Assignment #2

① $\binom{n+1}{k+1}$ = The number of $(k+1)$-element subsets of The set $\{1,2,3,\ldots\ldots,n+1\}$.

If S is a $(k+1)$-element subset of $\{1,2,3,\ldots,n+1\}$ Then The largest element in S is between $k+1$ and $n+1$.

The number of $(k+1)$-element subsets of $\{1,2,3,\ldots\ldots,i,i+1,\ldots\ldots,n+1\}$ whose largest element is $i+1$ is $\binom{i}{k}$.

Therefore

$$\binom{n+1}{k+1} = \sum_{i=k}^{n} \binom{i}{k} \qquad \blacksquare$$

(2) $(4x-2)^{10} = (4x)^{10} + \binom{10}{1}(4x)^9(-2)^1 + \binom{10}{2}(4x)^8(-2)^2 + \binom{10}{3}(4x)^7(-2)^3 + \binom{10}{4}(4x)^6(-2)^4 + \binom{10}{5}(4x)^5(-2)^5$

$+ \binom{10}{6}(4x)^4(-2)^6 + \binom{10}{7}(4x)^3(-2)^7 + \binom{10}{8}(4x)^2(-2)^8 + \binom{10}{9}(4x)^1(-2)^9 + (-2)^{10}$

$= 1{,}048{,}576x^{10} - 5{,}242{,}880x^9 + 11{,}796{,}480x^8 - 15{,}728{,}640x^7 + 13{,}762{,}560x^6$

$- 8{,}257{,}536x^5 + 3{,}440{,}640x^4 - 983{,}040x^3 + 184{,}320x^2 - 20{,}480x + 1024$

(3) $\binom{15}{8,3,4} x^8(-3y)^3(-2z)^4 = 225{,}225\, x^8(-27)y^3(16)z^4$

$= -97{,}297{,}200\, x^8y^3z^4$

(4) Assume n is odd. Then for every $k \in \{0, \ldots, n\}$, the parities of k and $n-k$ are different.

So now because $\binom{n}{k} = \binom{n}{n-k}$ we have that

$$(-1)^k \binom{n}{k} + (-1)^{n-k} \binom{n}{n-k} = 0.$$

Therefore

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 = 0$$

Assume $n = 2m$ is even. The Binomial Theorem now gives us that

$$(1-x^2)^n = \binom{n}{0} - \binom{n}{1}x^2 + \binom{n}{2}x^4 - \binom{n}{3}x^6 \cdots\cdots + \binom{n}{n}x^{2n}.$$

The coefficient of x^n in this polynomial is $(-1)^m \binom{2m}{m}$.

The Binomial Theorem also gives us that

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots + \binom{n}{n}x^n \quad \text{and}$$

$$(1-x)^n = \binom{n}{0} - \binom{n}{1}x + \binom{n}{2}x^2 - \binom{n}{3}x^3 + \cdots + \binom{n}{n}x^n.$$

Now the coefficient of x^n in $(1+x)^n(1-x)^n$ is

$$\binom{n}{0}\binom{n}{n} - \binom{n}{1}\binom{n}{n-1} + \binom{n}{2}\binom{n}{n-2} - \binom{n}{3}\binom{n}{n-3} + \cdots + \binom{n}{n}\binom{n}{0} =$$

$$\binom{n}{0}^2 - \binom{n}{1}^2 + \binom{n}{2}^2 - \binom{n}{3}^2 + \cdots + \binom{n}{n}^2$$

Because $(1-x^2)^n = ((1-x)(1+x))^n = (1-x)^n(1+x)^n$ we get that

$$(-1)^m\binom{2m}{m} = \binom{n}{0}^2 - \binom{n}{1}^2 + \binom{n}{2}^2 - \binom{n}{3}^2 + \cdots + \binom{n}{n}^2,$$

as required.