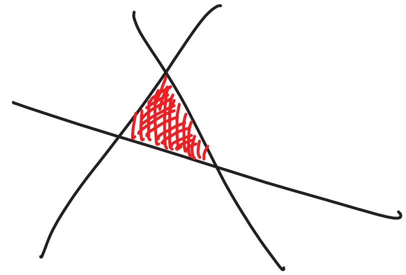
Solutions for assignment #4

① Base case $(n=3)$ 3 lines in general position creates ① bounded region



and $\frac{n^2-3n+2}{2} = \frac{9-9+2}{2} = 1$.

Inductive step

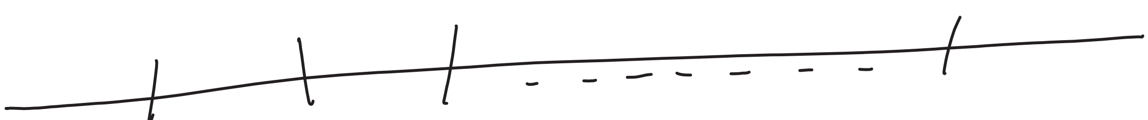
Assume that $k \in \{3,\dots,n\}$ lines in general position defines $\frac{k^2-3k+2}{2}$ bounded regions.

Show that $(n+1)$ lines in general position defines

$$\frac{(n+1)^2-3(n+1)+2}{2} = \frac{n^2-n}{2}$$

bounded regions.

Given an arrangement $\mathcal{L}$ of $(n+1)$ lines in general position, take out one line h. The arrangement $\mathcal{L}-h$ consists of n lines in general position. By the inductive hypothesis $\mathcal{L}-h$ defines $\frac{n^2-3n+2}{2}$ bounded regions. The line h intersects the n remaining lines in n points.



These n points divide h into $n-1$ bounded segments and 2 unbounded segments. The two unbounded segments of h must lie within an unbounded region for $\mathcal{L}-h$ and so cut each unbounded region into two unbounded regions. Each of the $n-1$ bounded segments either cuts a bounded region for $\mathcal{L}-h$ into two bounded regions or cuts an unbounded region for $\mathcal{L}-h$ into a bounded region and an unbounded region. Therefore the increase in the number of bounded regions from $\mathcal{L}-h$ to $\mathcal{L}$ is $n-1$. Therefore the number of bounded regions defined by $\mathcal{L}$ is

$$\frac{n^2-3n+2}{2} + (n-1) =$$

$$\frac{n^2-3n+2}{2} + \frac{2n-2}{2} =$$

$$\frac{n^2-n}{2},$$ as required.

Our Construction

Starting with the balanced sequence $+1,-1$ we can repeatedly use the following two concatenation operations to form more balanced sequences.

* If a is balanced, then $+1, a, -1$ is balanced.
* If a and b are balanced, then a, b is balanced.

Prove that every balanced sequence is built by Our Construction.

Use strong induction on the length $2n$ of a balanced sequence.

Base Case $(n=1)$

A balanced sequence of length 2 must be $+1,-1$.

This is the starting point of "Our Construction."

Inductive Step

Assume that for any $k \in \{1, \dots, n\}$ that every balanced sequence of length $2k$ is built by "Our construction."

Show that every balanced sequence of length $2n+2$ is built by our construction.

Let a be a balanced sequence of length $2n+2$.

The total of the terms of a is zero. Let i be the smallest possible integer such that the i^{th} partial sum is zero.

If $i=2n$, then $a=+1,b,-1$ where the partial sums of $+1,b$ must be strictly positive. Therefore the partial sums of b must be non-negative which means that b is a balanced sequence of length $2n$ which, by the inductive hypothesis, is built by "Our Construction." Therefore $a=+1,b,-1$ is built from "Our construction."

If $i<2n$, then $a=b,c$ where the sum of b is zero and c is nonempty. Since the partial sums of $a=b,c$ are non-negative we must then also have that the partial sums of c are non-negative as well. Therefore b and c are balanced sequences of length $2k$ and $2l$ for some $k,l \in \{1,\dots,n\}$. So by our inductive hypothesis, b and c are built by "Our construction" and therefore $a=b,c$ is built by "Our construction."