(1)

n	0	1	2	3	4	5	6	7
f_n	0	1	1	2	3	5	8	13

Base case ($n=6$)

$$f_6 = 8$$

$$(1.6)^6 = 6.55$$

and $8 > 6.55$.

Induction Step

Assume that ~~for~~ each $k \in \{6, 7, \ldots, n\}$ that $f_n > (1.6)^{n-2}$.

Show that $f_{n+1} > (1.6)^{n-1}$.

Well

$$f_{n+1} \underset{\substack{\text{definition}\\ \text{of } f_n}}{=} f_n + f_{n-1} \underset{\substack{\text{by the}\\ \text{induction}\\ \text{hypothesis}}}{>} (1.6)^{n-2} + (1.6)^{n-3} = (1.6)(1.6)^{n-3} + (1.6)^{n-3}$$

$$= (1.6+1)(1.6)^{n-3} = (2.6)(1.6)^{n-3} > (2.56)(1.6)^{n-3} = (1.6)^2(1.6)^{n-3}$$

$$= (1.6)^{n-1}, \text{ as required.}$$

②

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$$

$$\frac{1}{(1-x)^2}=\sum_{n=1}^{\infty}nx^{n-1}=\sum_{n=0}^{\infty}(n+1)x^n$$

$$\frac{x}{(1-x)^2}=\sum_{n=1}^{\infty}nx^n$$

$$p(x)=\sum_{n=0}^{\infty}r_n x^n$$

$$p(x)=1+2x+\sum_{n=2}^{\infty}r_n x^n$$

$$p(x)=1+2x+\sum_{n=2}^{\infty}\left(4r_{n-1}-4r_{n-2}+3n+1\right)x^n$$

$$p(x)=1+2x+\sum_{n=2}^{\infty}4r_{n-1}x^n-\sum_{n=2}^{\infty}4r_{n-2}x^n+\sum_{n=2}^{\infty}3nx^n+\sum_{n=2}^{\infty}x^n$$

$$p(x)=1+2x+4x\sum_{n=2}^{\infty}r_{n-1}x^{n-1}-4x^2\sum_{n=2}^{\infty}r_{n-2}x^{n-2}+3\sum_{n=2}^{\infty}nx^n+\sum_{n=2}^{\infty}x^n$$

$$p(x)=1+2x+4x\sum_{n=1}^{\infty}r_n x^n-4x^2\sum_{n=0}^{\infty}r_n x^n+3\sum_{n=2}^{\infty}nx^n+\sum_{n=2}^{\infty}x^n$$

$$p(x)=1+2x+4x\left(p(x)-1\right)-4x^2p(x)+3\left(\frac{x}{(1-x)^2}-x\right)+\left(\frac{1}{1-x}-1-x\right)$$

$$p(x)\left(1-4x+4x^2\right)=-6x+\frac{3x}{(1-x)^2}+\frac{1}{1-x}$$

$$p(x)=\frac{-6x}{(1-2x)^2}+\frac{3x}{(1-x)^2(1-2x)^2}+\frac{1}{(1-x)(1-2x)^2}$$

partial fractions using Wolfram Alpha

$$p(x)=\frac{3}{1-2x}-\frac{3}{(1-2x)^2}+\frac{9}{1-x}+\frac{3}{(1-x)^2}-\frac{18}{1-2x}+\frac{6}{(1-2x)^2}+\frac{1}{1-x}-\frac{2}{1-2x}+\frac{2}{(1-2x)^2}$$

$$p(x)=\frac{10}{1-x}+\frac{3}{(1-x)^2}-\frac{17}{1-2x}+\frac{5}{(1-2x)^2}$$

$$\frac{1}{1-2x}=\sum_{n=0}^{\infty}2^nx^n$$

$$\frac{2}{(1-2x)^2}=\sum_{n=1}^{\infty}n2^nx^{n-1}$$

$$\frac{1}{(1-2x)^2}=\sum_{n=0}^{\infty}(n+1)2^nx^n$$

$$p(x)=\sum_{n=0}^{\infty}\left(10+3(n+1)-17\cdot 2^n+5(n+1)2^n\right)x^n$$

So

$$r_n = 10 + 3(n+1) - 17 \cdot 2^n + 5(n+1)2^n$$

$$\boxed{r_n = (3n+13) + (5n-12)2^n}$$

Some checks

$\boxed{\begin{array}{l}\text{Calculate with} \\ r_n = (3n+13) + (5n-12)2^n\end{array}}$

$r_0 = 13 + (0-12) = 1$

$r_1 = 16 + (-7)2 = 2$

$r_2 = 19 + (-2)4 = 11$

$r_3 = 22 + (3)8 = 46$

$\boxed{\begin{array}{l}\text{Calculate with } r_0 = 1,\ r_1 = 2 \\ r_n = 4r_{n-1} - 4r_{n-2} + 3n + 1 \text{ for } n \geq 2\end{array}}$

$r_0 = 1$

$r_1 = 2$

$r_2 = 4(2) - 4(1) + 6 + 1 = 11$

$r_3 = 4(11) - 4(2) + 9 + 1 = 46$