Homework Assignment #10
Due Date, Wednesday 12/6

1. Let D be a directed graph and v a vertex in D. Denote the number of edges in D whose head endpoint is v by $d_+(v)$. This is called the *in-degree* of v in D. (The *out-degree*, $d_-(v)$, is the number of edges in D with v as its tail endpoint.) Prove that

$$\sum_{v \in V(D)} d_+(v) = |E(D)|$$

In other words, the sum of the in-degrees in D is exactly the number of edges in D.

2. Let T be a tournament and $A \subseteq V(D)$. Remember that $|E(K_m)| = \binom{m}{2}$.

(a) Prove that $\sum_{v \in A} d_+(v) \geq \binom{|A|}{2}$.

(b) Suppose that there is a proper and non-empty subset A of $V(D)$ for which $\sum_{v \in A} d_+(v) = \binom{|A|}{2}$. Prove that the bipartition A, $V(D) - A$ is a cut.

(c) Prove that T is strongly connected if and only if for every non-empty and proper subset A of $V(D)$ that $\sum_{v \in A} d_+(v) > \binom{|A|}{2}$.

3. Let D be a directed graph with a nonempty edge set and such that $d_+(v) = d_-(v)$ for each vertex v in D.

(a) Using a directed path of longest length in D, prove that D contains a directed cycle.

(b) Use mathematical induction on the number of edges in D to show that $E(D)$ partitions into pairwise disjoint edge sets of directed cycles.