X) = |N(X)| - b(X)$ where $N(X)$ is the collection of nodes incident to an edge in X and $b(X)$ is the number of balanced components of the subgraph of Ω whose edge set is X and whose node set is $N(X)$ (see [6, Thm 2.1]). As a convention we say that loose edges do not contribute to the number of balanced components. The *bicircular matroid* of a graph Γ is the bias matroid of the contrabalanced biased graph $(\Gamma, \emptyset)$.

Given the form of the rank function we find that addition and deletion of isolated nodes in Ω does not affect $G(\Omega)$, loose edges and balanced loops in Ω are both loops in $G(\Omega)$, and a half edge is indistinguishable in $G(\Omega)$ from an unbalanced loop. Note that if Ω is balanced, then $G(\Omega)$ is simply the ordinary graphic matroid of the underlying graph of Ω.

Given an edge e in $\Omega = (\Gamma, \mathcal{B})$, the deletion of e is defined in the obvious way as $\Omega \backslash e = (\Gamma \backslash e, \mathcal{B} \cap \mathcal{C}(\Gamma \backslash e))$, where $\mathcal{C}(\Gamma \backslash e)$ is the collection of circles in $\Gamma \backslash e$. Evidently $G(\Omega) \backslash e = G(\Omega \backslash e)$. A *balancing set* of Ω is a collection of edges whose removal leaves a balanced biased graph. A cocircuit of $G(\Omega)$ is a minimal edge set whose removal increases the number of balanced components by one (see [6, Thm 2.1]). Given a biased graph Ω we define a *node cocircuit* of the bias matroid $G(\Omega)$ to be a cocircuit that is exactly the collection of edges incident to some node of Ω. In general, the set of edges incident to a given node may not be a cocircuit.

The contraction of an edge e in $\Omega = (\Gamma, \mathcal{B})$ is defined for three cases. If e is a balanced loop or loose edge, then $\Omega / e = \Omega \backslash e$. If e is a link, then Ω / e is the biased graph with underlying graph Γ / e in which a circle C in Γ / e is balanced if $C \in \mathcal{B}$ or $C = C' / e$ for some $C' \in \mathcal{B}$. If e is an unbalanced loop or half edge with endpoint v, then Ω / e is the biased graph obtained from Ω by deleting e, detaching the ends incident to v of the remaining edges, then removing v. It is known that $G(\Omega)/e = G(\Omega / e)$ (see [6, Thm 2.5]).

3 The Proof of Theorem 2

Lemma 3. *If Ω is a connected biased graph, then the complementary cocircuit of a connected hyperplane of $G(\Omega)$ is either a minimal balancing set of Ω or a node cocircuit of Ω. Furthermore, the complementary cocircuit of a connected and nonbinary hyperplane of $G(\Omega)$ is a node cocircuit of Ω.*

Proof. Recall that a cocircuit of $G(\Omega)$ is a minimal set of edges whose removal increases the number of balanced components of Ω by one. Thus a cocircuit C can be written as a disjoint union $C = S \cup B$ where $S = \emptyset$ or S is a separating edge set of Ω and $B = \emptyset$ or B is a minimal balancing set of an unbalanced component of $\Omega \setminus S$. Now, if a biased graph has two components with nonempty edge sets, then its matroid cannot be connected. Since Ω is connected, the complementary cocircuit of a connected hyperplane of Ω must be either a node cocircuit or a minimal balancing set of Ω. Furthermore, since the bias matroid of a balanced biased graph is graphic (and thus binary), the complementary cocircuit of a connected and nonbinary hyperplane of $G(\Omega)$ must be a node cocircuit of Ω. □

Lemma 4. *Let Ω be a biased graph without balanced loops. If Ω is unbalanced, 2-connected, and without balancing nodes, then the edges of any node of Ω form a node cocircuit.*

Proof. Let v denote some node of Ω. By assumption $\Omega \backslash v$ is connected and unbalanced. Thus the rank of $E(\Omega \backslash v)$ is one less than the rank of $G(\Omega)$. Furthermore, if e is an edge incident to v, then e is a link, half edge, or unbalanced loop. Thus $E(\Omega \backslash v) \cup e$ has full rank in $G(\Omega)$. Thus the edges incident to v form a cocircuit. □

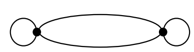
Lemma 5. *Let Ω be a biased graph without balanced loops or loose edges. If Ω is unbalanced, 2-connected, and without balancing nodes, then $G(\Omega)$ is connected.*

Proof. By way of contradiction, suppose that we can partition the edges of Ω into nonempty subsets X and Y such that every cocircuit of $G(\Omega)$ is contained entirely in X or entirely in Y. Let $N(X)$ denote the collection of nodes of Ω incident to some edge in X. Since Ω is 2-connected and contains no loose edges, $N(X) \cap N(Y) \neq \emptyset$. Let $v \in N(X) \cap N(Y)$. Since Ω is unbalanced and 2-connected and does not contain balancing nodes and balanced loops, Lemma 4 implies that the edges incident to v form a node cocircuit of $G(\Omega)$. This node cocircuit must intersect both X and Y, a contradiction. □

Lemma 6. *Let Ω be biased graph without balanced loops and loose edges and with all half edges replaced with unbalanced loops. If Ω is 3-connected and contains three node-disjoint unbalanced circles, at most one of which is a loop, then for any node v in Ω, $G(\Omega \setminus v)$ is nonbinary and connected.*

Proof. Since Ω is 3-connected, $\Omega \setminus v$ is 2-connected. Since Ω contains three node-disjoint unbalanced circles (at most one of which is a loop), $\Omega \setminus v$ contains two node-disjoint unbalanced circles (at most one of which is a loop). Thus Menger's Theorem implies that Ω contains a subdivision, call it S, of one of the following two graphs where the digons and loops are all unbalanced.

Because theta subgraphs of biased graphs do not contain exactly two balanced circles, S must contract to the contrabalanced biased graph shown below.



The matroid of this biased graph is the four-point line. Thus $G(\Omega \setminus v)$ is nonbinary. That $G(\Omega \setminus v)$ is connected follows from Lemma 5 and the fact that $\Omega \setminus v$ (which contains S) cannot contain a balancing node, has no balanced loops and loose edges, and is 2-connected. □

Proof of Theorem 2. If $\Omega \cong \Omega_0$, then we must have $G(\Omega) \cong G(\Omega_0)$. So now assume that $G(\Omega) \cong G(\Omega_0)$ and that Ω and Ω_0 are biased graphs satisfying the hypotheses of our theorem. By Lemma 3, every connected and nonbinary hyperplane of $G(\Omega)$ is the complement of a node cocircuit. Furthermore, Lemmas 4 and 6 imply that every node of Ω is incident to a node cocircuit whose complementary hyperplane is connected and nonbinary. Since Ω is connected and unbalanced, $G(\Omega)$ has exactly $|N(\Omega)| = \mathrm{rk}(G(\Omega))$ connected and nonbinary hyperplanes, each of which is the complement of a node cocircuit. Thus the connected and nonbinary hyperplanes of $G(\Omega)$ completely determine the incidences of nodes with edges in Ω.

Since $G(\Omega)$ is a connected matroid, $G(\Omega_0)$ is a connected matroid. Since Ω_0 has no isolated nodes, Ω_0 must be connected. Since $G(\Omega)$ is nonbinary, $G(\Omega_0)$ is nonbinary. Thus Ω_0 must be unbalanced. Since

Ω_0 is connected and unbalanced, $\mathrm{rk}(G(\Omega_0)) = |N(\Omega_0)|$. Thus $G(\Omega) \cong G(\Omega_0)$ implies that $|N(\Omega_0)| = \mathrm{rk}(G(\Omega_0)) = \mathrm{rk}(G(\Omega)) = |N(\Omega)|$. Since the number of connected and nonbinary hyperplanes of $G(\Omega)$ and $G(\Omega_0)$ must be the same, $G(\Omega_0)$ has exactly $|N(\Omega_0)|$ connected and nonbinary hyperplanes. Since Ω_0 is connected, Lemma 3 implies that all of these connected and nonbinary hyperplanes of $G(\Omega_0)$ are complements of node cocircuits. Furthermore, the number of these hyperplanes implies that every node of Ω_0 is incident to a node cocircuit whose complementary hyperplane is connected and nonbinary. So here too the connected and nonbinary hyperplanes of $G(\Omega_0)$ completely determine the incidences of nodes with edges in Ω_0.

The conclusions of the previous two paragraphs imply that the isomorphism between $G(\Omega)$ and $G(\Omega_0)$ is an isomorphism between the underlying graphs of Ω and Ω_0. Thus $\Omega \cong \Omega_0$ because two biased graphs with isomorphic underlying graphs represent isomorphic bias matroids iff they have corresponding lists of balanced circles. □

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