Combinatorics & Graph Theory – Exam 1 **Name**________________________________

(1) Consider a standard deck of 52 cards. We will be considering 6-card hands.

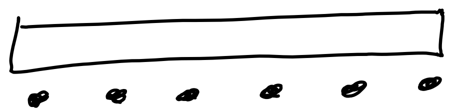
(a) How many 6-card hands are possible?

$$\binom{52}{6} = 20{,}358{,}520$$

(b) How many 6-card hands contain 3-of-a-kind but no better?

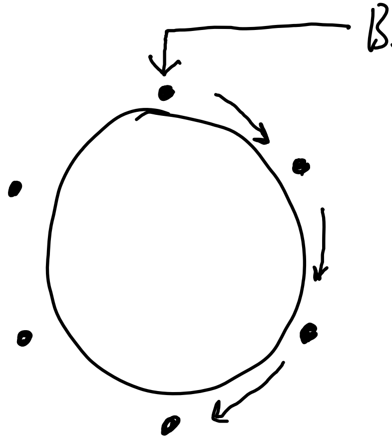
$$\binom{13}{4}\binom{4}{1}\binom{4}{3}\binom{4}{1}^3 = 732{,}160$$

(2) (a) Six people are seated, evenly spaced, from left to right on one side of a rectangular table. In how many ways can they be seated?



$6! = 720$

(b) A circular table is placed in a featureless white room. Six people are to be seated, evenly spaced, at the table. In how many ways can they be seated?



No seat is distinguishable from another so there is only one way to do this. The remaining 5 people sit in clockwise order from Bob in $5! = 120$ ways.

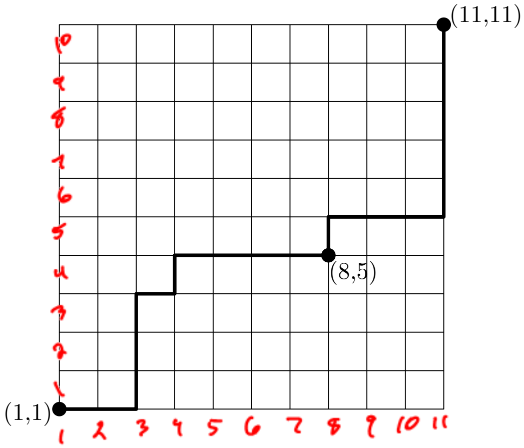
(c) A bracelet is to be made by stringing together six distinctly colored beads that will be evenly spaced. How many different bracelets can be made?

This is the same as the table problem, but we can no longer distinguish clockwise from counterclockwise. So half of the number of arrangements from before.

$\frac{1}{2} 5! = 60$

(3) Consider the labeled 11×11 grid shown below. A *lattice path* is a path from the (1,1) to (11,11) vertex in this grid in which each move is one segment to the right or one segment upward. An example of a lattice path is shown on the left-hand grid.

(a) How many lattice paths are there?



$$\binom{10+11-1}{11-1} = \binom{20}{10} = \boxed{184,756}$$

(b) Now many lattice paths pass though the (8,5) vertex?

$$\binom{4+8-1}{8-1}\binom{6+4-1}{4-1} = \binom{11}{7}\binom{9}{3} = 330 \cdot 84 = \boxed{27,720}$$

lattice paths from (1,1) to (8,5)

lattice paths from (8,5) to (11,11)

(4) Give a combinatorial proof of the fact that

$$k\binom{n}{k} = n\binom{n-1}{k-1}$$

No credit will be given for an algebraic proof, combinatorial proofs only.

$\binom{n}{k}$ = The number of ways to choose a k-element subset $S \subseteq \{1, \dots, n\}$.

$k = \binom{k}{1}$ = The number of ways to choose $x \in S$.

Therefore $k\binom{n}{k}$ = The number of ways to choose (x, S) where $x \in S$.

Now $n = \binom{n}{1}$ = The number of ways to chose $x \in \{1, \dots, n\}$

$\binom{n-1}{k-1}$ = The number of ways to choose the remaining $k-1$ elements of S from $\{1, \dots, n\} - \{x\}$.

Therefore $n\binom{n-1}{k-1}$ = The number of ways to choose (x, S) where $(x \in S)$.

Therefore $k\binom{n}{k} = n\binom{n-1}{k-1}$

(5) A party has n people at it. Each pair of persons at the party are either friends or not friends. (Do not consider a person to be a friend or not a friend with himself.) Use the Pigeonhole Principle to show that two people at the party have the same number of friends. Hint: You must split the proof into two cases: everyone at the party has at least one friend and there is someone at the party who is friends with nobody.

Case 1 If everyone has at least one friend, then the number of friends a person has is in $\{1, \ldots, n-1\}$. This is $n-1$ distinct possibilities for the number of friends that a person can have. Thus there are 2 of n people with the same number of friends.

Case 2 If someone has 0 friends, then the number of friends a person has is in $\{0, \ldots, n-2\}$. Again, this is $n-1$ distinct possibilities for the number of friends that a person can have. Thus there are 2 of n people with the same number of friends.

(6) Three sets A, B, and C satisfy

$$|A| = 11 \quad |B| = 11 \quad |C| = 7 \quad |A \cap B| = 6 \quad |A \cap C| = 3 \quad |B \cap C| = 1 \quad |A \cup B \cup C| = 20$$

Fill in the number of elements in each of the seven cells of a Venn Diagram for A, B, and C.

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$20 = 11 + 11 + 7 - 6 - 3 - 1 + |A \cap B \cap C|$$

$$1 = |A \cap B \cap C|$$