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Matroid duality from topological duality in surfaces of nonnegative Euler characteristic

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Let G be a connected graph that is 2-cell imbedded in surface S and let G^* be its topological dual graph. We will define and discuss several matroids whose element set is $E(G)$ for S homeomorphic to the plane, projective plane, or torus. We will also state and prove old and new results of the type that the dual matroid of G is the matroid of the topological dual G^*.

1. Introduction

One of the most basic examples of matroid duality is the following. Let G be a graph imbedded in the plane and let G^* be its topological dual graph. If $\mathcal{M}(G)$ is the cycle matroid of G, then the dual matroid $\mathcal{M}^*(G) = \mathcal{M}(G^*)$. If G is a connected graph that is 2-cell imbedded in a surface of demigenus $d > 0$ (the demigenus is equal to 2 minus the euler characteristic of the surface), then $\mathcal{M}^*(G) \neq \mathcal{M}(G^*)$ for the simple reason that

$$\begin{aligned}
|V(G)| - |E(G)| + |F(G)| &= 2 - d \\
(|V(G)| - 1) + (|V(G^*)| - 1) &= |E(G)| - d \\
rk(\mathcal{M}(G)) + rk(\mathcal{M}(G^*)) &= |E(G)| - d \\
&\neq |E(G)|
\end{aligned}$$

while the rank of a matroid and its dual need to sum to the size of their element set. In this work we will utilize biased graphs and their matroids from [4] and [5] to describe methods of constructing matroids from G imbedded in the projective plane and torus of rank $|V(G)|$ that will give the desirable result that matroid duality comes from the topological dual of G. This duality will also yield a connection between closed cuts on surfaces with matroid circuits and cocircuits.

2. Definitions and background

2.1. Graphs, imbeddings, and homology

A graph is a pair $(V(G), E(G))$ in which $V(G)$ is a collection of topological points called *vertices* and $E(G)$ is a collection of topological 1-cells called *edges*. Each edge has two ends: an edge with both ends identified to a single vertex is called a *loop* and an edge with its ends identified to distinct vertices is called a *link*. If $S \subseteq E(G)$, then $G{:}S$ will denote the subgraph of G consisting of the edges in S and the vertices of G incident to edges from S. Since, for any $H \subseteq G$ without vertices of degree zero, $H = G{:}E(H)$, we may, without being ambiguous, refer to a subgraph of G without vertices of degree zero simply by referring to its edge set. We do this often.

Given an edge e, we denote an orientation of e by $\vec{e}$ and the reverse orientation of $\vec{e}$ by $\vec{e}^{\,-1}$. A *path* is a graph that is a subdivision of a link. A *circle* is a graph that is a subdivision of a loop. We specify an orientation of a path γ by $\vec{\gamma}$ and an orientation of a circle C by $\vec{C}$. The reverse orientations are denoted by $\vec{\gamma}^{-1}$ and $\vec{C}^{-1}$, respectively. The edges of an oriented circle or path are considered to be oriented consistently with the oriented path or circle.

In this paper, all imbeddings of graphs in surfaces will be 2-cell imbeddings. That is, when G is imbedded in S, $\mathsf{S} \setminus G$ is a disjoint union of open 2-cells. We use $F(G)$ to denote the set of open 2-cells (which we call *faces*) into which G subdivides the surface S in which it is imbedded. When G is 2-cell imbedded in S, it is necessary that G is connected. The *topological dual* of G is a graph imbedded in S, denoted by G^*, which is constructed as follows. The vertex set $V(G^*)$ is in bijective correspondence with $F(G)$ where we denote the vertex corresponding to $f \in F(G)$ by f^*. The vertex f^* is imbedded in the interior of the face f. The edge set $E(G^*)$ is in bijective correspondence with $E(G)$ where the edge in $E(G^*)$ corresponding to $e \in E(G)$ is denoted by e^*. The edge e^* has its ends connected to vertices f_1^* and f_2^* iff $f_1^* \neq f_2^*$ and the boundaries of faces f_1 and f_2 intersect on the edge e or $f_1^* = f_2^*$ and e is a boundary edge of the face f_1 twice. The edge e^* is imbedded connecting vertices (or vertex) f_1^* and f_2^* such that e and e^* intersect transversely at a point and e^* does not intersect G at any other point. From this definition, G^* is connected when G is connected, G^* is 2-cell imbedded in S and $(G^*)^* = G$. Given a set $X \subseteq E(G)$ we denote the corresponding subset of $E(G^*)$ as X^*. To avoid confusion, subsets of $E(G^*)$ will always be denoted with a superscript $*$. Figure 1 shows an imbedding of K_4 in the torus with its topological dual constructed and imbedded as defined above. The dual is drawn with the thicker edges.

Let $C(G)$ be the group of formal $\mathbb{Z}$-linear combinations of the oriented edges of G modulo the relation $\vec{e} + \vec{e}^{\,-1} = 0$. We call this the group of *chains* of G over $\mathbb{Z}$. If $\vec{\omega}$ is an oriented walk in the graph G, then we also use the symbol $\vec{\omega}$ to denote the element of $C(G)$ consisting of the sum of the oriented edges of $\vec{\omega}$ all with coefficient $1 \in \mathbb{Z}$. Let $Z(G)$ be the subgroup of $C(G)$ generated by the oriented circles of G, and let $B(G)$ be the subgroup of $Z(G)$ generated by the oriented face boundaries of G. These, respectively, are called the group of *cycles* and the group of *boundaries* of G over $\mathbb{Z}$. It is well known result of algebraic topology that the quotient group $H_1(\mathsf{S}) = Z(G)/B(G)$ is invariant, up to group isomorphism, for any graph G that is 2-cell imbedded in S. Given G imbedded in S, let $i : Z(G) \to H_1(\mathsf{S})$ be the natural homomorphism given by the imbedding. Some immediate and important properties coming from the definition of the homomorphism $i : Z(G) \to H_1(\mathsf{S})$ are given in Proposition 2.1.

Proposition 2.1.

(1) If $\vec{C}$ is an oriented circle, then $i(\vec{C}^{-1}) = -i(\vec{C})$.

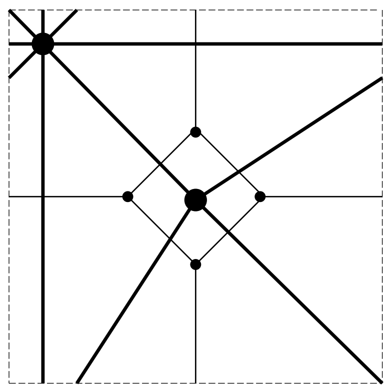
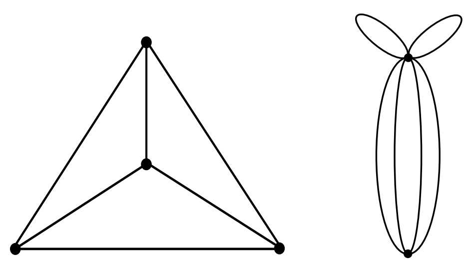


Figure 1 A depiction of K_4 (below and left) imbedded in the torus and drawn with the thinner lines and its corresponding dual graph (below and right) drawn in the torus with thicker lines.



(2) For any theta graph with circles oriented as shown in Figure 2, $i(\vec{C}_1) + i(\vec{C}_2) = i(\vec{C}_1 + \vec{C}_2) = i(\vec{C}_3)$.

It is well known that $H_1(\mathsf{T}) \cong \mathbb{Z} \times \mathbb{Z}$ (here T denotes the torus). Proposition 2.2 can be found in [2, page 214] and is used freely throughout this paper without further reference.

Proposition 2.2. *There exists an oriented circle C imbedded in T with $i(\vec{C}) = (m, n) \in H_1(\mathsf{T})$ iff (m, n) is a relatively prime pair of integers.*

Let $\vec{C}$ and $\vec{D}$ be two oriented circles imbedded in T such that $C \cap D$ is a finite collection of points. Each intersection point of $\vec{C}$ with $\vec{D}$ is one of three possible types: a clockwise transverse crossing, a counterclockwise transverse crossing, or a nontransverse crossing. Examples of clockwise and counterclockwise transverse crossings of $\vec{C}$ with $\vec{D}$ are shown in Figure 2.1. An example of a nontransverse crossing of two curves would be the intersection of the curves $y = 0$ and $y = x^2$ in the xy-plane. The *algebraic intersection* $\vec{C} \cdot \vec{D}$ is the number of clockwise crossings of $\vec{C}$ with $\vec{D}$ minus the number of counterclockwise crossings of $\vec{C}$ with $\vec{D}$. Because of Propositions 2.4 and 2.3, the algebraic intersection of curves on a surface can be especially useful when studying graphs imbedded in surfaces. Proposition 2.3 can be found in [2, section 6.4.3].

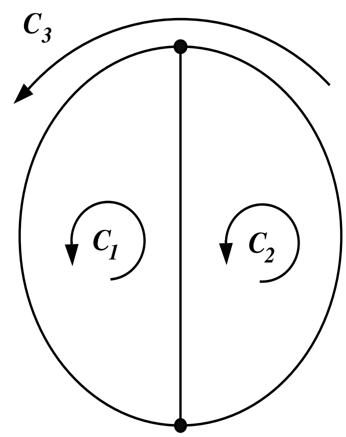


Figure 2 The desired orientations of the circles in the theta graph of Proposition 2.1.

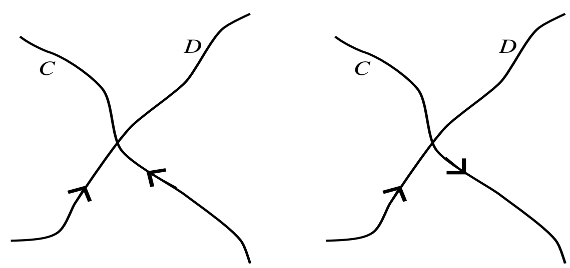


Figure 3 The left figure displays a clockwise transverse crossing of $\vec{C}$ with $\vec{D}$. The right is a counterclockwise transverse crossing of $\vec{C}$ with $\vec{D}$.

Proposition 2.3. *Let $\vec{C}$ and $\vec{D}$ are oriented circles imbedded in T that intersect in a finite number of points. If $i(\vec{C}) = (m_1, n_1)$ and $i(\vec{D}) = (m_2, n_2)$, then*

$$\vec{C} \cdot \vec{D} = \pm \begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix}$$

Proposition 2.4. *If G is a graph imbedded in surface S, C is the edge set of a circle in G, and D^* is the edge set of a circle in G^*, then the following are true.*

(1) $(G{:}C) \cap (G^*{:}D^*) = C \cap D^*$ *is a finite collection of points, each point being a transverse crossing of a pair of dual edges* (e, e^*).
(2) $|C \cap D^*| = |C \cap D|$.

In addition to the geometric meaning of the algebraic intersection of two oriented circles, Proposition 2.3 is even more useful in that one may impose a multiplicative structure on $\mathbb{Z} \times \mathbb{Z} \cong H_1(\mathsf{T})$ by defining the product $(m_1, n_1) \cdot (m_2, n_2)$ to be the 2×2

determinant in Proposition 2.3. Now the algebraic properties listed in Proposition 2.5 may be brought to bear on the elements of $Z(G)$ which include the oriented circles of G.

Proposition 2.5. *For all $C, D, D' \in Z(G)$, the following are true.*
(1) $i(C) \cdot i(D) = -i(D) \cdot i(C)$
(2) $i(C) \cdot i(D + D') = i(C) \cdot [i(D) + i(D')] = [i(C) \cdot i(D)] + [i(C) \cdot i(D')]$.

2.2. Biased graphs and their matroids coming from graphs imbedded in the torus

Biased graphs are discussed in [4]. We will state the relevant definitions. A collection of circles L in a graph G is called a *linear class* of circles if each theta subgraph of G contains 0, 1, or 3 circles from L. A *biased graph* is a pair (G, L) where G is a graph and L is a linear class of circles of G. Circles in L are called *balanced*. A subgraph H of G is called *balanced* if all circles of H are balanced; H is called *contrabalanced* if all circles of H are unbalanced.

A special case of a biased graph is an *additively* biased graph. This is a biased graph in which each theta subgraph contains either 1 or 3 balanced circles. A *signed graph* is a pair (G, σ) in which $\sigma : E(G) \to \{+, -\}$. Given a signed graph (G, σ), let C_+ be the collection of circles in G for which the product of signs on its edges is positive. In [3, Section 2] it is shown that the pair (G, C_+) is an additively biased graph; furthermore, any additively biased graph is of the form (G, C_+) for some signed graph (G, σ).

We will define two biased graphs associated with G. First, let $H(G)$ be the collection of circles C in G such that $i(\vec{C}) = -i(\vec{C}^{-1}) = \mathbf{0}$. The pair $\mathcal{H}(G) = (G, H(G))$ is a biased graph because Proposition 2.1(2) guarantees that a theta graph cannot have exactly two circles in homology class $\mathbf{0} \in H_1(\mathsf{T})$.

Second, let $A(G)$ be the collection of circles C such that the vector dot product $i(\vec{C}) \cdot (1, 1) \equiv 0 \mod 2$ (which happens iff $i(\vec{C}) = (m, n)$ and m and n have the same parity). The pair $\mathcal{A}(G) = (G, A(G))$ is an additively biased graph because Proposition 2.1(2) and the fact that $i(\vec{C}_1 + \vec{C}_2) \cdot (1, 1) = (i(\vec{C}_1) + i(\vec{C}_2)) \cdot (1, 1) = i(\vec{C}_1) \cdot (1, 1) + i(\vec{C}_2) \cdot (1, 1)$ guarantees that a theta graph may only have 1 or 3 circles in $A(G)$.

Proposition 2.6. *If G is 2-cell imbedded in T, then $\mathcal{H}(G)$ and $\mathcal{A}(G)$ are both unbalanced biased graphs.*

Proof. Since $H_1(\mathsf{T}) = Z(G)/B(G)$, the natural homomorphism $i : Z(G) \to H_1(\mathsf{T}) \cong \mathbb{Z} \times \mathbb{Z}$ is a surjection. So for each $(x, y) \in \mathbb{Z} \times \mathbb{Z}$, there is $z \in Z(G)$ such that $i(z) = (a, b)$. By the definition of $Z(G)$, there are $\lambda_1, \ldots, \lambda_t \in \mathbb{Z}$ and oriented circles $\vec{C}_1, \ldots, \vec{C}_t$ in G such that $z = \lambda_1 \vec{C}_1 + \ldots \lambda_t \vec{C}_t$. So if we insist that $i(z) \neq \mathbf{0}$ or that $i(z) \cdot (1, 1) = \lambda_1 i(\vec{C}_1) \cdot (1, 1) + \cdots + \lambda_t i(\vec{C}_t) \cdot (1, 1)$ is not even, then there must be some $i(\vec{C}_j) \neq \mathbf{0}$ or $i(\vec{C}_j) \cdot (1, 1)$ that is not even. Thus there is an unbalanced circle in each of $\mathcal{H}(G)$ and $\mathcal{A}(G)$. □

We will consider three matroids associated with a biased graph (G, L). Two of these matroids have element set $E(G)$ and the third has element set $E_0(G) = E(G) \cup e_0$. They are, respectively, the *lift* matroid $\mathcal{L}(G, L)$, the *bias* matroid $\mathcal{B}(G, L)$, and the complete lift matroid $\mathcal{L}_0(G, L)$.

These matroids are discussed in detail in [5]. We will review them in the next two paragraphs. The definitions of these matroids utilize the *cyclomatic number* of a graph G. It is the number of edges G has outside of a maximal forest; that is, $|E(G)|-|V(G)|+c(G)$ where $c(G)$ is the number of connected components of G. Also, in a matroid M, we will denote the rank of $X \subseteq E(M)$ by $rk_M(X)$. The rank of a matroid M is defined to be $rk_M(E(M))$ but is denoted more simply by $rk(M)$.

The circuits of the lift matroid $\mathcal{L}(G, L)$ are the edge sets of minimal balanced subgraphs of cyclomatic number 1 (i.e., balanced circles) and minimal contrabalanced subgraphs of cyclomatic number 2 (i.e., the union of two vertex-disjoint unbalanced circles or the union of two unbalanced circles that intersect in a vertex or path and whose union is contrabalanced). See Figure 4. Proposition 2.7 is from [5, Theorem 3.1(j)].

Proposition 2.7. *Given $X \subseteq E(G)$,*

$$rk_{\mathcal{L}(G,L)}(X) = \left\{ \begin{array}{ll} |V(G{:}X)| - c(G{:}X) & \textit{if } G{:}X \textit{ is unbalanced} \\ |V(G{:}X)| - c(G{:}X) - 1 & \textit{if } G{:}X \textit{ is balanced} \end{array} \right\}.$$

The complete lift matroid $\mathcal{L}_0(G, L)$ is defined to be the lift matroid of the biased graph consisting of $(G, L)_0$ which is (G, L) along with a new vertex v_0 attached to unbalanced loop e_0. Proposition 2.8 is apparent from the definitions of $\mathcal{L}(G, L)$ and $\mathcal{L}_0(G, L)$.

Proposition 2.8.

(1) $\mathcal{L}_0(G, L) \backslash e_0 = \mathcal{L}(G, L)$
(2) $\mathcal{L}_0(G, L)/e_0 = \mathcal{M}(G)$

The circuits of the bias matroid $\mathcal{B}(G, L)$ are the edge sets of minimal balanced subgraphs of cyclomatic number 1 (i.e., balanced circles) and minimal connected contrabalanced subgraphs of cyclomatic number 2 (i.e., the union of two vertex-disjoint unbalanced circles and a connecting path between them or the union of two unbalanced circles that intersect in a vertex or path and whose union is contrabalanced). See Figure 5. Proposition 2.9 is from [5, Theorem 2.1(j)].

Proposition 2.9. *Given $X \subseteq E(G)$,*

$$rk_{\mathcal{B}(G,L)}(X) = |V(G{:}X)| - b(G{:}X)$$

where $b(G{:}X)$ is the number of connected components of $G{:}X$ that are balanced.

Proposition 2.10 is evident from the definitions of $\mathcal{B}(G, L)$ and $\mathcal{L}(G, L)$.

Proposition 2.10. *If (G, L) is a biased graph, then $\mathcal{B}(G, L) = \mathcal{L}(G, L)$ iff there do not exist 2 vertex-disjoint unbalanced circles in (G, L).*

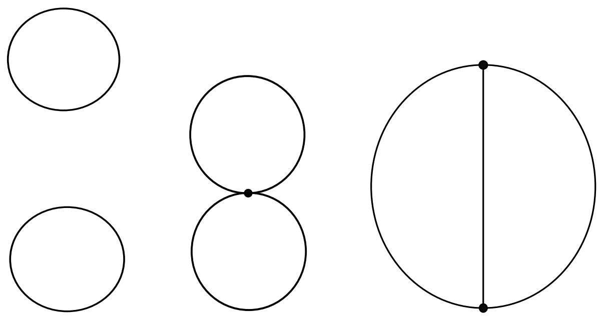


Figure 4 Lift matroid circuits are edge sets of balanced circles and edge sets of contrabalanced subgraphs that are subdivisions of one of the above graphs.

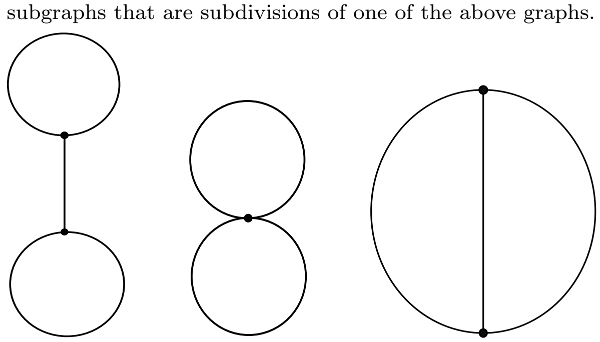


Figure 5 Bias matroid circuits are edge sets of balanced circles and edge sets of contrabalanced subgraphs that are subdivisions of one of the above graphs.

2.3. Matroid Duality

Let M be a matroid on element set E. If $\mathfrak{B}$ is the set of bases of M then the set of cobases of M is $E \setminus \mathfrak{B} = \{E \setminus B : B \in \mathfrak{B}\}$. The dual matroid M^* is normally defined to be the matroid on E whose set of bases is $E \setminus \mathfrak{B}$ and whose set of cobases is $\mathfrak{B}$. For our purposes, it will be more convenient to do the following. Let E^* be a set in bijective correspondence with E by the map $e \mapsto e^*$. The subset of E^* corresponding to $X \subseteq E$ will be written as X^*; additionally, subsets of E^* will always be written with a $*$ superscript. Define M^* to be the matroid on E^* whose set of bases is $(E \setminus \mathfrak{B})^*$ and whose set of cobases is $\mathfrak{B}^*$. Proposition 2.11 is a characterization of matroid duality.

Proposition 2.11. *If M is a matroid on E and N is a matroid on E^*, then $M^* = N$ iff $rk(M)+rk(N) = |E|$ and, for each circuit C of M and each circuit D^* of N, $|C \cap D| \neq 1$.*

Proof. The conditions on rank and circuit intersections are well known necessary conditions for $M^* = N$. Conversely, let B be a basis of M. Since $rk(M) + rk(N) = |E|$, showing that $(E \setminus B)^*$ is independent in N will imply that $(E \setminus B)^*$ is a basis of N. This will prove that $M^* = N$.

By way of contradiction, if we assume that $(E \setminus B)^*$ is a dependent set in N, there is $D^* \subseteq (E \setminus B)^*$ that is a circuit of N. Pick $e \in D$. Since B is a basis of M and $e \notin B$, there is $C \subseteq B \cup \{e\}$ that is a circuit of M and contains e. This gives us the contradiction $|C \cap D| = 1$. □

2.4. Matroid duality from graphs imbedded in the plane and projective plane

Proposition 2.11 along with Proposition 2.4 are excellent tools for proving results about matroid duality from graphs imbedded in surfaces. Using them me will prove two known results (Theorem 2.1 and Corollary 2.2) and one closely related result (Theorem 2.2).

Theorem 2.1. *If G be a connected graph imbedded in the plane, then $\mathcal{M}^*(G) = \mathcal{M}(G^*)$.*

Proof. Since G is connected and imbedded in the plane, $|V(G)| - |E(G)| + |F(G)| = 2$. By the definition of G^*, $|V(G)| - |E(G)| + |V(G^*)| = 2$. Thus

$$\begin{array}{lcl} (|V(G)| - 1) & + & (|V(G^*)| - 1) & = & |E(G)| \\ rk(\mathcal{M}(G)) & + & rk(\mathcal{M}(G^*)) & = & |E(G)|, \end{array}$$

which is the first condition from Proposition 2.11 needed to prove that $\mathcal{M}^*(G) = \mathcal{M}(G^*)$. The second and last condition of Proposition 2.11 is that, for each circle C in G and circle D^* in G^*, $|E(C) \cap E(D)| \neq 1$. By Proposition 2.4, $|E(C) \cap E(D)| \neq 1$ iff $|C \cap D^*| \neq 1$. Well, then Jordan curve theorem states that every simple closed curve in the plane separates the plane into two regions. Since every intersection of C and D^* is transverse, $|C \cap D^*|$ is even. Thus $|C \cap D| \neq 1$, as required. □

Corollary 2.1. *If G is a graph imbedded in the plane and $X \subseteq E(G)$, then $G{:}X$ separates the plane iff X^* separates G^*.*

Proof. It is known that the cocircuits of $\mathcal{M}(G)$ are the bonds of G. Since $\mathcal{M}^*(G) = \mathcal{M}(G^*)$, C is a circuit of $\mathcal{M}(G)$ iff C^* is a cocircuit of $\mathcal{M}(G^*)$. Thus $G{:}X$ separates the plane iff $G{:}X$ contains a circle iff X^* contains a bond of G^* iff X^* separates G^*. □

Corollary 2.2 below is an unpublished result of Thomas Zaslavsky from [6]. It follows from Theorem 2.2, Proposition 2.8, and the fact that, for any matroid M and $e \in E(M)$, $(M/e)^* = M^* \backslash e^*$. Before presenting these results, we note a few facts about graphs imbedded in the projective plane, which we denote by P_2.

Proposition 2.12. *If G is a connected graph that is 2-cell imbedded in the projective plane, then*

(1) $H_1(\mathsf{P}_2) = Z(G)/B(G) \cong \mathbb{Z}_2$,
(2) C is a circle in G that separates P_2 iff $i(C) = 0$, and
(3) if C and C' are circles such that $i(C) = i(C') = 1$, then $C \cap C' \neq \emptyset$.

Now let $H(G)$ be the collection of circles in G such that $i(C) = 0$ (i.e., $H(G)$ is the collection of circles in G that separate P_2). It follows from Propositions 2.1, 2.12, and 2.10 that $\mathcal{H}(G) = (G, H(G))$ is an additively biased graph for which $\mathcal{B}(\mathcal{H}(G)) = \mathcal{L}(\mathcal{H}(G))$.

Theorem 2.2. *If G is a graph that is 2-cell imbedded in the projective plane, then* $\mathcal{L}_0^*(\mathcal{H}(G)) = \mathcal{L}_0(\mathcal{H}(G^*))$.

Proof. Since G^* is 2-cell imbedded in the projective plane, G is connected and $i : Z(G) \to H_1(\mathsf{P}_2) = Z(G)/B(G)$ is onto. Thus there must be an unbalanced circle in $\mathcal{H}(G)$. Thus $rk(\mathcal{L}_0(\mathcal{H}(G))) = |V(G)|$. Similarly, $rk(\mathcal{L}_0(\mathcal{H}(G^*))) = |V(G^*)|$. Since G is connected and is 2-cell imbedded in the projective plane, we have $|V(G)| - |E(G)| + |F(G)| = 1$. Thus

$$\begin{array}{lclcl} |V(G)| & + & |V(G^*)| & = & |E(G)| + 1 \\ rk(\mathcal{L}_0(\mathcal{H}(G))) & + & rk(\mathcal{L}_0(\mathcal{H}(G^*))) & = & |E_0(G)|. \end{array}$$

Our result will now follow by showing that, for any circuit C in $\mathcal{L}_0(\mathcal{H}(G))$ and any circuit D^* in $\mathcal{L}_0(\mathcal{H}(G^*))$, we have $|C \cap D^*| \neq 1$. Now, without loss of generality, either C is a balanced circle or neither C nor D is a balanced circle.

Case 1: Recall that C is a balanced circle because $i(C) = 0$. Thus C separates P_2. Since D is either a circle or the union of two edge-disjoint circles and since G and G^* only intersect in transversely crossing dual edge pairs, $|C \cap D^*|$ is even. In particular $|C \cap D^*| \neq 1$.

Case 2: Since both C and D^* are not balanced circles, C is the union of two edge-disjoint circles in G and D^* is the union of two edge-disjoint circles in G^*. Since G and G^* only intersect in transversely crossing dual edge pairs, we get the following: $|C \cap D^*| \geq 4$ when $e_0 \notin C$ and $e_0^* \notin D^*$, $|C \cap D^*| \geq 2$ when $e_0 \notin C$ iff $e_0^* \in D^*$, and $|C \cap D^*| \geq 2$ when $e_0 \in C$ and $e_0^* \in D^*$. In all cases $|C \cap D^*| \neq 1$. □

Corollary 2.2. *If G is a graph that is 2-cell imbedded in the projective plane, then* $\mathcal{M}^*(G) = \mathcal{L}(\mathcal{H}(G^*))$.

3. Lift matroid duality from topological duality

Theorems 3.1 and 3.2 are both similar to the well-known result that $\mathcal{M}^*(G) = \mathcal{M}(G^*)$ for a graph G imbedded in the plane.

Theorem 3.1. *If G is a connected graph that is 2-cell imbedded in the torus, then* $\mathcal{L}^*(\mathcal{H}(G)) = \mathcal{L}(\mathcal{H}(G^*))$.

Theorem 3.2. *If G is a connected graph that is 2-cell imbedded in the torus, then* $\mathcal{L}^*(\mathcal{A}(G)) = \mathcal{L}(\mathcal{A}(G^*))$.

Before the proofs of Theorems 3.1–3.2 we present Lemmas 3.1–3.3.

Lemma 3.1. *If G is a connected graph that is 2-cell imbedded in the torus, then*

(1) the matroids $\mathcal{B}(\mathcal{H}(G))$, $\mathcal{L}(\mathcal{H}(G))$, $\mathcal{B}(\mathcal{A}(G))$, and $\mathcal{L}(\mathcal{A}(G))$ all have rank $|V(G)|$, and

(2) $|V(G)| - |E(G)| + |F(G)| = 0$.

Proof. Part (1) follows from Proposition 2.6 and the definition of rank in the bias and lift matroids. Part (2) is Euler's formula. □

Lemma 3.2. *Let G be a graph that is 2-cell imbedded in the torus.*

(1) If C and D are vertex disjoint circles in G with each $i(\vec{C}), i(\vec{D}) \neq \mathbf{0}$, then $i(\vec{C}) = \pm i(\vec{D})$.
(2) If Θ is a contrabalanced theta graph in $\mathcal{H}(\mathcal{G})$ with circles C_1, C_2, C_3, then $i(\vec{C}_j) \neq \pm i(\vec{C}_k)$ for each $j \neq k$.

Proof. (1) Since C and D are vertex disjoint, $\vec{C} \cdot \vec{D} = 0$. A 2×2 determinant with nonzero rows is zero is iff the first row is a constant multiple of the second. If this multiple is anything other than ± 1, then either $i(\vec{C})$ or $i(\vec{D})$ is not a relatively prime pair.
(2) By way of contradiction, say that $i(\vec{C}_1) = i(\vec{C}_2)$. Thus $i(\vec{C}_3) = \pm i(\vec{C}_1) \pm i(\vec{C}_2) = 0$ or $\pm 2i(\vec{C}_1)$; however, $i(\vec{C}_3) \neq \mathbf{0}$ because Θ is contrabalanced and $i(\vec{C}_3) \neq \pm 2i(\vec{C}_1)$ because $i(\vec{C}_3)$ is a relatively prime pair of integers. □

Lemma 3.3. *If G is a graph imbedded in S, C is a circle in G, and D^* is a circle in G^*, then*

(1) $|C \cap D^*| = 1$ implies that $\vec{C} \cdot \vec{D} = \pm 1$, and
(2) $|C \cap D^*|$ is even iff $\vec{C} \cdot \vec{D}$ is even.

Proof. Both parts follow from the fact that G and G^* only intersect in transversely crossing edge pairs e, e^*. □

Proof of Theorem 3.1 To prove our theorem we will use Proposition 2.11. First, Euler's Formula states that $|V(G)| - |E(G)| + |F(G)| = 0$ when G is a 2-cell imbedding. Since G and G^* are unbalanced (by Proposition 2.6), $rk(\mathcal{L}(\mathcal{H}(G))) = |V(G)|$ and $rk(\mathcal{L}(\mathcal{H}(G^*))) = |F(G)|$. Thus $rk(\mathcal{L}(\mathcal{H}(G))) + rk(\mathcal{L}(\mathcal{H}(G^*))) = |E(G)|$. So to complete the proof we need only show, for any lift circuit C in $\mathcal{H}(G)$ and lift circuit D^* in $\mathcal{H}(G^*)$, that $|C \cap D^*| \neq 1$. We break the proof into two cases. In the first case, C is a balanced circle and, in the second case, both C and D are not balanced circles.
Case 1: Since C is a balanced circle in $\mathcal{H}$, $i(\vec{C}) = \mathbf{0}$. Thus, for any circle D_0^* in D^*, $\vec{C} \cdot \vec{D}_0^*$ is even. Thus $|E(C) \cap E(D_0)| \neq 1$.

Case 2: Since C and D^* are both not balanced circles we may write $C = C_1 \cup C_2$ and $D^* = D_1^* \cup D_2^*$ where (C_1, C_2) and (D_1^*, D_2^*) are modular pairs of circles in nonzero homology classes. Without loss of generality, we may split the remainder of the proof into the following four subcases:

$$i(\vec{C}_1) = \pm i(\vec{C}_2) = \pm i(\vec{D}_1) = \pm i(\vec{D}_2),$$
$$i(\vec{C}_1) = \pm i(\vec{C}_2) \neq \pm i(\vec{D}_1) = \pm i(\vec{D}_2),$$
$$i(\vec{C}_1) = \pm i(\vec{C}_2) \text{ and } \pm i(\vec{D}_1) \neq \pm i(\vec{D}_2), \text{ and}$$
$$i(\vec{C}_1) \neq \pm i(\vec{C}_2) \text{ and } \pm i(\vec{D}_1) \neq \pm i(\vec{D}_2).$$

Case 2.1: By Lemma 3.2, $E(C_1) \cap E(C_2) = E(D_1^*) \cap E(D_2^*) = \emptyset$. Since each $\vec{C}_j \cdot \vec{D}_k^* = 0$, each $|E(C_j)\cap E(D_k)|$ is even. Thus $|E(C)\cap E(D)| = \sum_{j,k\in\{1,2\}} |E(C_j)\cap E(D_k)|$ is even (in particular, $|E(C)\cap E(D)| \neq 1$).

Case 2.2: By Lemma 3.2, $E(C_1)\cap E(C_2) = E(D_1^*)\cap E(D_2^*) = \emptyset$. Since each $\vec{C}_j \cdot \vec{D}_k^* \neq 0$, each $|E(C_j)\cap E(D_k)| \geq 1$. Thus $|E(C)\cap E(D)| = \sum_{j,k\in\{1,2\}} |E(C_j)\cap E(D_k)| \geq 4$.

Case 2.3: By Lemma 3.2, $E(C_1)\cap E(C_2) = \emptyset$. Without loss of generality, $i(\vec{C}_1) \neq \pm i(\vec{D}_1^*)$. Thus each $\vec{C}_j \cdot \vec{D}_1^* \neq 0$. Thus $|E(C)\cap E(D)| \geq \sum_{j\in\{1,2\}} |E(C_j)\cap E(D_1)| \geq 2$.

Case 2.4: Without loss of generality $\vec{C}_1 \cdot \vec{D}_1^* \neq 0$ and $\vec{C}_2 \cdot \vec{D}_2^* \neq 0$. We may now conclude that $|E(C)\cap E(D)| \geq 2$ unless we have that $E(C_1)\cap E(D_1) = E(C_2)\cap E(D_2) = \{e\}$. In the latter case, C must be a contrabalanced theta graph in $\mathcal{H}$ because $E(C_1)\cap E(C_2) \neq \emptyset$. Consider C_1+C_2, which is the third circle in C. We cannot have that $(\vec{C}_1+\vec{C}_2)\cdot\vec{D}_1^* = (\vec{C}_1+\vec{C}_2)\cdot \vec{D}_2^* = 0$ or else $i(\vec{D}_1^*) = \pm i(\vec{C}_1+\vec{C}_2) = \pm i(\vec{D}_2^*)$ while we assumed $i(\vec{D}_1^*) \neq \pm i(\vec{D}_2^*)$. Thus $|E(C_1+C_2)\cap E(D_1)| \geq 1$ or $|E(C_1+C_2)\cap E(D_2)| \geq 1$ and, since $e \in E(C_1)\cap E(C_2)$, we may now conclude that $|E(C)\cap E(D)| \geq 2$. □

Proof of Theorem 3.2 First, $rk(\mathcal{L}(\mathcal{A}(G)))+rk(\mathcal{L}(\mathcal{A}(G^*))) = |E(G)|$ follows from the same argument as in the first paragraph of the proof of Theorem 3.1. So let C be a lift circuit of $\mathcal{A}(G)$ and D^* be a lift circuit of $\mathcal{A}(G^*)$. We will show that $|E(C)^*\cap E(D)| \neq 1$. Recall that an additively biased graph contains no contrabalanced theta graphs. Thus a lift circuit is either a balanced circle or the union of two edge-disjoint unbalanced circles. We divide the remainder of the proof into three cases: C and D^* are both balanced circles, C is a balanced circle and D^* is not, C and D^* are both not balanced circles.

Case 1: Here $i(\vec{C})\cdot(1,1)$ and $i(\vec{D}^*)\cdot(1,1)$ are both even. Thus $i(\vec{C})$ and $i(\vec{D}^*)$ are both pairs of odd integers. Thus the 2×2 determinant $\vec{C}\cdot\vec{D}^*$ is even. Thus $|E(C)\cap E(D)|$ is even. Thus $|E(C)\cap E(D)| \neq 1$.

Case 2: Here $i(\vec{C})$ is a pair of odd integers and D^* is the union of two edge-disjoint circles D_1^* and D_2^* such that each $i(\vec{D}_j^*)$ is a pair of integers of different parity. Thus the 2×2 determinant $\vec{C}\cdot\vec{D}_j^*$ is odd. Thus each $|E(C)\cap E(D_j)| \geq 1$. Thus $|E(C)\cap E(D)| \geq 2$.

Case 3: Here C is the union of edge-disjoint circles C_1 and C_2 and D^* is the union of two edge-disjoint circles D_1^* and D_2^*. The details of the remainder of the proof are contained in the proof of Theorem 3.1 *Case 2.* □

3.1. The topological action of lift circuits on the torus

Theorem 3.1 tells us that if C is a lift circuit of $\mathcal{H}(G)$, then C^* is a lift cocircuit of $\mathcal{H}(G^*)$. In a matroid M, $H\subseteq E$ is a hyperplane iff $E\setminus H$ is a cocircuit. Thus a lift cocircuit in a biased graph is a minimal edge set D such that $rk_{\mathcal{L}(G,L)}(E(G)\setminus D) = rk_{\mathcal{L}(G,L)}(E(G))-1$. Thus D is minimal edge set that either disconnects G or whose removal leaves a balanced subgraph of (G,L). Given this and Proposition 2.7, we have Proposition 3.1.

Proposition 3.1. *If C is a lift cocircuit of (G,L), then*

(1) $G\setminus C$ is a connected and balanced subgraph of (G,L) or

(2) C is a bond of G separating connected subgraphs G_1 and G_2, no more than one of which is balanced.

Combining Theorem 3.1 and Proposition 3.1 yields Theorem 3.3.

Theorem 3.3. *If G is 2-cell imbedded in the torus T and C is a lift circuit of $\mathcal{H}(G)$, then either $\mathsf{T} \setminus C$ has two connected components, no more than one of which is homeomorphic to a disk, or $\mathsf{T} \setminus C$ is homeomorphic to a disk; furthermore, $\mathsf{T} \setminus C$*

(1) has two connected components iff C is a balanced circle or C is the union of two unbalanced circles C_1 and C_2 with $i(\vec{C}_1) = \pm i(\vec{C}_2)$ and

(2) is homeomorphic to a disk iff C is the union of two unbalanced circles C_1 and C_2 with $\vec{C}_1 \cdot \vec{C}_2 = \pm 1$.

Proof. The collection of 2-cells $F(G)$ is a collection of open polygons whose boundary edges are identified in distinct pairs to the edges of G which then yields a torus. Each $e^* \in E(G^*)$ represents an identification of one distinct pair of boundary edges of faces f_1 and f_2 (which are not necessarily distinct) to the edge $e \in E(G)$. The action of removing e^* from G^* is thus equivalent to removing the identification of the boundary edges of f_1 and f_2 to $e \in E(G)$ (i.e., removing e^* from G^* is equivalent to cutting the torus along e from G). Also, if $X^* \subseteq E(G^*)$, then each connected component $K^*{:}G^*$ of $G^* \setminus X^*$ will correspond to a connected component, call it T_K, obtained from cutting the torus along $X \subseteq E(G)$; furthermore T_K is constructed from the faces of $F(G)$ corresponding to the vertices of $K^*{:}G^*$ with the boundary identifications represented by the edges of K^*. Since a 2-cell is the only surface with boundary up to homeomorphism that has trivial first homology group, T_K will be a 2-cell iff K^* is a balanced subgraph of $\mathcal{H}(G^*)$. This all along with Theorem 3.1 and Proposition 3.1 imply our desired results. □

4. Bias matroid duality from topological duality

Theorem 4.1. *If G is 2-cell imbedded in the torus, then $\mathcal{B}^*(\mathcal{H}(G)) = \mathcal{B}(\mathcal{H}(G^*))$.*

Proof. First, $rk(\mathcal{B}(\mathcal{H}(G))) + rk(\mathcal{B}(\mathcal{H}(G^*))) = |E(G)|$ follows from the same argument as in the first paragraph of the proof of Theorem 3.1.

Second, let C be a circuit of $\mathcal{B}(\mathcal{H}(G))$ and let D^* be a circuit of $\mathcal{B}(\mathcal{H}(G^*))$. A circuit C' in a bias matroid $\mathcal{B}(G, L)$ is also a circuit in the lift matroid $\mathcal{L}(G, L)$ except when C' consists of two vertex disjoint unbalanced circles an a connecting path (called a *loose handcuff*). In this case C' is the union of a circuit in $\mathcal{L}(G, L)$ along with a connecting path. Thus we may use the exact same arguments as in the proof of Theorem 3.1 to show that $|E(C) \cap E(D)| \neq 1$ except when either C or D^* is a loose handcuff. In the case where either C or D^* is a loose handcuff let $C_{\mathcal{L}}$ and $D^*_{\mathcal{L}}$ be the unique lift circuits contained in C and D, respectively. By Theorem 3.1, $|E(C_{\mathcal{L}}) \cap E(D_{\mathcal{L}})| \neq 1$. So we may now conclude that $|E(C) \cap E(D)| \neq 1$ unless $|E(C_{\mathcal{L}}) \cap E(D_{\mathcal{L}})| = 0$. Without loss of generality we split the remainder of the proof into the following three cases where, in each one, C is a loose handcuff and $|E(C_{\mathcal{L}}) \cap E(D_{\mathcal{L}})| = 0$: in *Case 1*, D^* is a balanced circle; in *Case 2*, D^* contains two circles D^*_1 and D^*_2 for which $i(\vec{D}^*_1) \neq \pm i(\vec{D}^*_2)$; in *Case 3*, D^* contains two circles D^*_1 and D^*_2 for which $i(\vec{D}^*_1) = \pm i(\vec{D}^*_2)$. In each of the three cases, write $C = C_1 \cup C_2 \cup \gamma$ in which $C_{\mathcal{L}} = C_1 \cup C_2$ and γ is the connecting path of the loose handcuff. By Lemma 3.2(1), $i(\vec{C}_1) = \pm i(\vec{C}_2)$.

Case 1: Since D is a balanced circle, D bounds a disk in the torus. Since $|E(C_{\mathcal{L}}) \cap E(D_{\mathcal{L}})| = 0$, both endpoints of the path γ are not in the interior of the disk bounded by D. Thus $|E(\gamma) \cap E(D)|$ is even. Thus $|E(C) \cap E(D)|$ is even. In particular, $|E(C) \cap E(D)| \neq 1$.

Case 2: Since $i(\vec{D}_1^*) \neq \pm i(\vec{D}_2^*)$, there is j and k such that $\vec{C}_j \cdot \vec{D}_k^*$ is odd, which cannot happen when $|E(C_{\mathcal{L}}) \cap E(D_{\mathcal{L}})| = 0$. Thus this case does not occur when $|E(C_{\mathcal{L}}) \cap E(D_{\mathcal{L}})| = 0$.

Case 3: Since $i(\vec{D}_1^*) = \pm i(\vec{D}_2^*)$ and $|E(C_{\mathcal{L}}) \cap E(D_{\mathcal{L}})| = 0$, we get that each $\vec{D}_j^* \cdot \vec{C}_1 = 0$. Thus $i(\vec{D}_1^*) = \pm i(\vec{D}_2^*) = \pm i(\vec{C}_1) = \pm i(\vec{C}_2)$. By Theorem 3.3, $C_{\mathcal{L}}$ separates the torus into two connected components, call them T_1 and T_2, both of whose boundaries are the two vertex disjoint circles C_1 and C_2. Without loss of generality, the connecting path γ is contained in T_1 with its endpoints connecting the boundary components C_1 and C_2. Since $|E(C_{\mathcal{L}}) \cap E(D_{\mathcal{L}})| = 0$, each circle D_1^* and D_2^* and is contained in either T_1 or T_2. We separate the remainder of the proof into three subcases: in *Case 3.1*, D_1^* and D_2^* are both contained in T_1; in *Case 3.2*, D_1^* and D_2^* are both contained in T_2; and in *Case 3.3*, D_1^* is contained in T_1 and D_2^* is contained in T_2. In all three of these subcases let C_γ be a circle formed by connecting the endpoints of γ in T_2. Since C_γ intersects each of C_1 and C_2 transversely at one point, $\vec{C}_\gamma \cdot \vec{C}_1 = \pm \vec{C}_\gamma \cdot \vec{C}_2 = \pm 1$; furthermore, since there can only be one relatively prime pair of integers, up to negation, satisfying this dot product relation, the homology class of $\vec{C}_\gamma$ is uniquely determined, up to negation.

Case 3.1: Here $\vec{C}_\gamma \cdot \vec{D}_1^* = \pm \vec{C}_\gamma \cdot \vec{D}_2^* = \pm 1$. Thus, since D_1^* and D_2^* are both contained in T_1, $\sum_{j=1}^{2} |E(\gamma) \cap E(D_j)| \geq 2$. Thus $|E(C) \cap E(D)| \geq 2$.

Case 3.2: Since γ is contained in T_1, $\gamma \cap D_{\mathcal{L}}^* = \emptyset$. Thus $E(C) \cap E(D_{\mathcal{L}} = \emptyset$. Now either $D = D_{\mathcal{L}}$ or D has a connecting path of γ_D^*. If $D = D_{\mathcal{L}}$, then we are done; otherwise, since γ_D^* has both endpoints in T_1, $|\gamma_D^* \cap C| \neq 1$. Thus $|E(C) \cap E(D)| \neq 1$.

Case 3.2: In this case, D_1^* and D_2^* must be vertex disjoint. Thus D is a loose handcuff with connecting path γ_D^* that has one endpoint in T_1 and the other in T_2. Thus $|\gamma_D^* \cap C_{\mathcal{L}}| \geq 1$. Similar to $C_{\mathcal{L}}$, $D_{\mathcal{L}}^*$ separates the torus into two connected components, call them T_1' and T_2'. Since D_1^* is contained in T_1 and D_2^* is contained in T_2, one can show that C_1 is contained in T_j' and C_2 is contained in T_k' where $j \neq k$. Thus the connecting path γ has one endpoint in T_1' and the other in T_2'. Thus $|\gamma \cap D_{\mathcal{L}}^*| \geq 1$. Thus $|E(C) \cap E(D)| \geq |\gamma_D^* \cap C_{\mathcal{L}}| + |\gamma \cap D_{\mathcal{L}}^*| \geq 2$. □

Theorem 4.2. *If G is 2-cell imbedded in the torus, then $\mathcal{B}^*(\mathcal{A}(G)) = \mathcal{B}(\mathcal{A}(G^*))$.*

Proof. First, $rk(\mathcal{B}(\mathcal{A}(G))) + rk(\mathcal{B}(\mathcal{A}(G^*))) = |E(G)|$ follows from the same argument as in the first paragraph of the proof of Theorem 3.1.

Second, if C is a bias circuit $\mathcal{A}(G)$ and D^* is a bias circuit of $\mathcal{A}(G^*)$, then we can show that $|C \cap D| \neq 1$ by the same arguments as in the proof of Theorem 4.1 because a bias circuit in $\mathcal{A}(G)$ is a bias circuit in $\mathcal{H}(G)$ unless it is a balanced circle. In the case where C is a balanced circle, $i(\vec{C}) \cdot (1,1)$ is even but $i(\vec{C}) \neq \mathbf{0}$. This implies that $|C \cap D_{\mathcal{L}}^*| \geq 2$

unless D^* is a balanced circle in $\mathcal{A}(G)$. But when D is a balanced circle, this implies that C and D^* are both lift circuits. Thus $|C \cap D^*| \neq 1$. □

5. Concluding remarks

Similar matroid duality results should also hold for connected graphs that are 2-cell imbedded in the klein bottle. For duality results coming from graphs imbedded in surfaces S of negative Euler characteristic, new matroids of rank $|V(G)| + \lceil \frac{2-\chi(\mathsf{S})}{2} \rceil$ need to be identified.

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