An algebraic characterization of projective-planar graphs

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Abstract

We give a detailed algebraic characterization of when a graph G can be imbedded in the projective plane. The characterization is in terms of the existence of a dual graph G^* on the same edge set as G which satisfies algebraic conditions inspired by homology groups and intersection products in homology groups.

1 Introduction

Theorem 29 of [15] (shown below) is a classic result of H. Whitney characterizing planar graphs in terms of the existence of a *dual* (defined in [15, §II]). Let $c(G)$ be the number of components of a graph G. A *dual* of a graph G is a graph G' on the same edge set as G such that, for any $H \subseteq G$, the subgraph of $H' \subseteq G'$ with $E(H') = E(G) \setminus E(H)$ satisfies

$$|E(H)| - |V(H)| + c(H) = |V(G')| - c(G') - |V(H')| + c(H').$$

Theorem 1.1 (Whitney). *A graph is planar iff it has a dual.*

Similar to Whitney's result is that of S. MacLane from [10]. There he uses the idea of a *2-basis* to algebraically characterize planar graphs. Let V be a binary vectorspace with standard basis $e_1, \ldots, e_k$ and let W be a subspace of V. A 2-*basis* B of W is a basis of W in which each e_i is a summand of at most two elements of B.

Theorem 1.2 (MacLane). *A graph G is planar iff the cycle space of G has a 2-basis.*

Whitney's and MacLane's theorems are similar in that the 2-basis for the cycle space of G can be used as the face boundaries of an imbedding of G in the plane, so the 2-basis can also be viewed as the vertex set of a dual of G. Thus both theorems classify planarity in terms of the existence of a dual graph.

There are two notable generalizations of the theorems of Whitney and MacLane: one by S. Lefschetz in [8] and another by J. Edmonds in [6]. The result of Lefschetz uses the combinatorial language of rotation systems of graphs imbedded in surfaces. Edmonds' Theorems (shown below) are elegant and simply-stated combinatorial results using the idea of duality.

Theorem 1.3 (Edmonds). *A necessary and sufficient condition for a graph G to have a polyhedral surface imbedding in a surface S of Euler characteristic $\chi(S)$ is that it has an edge correspondence with another graph G^* for which*

(1) the conditions of Theorem 1.4 are satisfied and

(2) $|V(G)| - |E(G)| + |V(G^*)| = \chi(S)$.

Theorem 1.4 (Edmonds). *A one-to-one correspondence between the edges of two connected graphs is a duality with respect to some surface S if and only if, for each vertex v of each graph, the edges which meet v in the graph of v form in the other graph a subgraph which is connected and has an even number of edge ends to each of its vertices (where if an edge meets v at both ends, its image in H is counted twice).*

In this paper we provide an algebraic reformulation of Whitney's characterization, together with a new proof of that result, and then generalize it to algebraically characterize projective-planar graphs. We believe that our approach may provide a framework for a general theorem characterizing the graphs that imbed in any given surface or pseudosurface. Such a result is useful because, as Edmonds comments in the conclusion of [6], his theorems cannot be generalized using oriented edges in order to distinguish between orientable and nonorientable surfaces. He does not comment on characterizing imbeddability in pseudosurfaces.

Some other algebraic approaches to imbeddability are of interest in this context. Archdeacon, Bonnington, and Little provide a novel algebraic characterization of planarity in [2]. The topological content of that result is clarified by the approach of Richter and Keir in [12] and [13]. The latter approach essentially deals with homology and the former with cohomology, although neither is expressed in these terms.

We begin the main body of this paper in Section 2 by describing various vector spaces coming from graphs and the relevant notions of cellular homology, generally laying the ground-work for the approach of this paper. Most of the material in Section 2 is well known and can also be found in works such as [3], [4], and [14]. We include the discussion in an effort to keep this paper more self-contained.

In Section 3 we define an *algebraic dual*, which is a weakening of the notion of a dual as first defined by Whitney. Section 3 also describes our main construction: given a graph G and a loopless algebraic dual G^*, we construct a cellular 2-complex whose 1-skeleton is G and whose 2-cell structure depends on G^*.

Section 4 contains our reformulation of Whitney's theorem in the language of algebraic duals, as well as a new proof of that theorem based on ideas from homology theory. The techniques that we use are similar to those used in [9, Section 3.2], but the proofs there contain some hidden assumptions that lead to errors in the presentation. We identify some of these in Section 6.

In Section 5 is our main theorem characterizing projective planarity. Again, the statement and proof of the theorem are inspired by homology theory; in particular we use the well-known notion of the intersection product (or intersection index) in homology groups. See [5, pp. 220–221] for a discussion of intersection products.

2 Definitions

2.1 Vector spaces coming from graphs

Given a graph G with vertex-set $V(G)$ and edge-set $E(G)$, let $C_0(G)$ and $C_1(G)$ denote the $\mathbb{Z}_2$-vector space of formal linear combinations of elements of $V(G)$ and $E(G)$, respectively. We call $C_0(G)$ the *space of 0-chains* and $C_1(G)$ the *space of 1-chains*. If $c \in C_i(G)$, then we let c also denote the collection of elements with nonzero coefficients. This should not cause confusion, since the nonzero support of the sum of two vectors in a binary vector space is exactly the symmetric difference of the nonzero supports of the two vectors. If $X \subseteq E(G)$, then we denote by $G{:}X$ the subgraph of G consisting of the edges in X and the vertices in G incident to edges in X.

A *circle* in G is a connected two-regular subgraph of G. A *bond* in G is a minimal set of edges whose removal increases the number of components of G. Let

$$Z_1(G) = \langle c \in C_1(G) : c \text{ is the edge set of a circle in } G\rangle.$$

(Here $\langle v_1, \dots, v_t\rangle$ denotes the subspace generated by $v_1, \dots, v_t$.) The subspace $Z_1(G)$ is called the *cycle space* of G and its elements are called *cycles*. Let $B^1(G) = \langle b \subseteq E(G) : b \text{ is a bond of } G\rangle$. The subspace $B^1(G)$ is called the *coboundary space* of G and its elements are called *coboundaries*. (These are sometimes called "cocycles.") The set of links incident to a vertex v in the vertex set of G is a coboundary called the *vertex star* of v.

A graph is called *Eulerian* if it has a closed walk passing through each edge exactly one time. Recall that a graph is Eulerian if and only if it is connected and each vertex has even degree. We let $c(G)$ be the number of components of G. Proposition 1 contains well known facts; see [3], [4], or [14] for a detailed exposition.

Proposition 1.

(1) *The dimension of* $Z_1(K)$ *is* $|E(G)| - |V(G)| + c(G)$

(2) *The dimension of* $B^1(K)$ *is* $|V(G)| - c(G)$

(3) *If* F *is a maximal forest of* G, $E(G) \setminus E(F) = \{e_1, \dots e_\beta\}$, *and* c_i *is the edge set of the unique circle in* $F \cup e_i$, *then* $\{c_1, \dots, c_\beta\}$ *is a basis for* $Z_1(G)$.

(4) $z \in Z_1(G)$ *iff* $G{:}z$ *is an edge-disjoint union of circles of* G.

(5) $z \in Z_1(G)$ *iff the components of* $G{:}z$ *are Eulerian.*

(6) $z \in B^1(G)$ *iff there is* $X \subseteq V(G)$ *such that* z *is the collection of links in* G *with one endpoint in* X *and the other in* $V(G) \setminus X$.

(7) *If* V' *is obtained from* $V(G)$ *by removing exactly one vertex from the collection of vertices of each connected component of* G, *then the collection of vertex stars from the vertices in* V' *is a 2-basis for* $B^1(G)$.

(8) *Under the bilinear form such that the edges of* G *form an orthonormal basis for* $C_1(G)$, *we have* $Z_1(G)^\perp = B^1(G)$.

2.2 Cellular complexes and homology

Let K be a 2-dimensional cellular complex (or 2-complex, for brevity) with vertices (*i.e.,* 0-cells) $V(K)$, edges (*i.e.,* 1-cells) $E(K)$ and faces (*i.e.,* 2-cells) $F(K)$. Note that the 1-skeleton of K is a graph. If one walks along the boundary of a face $f \in F(K)$, then one obtains a closed walk $v_1e_1v_2e_2\ldots v_ne_nv_1$ in the 1-skeleton of K. The *boundary* of f, denoted by ∂f, is $\sum_{i=1}^{n} e_i$ (*i.e.,* ∂f is the collection of edges incident to f an odd number of times). Since ∂f is the sum of edges in a closed walk, it can be shown that $\partial f \in Z_1(G)$. Let $C_2(K)$ denote the $\mathbb{Z}_2$-vector space with basis $F(K)$, and let $B_1(K)$ be the subspace of $Z_1(G)$ that is generated by the boundaries of faces of K (*i.e.,* $B_1(K) = \langle \partial C_2(K) \rangle$). The first homology group of K is the quotient space $Z_1(K)/B_1(K)$ and is denoted by $H_1(K)$. Since $Z_1(K)$ and $B_1(K)$ are both binary vector spaces, $H_1(K)$ is also a binary vector space. Let $|K|$ denote the geometric realization of K. Theorem 2.1 is well known, see *e.g.* [11, §18].

Theorem 2.1 (Invariance of Homology). *If K_1 and K_2 are 2-complexes and $|K_1|$ is homeomorphic to $|K_2|$, then $H_1(K_1) \cong H_1(K_2)$.*

We say that K is *connected* if its 1-skeleton is a connected graph. We say that K is *face-connected* if for any two faces f and $f' \in F(K)$ there is a sequence of faces $f = f_1, \ldots, f_n = f'$ such that, for each i, f_i and f_{i-1} share a common boundary edge. Evidently, a face-connected 2-complex is connected, but a connected 2-complex need not be face-connected. Denote the number of components of K by $c(K)$ and the number of face-connected components of K by $f(K)$. A 2-complex is 2-*regular* if each edge is either attached to exactly two faces or is attached to one face twice, *i.e.,* an edge of K either appears in two distinct boundary walks once or twice in one boundary walk.

Proposition 2. *A 2-complex K is not face-connected iff $|K|$ is separable by the removal of a finite number of points.*

Proof. If $F_1, \ldots, F_k$ are the face-connected components of K, then the only intersection two face-connected components may have is at the vertices of K. Thus the removal of some subset of the vertices of K will separate K. Conversely, if $|K|$ is separable by the removal of a finite number of points, then there must be 2 faces not connected by a face path because the geometric realization of a face path cannot be separated by the removal of a finite number of points. □

Proposition 3. *If K is a 2-regular 2-complex, then $H_1(K) \cong \mathbb{Z}_2^d$ in which*

$$d = f(K) + c(K) - |V(K)| + |E(K)| - |F(K)|.$$

Proof. The cycle space $Z_1(K)$ is a binary vector space of dimension $|E(K)| - |V(K)| + c(K)$ (see Proposition 1). Since each edge is attached to 2 distinct faces or attached twice to one face, the dimension of $B_1(K)$ is $|F(K)| - f(K)$. Thus, the dimension of $H_1(K) = Z_1(K)/B_1(K)$ is

$$|E(K)| - |V(K)| + c(K) - (|F(K)| - f(K)), \text{ as required.}$$

□

If K is a 2-regular 2-complex and x is a point in $|K| \setminus V(K)$, then there is a small neighborhood about x homeomorphic to a disk. If $x \in V(G)$, then every sufficiently small open neighborhood in $|K|$ around x is homeomorphic to a collection of $d_x \geq 1$ disks joined at the point x. If $d_x \geq 2$, then x is called a *pinchpoint.* If K has no pinchpoints, then $|K|$ is a disjoint union of surfaces. A *pseudosurface* is a topological space $P = |K|$ for some 2-regular 2-complex K. Note that every pseudosurface may be obtained from a disjoint union of surfaces by making a finite number of point identifications. If a graph G is the 1-skeleton of a 2-regular 2-complex K with $|K|$ homeomorphic to a pseudosurface P, then we say that G is *properly imbedded in* P. Thus all pinchpoints of P correspond to vertices of K.

The *demigenus* of a cellular complex K is $2-\chi(S)$, in which $\chi(S)$ is the Euler characteristic $|V(K)|-|E(K)|+|F(K)|$. (This is also called the "Euler genus.") If $|K|$ is an orientable surface, then the demigenus is equal to twice the genus. If $|K|$ is a nonorientable surface, then the demigenus is the crosscap number of the surface.

Proposition 4. *If S is a surface of demigenus d, then $H_1(S) = \mathbb{Z}_2^d$.*

Proposition 5.

(1) If P is a connected pseudosurface and P' is obtained from P by identifying two distinct points of P, then $H_1(P') \cong H_1(P) \times \mathbb{Z}_2$.

(2) If P_3 is the disjoint union of pseudosurfaces P_1 and P_2, then $H_1(P_3) \cong H_1(P_1) \times H_1(P_2)$.

(3) If P_3 is the "wedge" of pseudosurfaces P_1 and P_2, i.e., is obtained from the disjoint union of P_1 and P_2 by identifying a distinguished point of P_1 with a distinguished point of P_2, then $H_1(P_3) \cong H_1(P_1) \times H_1(P_2)$.

Propositions 4 and 5 are easy calculations. Using Propositions 2–5 and invariance of homology one may calculate the first homology group of any pseudosurface by starting with a disjoint union of surfaces and creating the appropriate pinch points.

3 Constructing a cellular complex from G and an algebraic dual.

Given a 2-regular 2-complex K with 1-skeleton G, there exists a topological dual graph $G^{\perp}$ constructed as follows. Let $V(G^{\perp}) = F(K)$ and $E(G^{\perp}) = E(G)$. The edge e is a link connecting distinct vertices f_1 and f_2 in $G^{\perp}$ when e is an edge in the boundary walks of distinct faces f_1 and f_2 in G. The edge e is a loop on vertex f in $G^{\perp}$ when e appears twice in the boundary walk of face f in G. When the imbedding of G is understood, we write $F(G)$ for $F(K)$. When viewing G and $G^{\perp}$ as subsets of $|K|$, we will presume that each point corresponding to a vertex of $G^{\perp}$ lies in the interior of the appropriate face of G, and that the two curves in $|K|$ corresponding to an edge of G and $G^{\perp}$, respectively, cross transversely and only at a single point. Finally, when K is a surface it is well known that $(G^{\perp})^{\perp} = G$.

Given a graph G, an *algebraic dual* to G is a graph G^* with $E(G^*) = E(G)$ and $B^1(G^*) \subseteq Z_1(G)$. Proposition 6 is evident.

Proposition 6. *Given a 2-regular 2-complex K with 1-skeleton G, the topological dual $G^{\perp}$ is an algebraic dual of G.*

Consider a graph G and a loopless algebraic dual G^*. Construction 1 describes a method for constructing a 2-dimensional, 2-regular cellular complex $K(G, G^*)$ using G and G^* which has 1-skeleton G. Note that G is properly imbedded in $K(G, G^*)$ and that each edge of $K(G, G^*)$ is on two distinct faces.

An important fact used in Construction 1 is that, if $z \in Z_1(G)$, then the components of $G{:}z$ are Eulerian (see Proposition 1). So in each connected component of $G{:}z$, the boundary of a disk may be attached to an Eulerian walk in that connected component.

Construction 1. This construction uses as input a graph G and a loopless algebraic dual G^*. Take the 1-skeleton of $K(G, G^*)$ to be the graph G. If $z_1, \ldots, z_n \in Z_1(G^*)$ are the vertex stars of G^*, then each $z_i \in Z_1(G)$ and each connected component of $G{:}z_i$ is Eulerian.

Identify the boundaries of 2-cells $F_1^i, \dots, F_{k_i}^i$ with Eulerian walks in the components $C_1^i, \dots, C_{k_i}^i$ of $G{:}z_i$. Let the faces of $K(G, G^*)$ be the cells F_j^i glued to the edges of G.

Proposition 7 is a natural consequence of Construction 1.

Proposition 7. *Let G^* be a loopless algebraic dual of G and let $G^\perp$ be the topological dual of G in $K(G, G^*)$. If for each vertex star s of G^*, $G{:}s$ is connected, then $G^\perp = G^*$ (after renaming of vertices).*

4 A proof of Whitney's planarity criterion

Theorem 4.1 is a reformulation of a theorem of H. Whitney from [15, Theorem 29] for which we present our own proof. This proof is not entirely new (see [9, §3.2]) but we include it to highlight our technique and point of view on the subject.

Theorem 4.1 (Whitney). *A graph G is planar iff there exists an algebraic dual G^* satisfying $Z_1(G) = B^1(G^*)$.*

Proof. A connected graph G is *separable* if there is a partition (E_1, E_2) of $E(G)$ such that $G{:}E_1$ and $G{:}E_2$ intersect in a single vertex. We say that a connected graph is 2-connected iff it is nonseparable. A *block* of a graph G is maximal 2-connected subgraph.

Assume first that G is 2-connected. If G is a planar graph, then we may imbed G in the sphere and obtain a 2-regular 2-complex K with topological dual graph $G^\perp$. It must be that $G^\perp$ is connected; otherwise K is not face connected and so may be separated by the removal of a finite number of vertices, contradicting the fact that $|K|$ is a sphere. By Proposition 6, $G^\perp$ is an algebraic dual of G, so $B^1(G^\perp) \subseteq Z_1(G)$. Furthermore, since $|V(G)| - |E(G)| + |F(G)| = 2$, we get that $|E(G)| - |V(G)| + 1 = |V(G^\perp)| - 1 = \dim(B^1(G^\perp))$. Thus $\dim(Z_1(G)) = \dim(B^1(G^\perp))$, forcing $B^1(G^\perp) = Z_1(G)$.

Conversely, assume there exists an algebraic dual G^* satisfying $Z_1(G) = B^1(G^*)$. Because G is 2-connected, every $e \in E(G)$ appears in some cycle $z \in Z_1(G)$. This makes G^* loopless because if e is a loop in G, then e does not appear in any coboundary $b \in B^1(G^*)$. So we can use Construction 1 on G and G^*. Let $G^\perp$ be the topological dual of G in $K := K(G, G^*)$. The construction of K yields

$B^1(G^*) \subseteq B^1(G^\perp) \subseteq Z_1(G)$, and thus $B^1(G^\perp) = Z_1(G)$. Also since G^* is loopless, Construction 1 forces $G^\perp$ to be loopless; so $G^\perp{:}s$ is a vertex star iff s is the set of edges in a boundary walk of some face $f \in F(K)$. Thus $B^1(G^\perp) = B_1(K)$. These equalities yield

$$H_1(K) = Z_1(K)/B_1(K) = Z_1(G)/B^1(G^\perp) = 0.$$

We know from Propositions 4 and 5 that a pseudosurface P with $H_1(P) = 0$ is either a sphere (up to homeomorphism), is not connected, or is separable by the removal of one point. Since G is 2-connected and is the 1-skeleton of K, $|K|$ is connected and cannot be separated by the removal of a point. Thus $|K|$ is a sphere in which G is imbedded, *i.e.,* G is planar.

Suppose now that G is not 2-connected, and recall that a graph is planar if and only if each of its blocks is planar. Let $B_1, \dots, B_k$ denote the blocks of G; we have $Z_1(G) = Z_1(B_1) + \cdots + Z_1(B_k)$. If G is planar, then applying the theorem to the blocks shows that there are algebraic duals $B_1^*, \dots, B_k^*$, such that $Z_1(B_1) + \cdots + Z_1(B_k) = B^1(B_1^*) + \cdots + B^1(B_k^*)$. Letting G^* be the vertex-disjoint union of $B_1^*, \dots, B_k^*$, we get that $B^1(G^*) = B^1(B_1^*) + \cdots + B^1(B_k^*)$, yielding $Z_1(G) = B^1(G^*)$.

Conversely, assume that there is an algebraic dual G^* such that $Z_1(G) = B^1(G^*)$. For each block B_i we have

$$Z_1(B_i) = Z_1(G) \cap C_1(B_i) = B^1(G^*) \cap C_1(B_i) = B^1\left(G^*{:}E(B_i)\right).$$

It follows that $G^*{:}E(B_i)$ is an algebraic dual to B_i, and moreover we may conclude from the 2-connected case of the theorem that B_i is planar. Since this is true for all i, G is planar. □

5 Projective planarity

An interesting combinatorial property of a 2-connected nonplanar graph that is imbedded in the projective plane is that its topological dual does not have loops. We prove this statement below.

Lemma 1. *Let G be a graph properly imbedded in the projective plane. If G is 2-connected and nonplanar, then $G^\perp$ is loopless.*

Proof. Let P denote the projective plane. We view G and $G^\perp$ as subsets of P in the manner described in Section 3. Suppose that e

is a loop in $G^\perp$. Either $G^\perp{:}e$ is a separating or nonseparating curve in P. If $G^\perp{:}e$ is separating, then e must be a separating edge of G since G only intersects $G^\perp$ in P at the transverse crossings of curves corresponding to the same edge. This contradicts the assumption that G is 2-connected, so it must be that $G^\perp{:}e$ is a nonseparating curve in the P. Cutting P along $G^\perp{:}e$ yields a disk in which $G \setminus e$ is imbedded; furthermore, the endpoints of e in G must be on the boundary of the outer face of $G \setminus e$ in the disk. Thus we can redraw e on a disk without crossing any edges of $G \setminus e$, showing that G is planar. This contradicts our assumption that G is not planar. Thus $G^\perp$ is loopless. □

Theorem 5.1. *If G is a nonplanar 2-connected graph, then G imbeds in the projective plane iff there exists an algebraic dual G^* satisfying the following conditions.*

(1) G^* is loopless.

(2) If $G^*{:}z$ is a vertex star, then $G{:}z$ is connected.

(3) $\dfrac{Z_1(G)}{B^1(G^*)} \cong \mathbb{Z}_2$.

(4) If z_1, z_2 are nonzero elements of $\dfrac{Z_1(G^*)}{B^1(G)}$, *then* $G^*{:}z_1 \cap G^*{:}z_2 \neq \emptyset$.

Propositions 8 and 1 and the fact that G^* is an algebraic dual of G guarantee that the quotients in parts (3) and (4) above are well defined and isomorphic.

Proposition 8. *If $V \subseteq W \subseteq \mathbb{Z}_2^n$, then* $\dfrac{W}{V} \cong \dfrac{V^\perp}{W^\perp}$.

Proof of Theorem 5.1. Suppose that G is projective planar and is imbedded in the projective plane P. Then G has a topological dual $G^\perp$ that, by Proposition 6, is also an algebraic dual of G. We will show that $G^\perp$ satisfies Conditions (1)–(4).
(1) Since G is 2-connected and not planar, Lemma 1 states that $G^\perp$ is loopless.
(2) Let $G^\perp{:}z$ be the vertex star for $f \in V(G^\perp)$, *i.e.,* the set of links in $E(G^\perp)$ incident to f. Since $G^\perp$ is loopless, z is the collection of edges incident to f in $G^\perp$. View f as the face in $F(G)$ corresponding to the vertex $f \in V(G^\perp)$, and note that each edge in the boundary walk of f appears only once, so $z = \partial f$. Also, since the boundary walk of f is connected, $G{:}z = G{:}\partial f$ is connected as well.

(3) Let $K := K(G, G^\perp)$. Evidently K is 2-regular and $B_1(K)$ is generated by the collection of face boundaries of G. Since the vertices of $G^\perp$ correspond to the faces of G, Proposition 1 shows that $B_1(K) = B^1(G^\perp)$. Thus, by Proposition 4, $Z_1(G)/B_1(K) = Z_1(G)/B^1(G^\perp) \cong \mathbb{Z}_2$.
(4) Let $K^\perp$ be the 2-regular 2-complex given by the cellular decomposition of $G^\perp$ in P. Since $(G^\perp)^\perp = G$ (because $G^\perp$ is imbedded in a surface) and G is loopless (by virtue of being 2-connected), we have $B_1(K^\perp) = B^1(G)$. Thus, by Proposition 4, $Z_1(G^\perp)/B^1(G) \cong H_1(\mathsf{P}) \cong \mathbb{Z}_2$. To prove the necessity of Condition (4), consider paragraph (c) in [5, p 221], which states that the intersection index for two 1-cycles on P that are nonzero in $H_1(\mathsf{P}) \cong \mathbb{Z}_2$ is equal to 1 mod 2. That is, two 1-cycles in general position which are nonzero in $H_1(\mathsf{P})$ intersect transversely an odd number of times. Thus if $z_1, z_2 \in Z_1(G^\perp)$ are nonzero in $Z_1(G^\perp)/B^1(G) \cong H_1(\mathsf{P}) \cong \mathbb{Z}_2$, then $G^\perp{:}z_1 \cap G^\perp{:}z_2 \neq \emptyset$.

Conversely, suppose that G has an algebraic dual G^* satisfying Conditions (1)–(4). Since G^* is loopless we can construct the 2-regular 2-complex $K := K(G, G^*)$, as per Construction 1. Let $G^\perp$ be the topological dual graph of G in K. By condition (2) and Proposition 7, $G^\perp = G^*$. Thus we have that $B^1(G^*) = B^1(G^\perp) = B_1(K)$, so Condition (3) gives $H_1(K) = Z_1(G)/B_1(K) = Z_1(G)/B^1(G) \cong \mathbb{Z}_2$. To complete the proof we need to show that $|K| \cong \mathsf{P}$. By way of contradiction, assume that $|K| \ncong \mathsf{P}$. We divide the rest of the proof into two cases; in the first, K is face connected and in the second, K is not face-connected.
Case 1: By Propositions 4 and 5, the only possibility for K when K is face-connected and has $H_1(K) \cong \mathbb{Z}_2$ is that $|K|$ is homeomorphic to the sphere with one pinchpoint, call it P_1.

Let $p \in V(G)$ be the pinchpoint of K and let z_p be the vertex star in G of p. Because G is loopless, the cycle $G^*{:}z_p$ is an edge-disjoint union of two circles with edge sets E_1 and E_2. Let G' be the graph obtained by splitting the pinchpoint vertex p into two vertices p_1 and p_2 where, for each i, the edges in E_i are incident to the vertex p_i. The new graph G' is the 1-skeleton of the cellular complex K' obtained by separating the pinchpoint p into p_1 and p_2 while leaving the incidences of edges with faces in K unchanged. Evidently $|K'|$ is homeomorphic to the sphere and, since the incidence of edges and faces is unchanged, the definition of topological dual gives that the topological dual of G' in K' equals $G^\perp = G^*$ and so, since $|K'|$ is a surface, the dual in K' of $G^\perp$ is G'.

Since $Z_1(G^\perp) = B^1(G')$, we get $Z_1(G^\perp)/B^1(G') = 0$. Identifying p_1 with p_2 (to get K from K') introduces the relation $E_1 = E_2$ in $B^1(G')$. Thus $\dim(B^1(G)) = \dim(B^1(G'))-1$, making $Z_1(G^\perp)/B^1(G) \cong \mathbb{Z}_2$ with E_1 and E_2 being nonzero elements in this factor group. We now show that $G^\perp{:}E_1$ and $G^\perp{:}E_2$ are vertex-disjoint, contrary to condition (4).

If $f \in V(G^\perp)$ is a vertex in both $G^\perp{:}E_1$ and $G^\perp{:}E_2$, then $f \in F(K')$ is a face of K' that has both p_1 and p_2 on its boundary walk. Since G is obtained from planar graph G' by identifying p_1 and p_2 which are on the same face of K', G must be planar, contrary to the assumption that G is nonplanar.

Case 2: Since G is 2-connected, K is connected and is not separable by one point. Since $H_1(K) \cong \mathbb{Z}_2$, Proposition 5 implies that K is separated into two components, call them C_1 and C_2, at exactly two pinchpoints, call them p and q, and that each $H_1(\mathsf{C}_i) = 0$.

Let C_i denote the 1-skeleton of C_i. Evidently $G^* = G^\perp$ is the disjoint union of C_1 and C_2. If $z_p \in B^1(G)$ is the vertex star of p, then $G^\perp{:}z_p$ is a vertex-disjoint union of two circles with edge sets $P_1 \subseteq \mathsf{C}_1$ and $P_2 \subseteq \mathsf{C}_2$. We arrive at a contradiction of condition (4) by showing that P_1 and P_2 are nonzero elements of $Z_1(G^\perp)/B^1(G)$. If K' is the 2-regular 2-complex obtained by separating the pinchpoint p into two vertices p_1 and p_2 while leaving the edge and face incidences unchanged, then K' is obtained by joining two homology-zero pseudosurfaces at a vertex. By Proposition 5, $Z_1(G^\perp)/B^1(K') = 0$ and by identifying p_1 and p_2 we obtain $Z_1(G^\perp)/B^1(G) \cong \mathbb{Z}_2$ in which $P_1, P_2 \in Z_1(G^\perp)$ are nonzero in this quotient, as required. □

Corollary 1. *If G is 2-connected and nonplanar and G^* is an algebraic dual satisfying the conditions in Theorem 5.1, then G^* is a geometric dual of an imbedding of G in the projective plane.*

Proof. In the proof of Theorem 5.1, $G^* = G^\perp$ in the construction of $K = K(G, G^*) \cong \mathsf{P}$. □

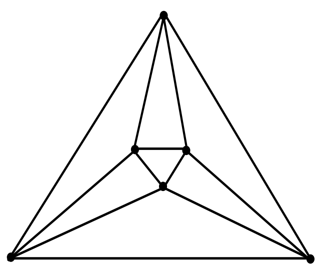
6 Some Errors in [9, §3.2]

Given a 2-regular 2-complex K with 1-skeleton G, write $\chi(G)$ for the Euler characteristic. Corollary 3.2.1 in [9] states that a graph G is planar iff there exists a 2-regular 2-complex K with 1-skeleton G satisfying $\chi(G) = 2$. This is not necessarily true when K is not face-connected. Consider the following example.

Let G_1 be a connected planar graph imbedded in a sphere containing $K_{3,3} \backslash e$. Label the endpoints of e as x and y. Let G_2 be any other connected planar graph imbedded in another sphere. Identify x and y with any two distinct vertices of G_2. This yields a 2-regular 2-complex with a nonplanar 1-skeleton G satisfying $\chi(G) = \chi(G_1)+\chi(G_2)-2 = 2$.

Corollary 3.2.3 in [9] states that a connected graph G can be imbedded in surface S of Euler characteristic $\chi(S)$ iff there exists a 2-regular 2-complex K satisfying $\chi(G) = \chi(S)$. Even if K is face-connected this is not necessarily true because pinchpoints can cause problems as in the following example.

Consider G, the octahedron with two subdivided faces, shown below and to the right. The graph G is shown imbedded in the plane and thus satisfies $\chi(G) = 2$.



The octahedron and the graph G being the octahedron with a subdivision of a pair of antipodal faces.

By identifying the two trivalent vertices x and y we get the multipartite graph $K_{2,2,2,1}$ properly imbedded in the one-pinchpoint sphere. Since all we have done is reduce the number of vertices by one, we have $\chi(K_{2,2,2,1}) = 1$. According to [9, Corollary 3.2.3], then, $K_{2,2,2,1}$ should imbed in the projective plane, which has Euler characteristic 1. This contradicts the fact that $K_{2,2,2,1}$ is one of the 35 minor-minimal graphs that does not imbed in the projective plane (see [1]).

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