Assignment #5   5.1, 5.9

A. Write down a generator matrix and a parity-check matrix for the ternary Hamming code of length $\frac{3^2-1}{3-1}=4$.

B. Write down a generator matrix and a parity-check matrix for the binary Hamming code of length $2^4-1=15$

C. Calculate the spherical bounds (both of them), the Plotkin Bound, and the singleton bound for $A_2(5,3)$. Use these to construct an optimal binary code of length $=5$, $d=3$.

Assignment #6   7.6, 7.7, 7.13

Exam 2   Covers Chapters 4, 5, 7-part 1.

Scheduled for Monday 3/27/23

$\boxed{B}$ Write down a generator matrix and a parity-check matrix for the binary Hamming code of length $2^4-1=15$

Remember $\mathcal{H}_4$ is a $[2^4-1, 2^4-1-4, 3]_2$-linear code

$[15, 11, 3]_2$-linear code

Generated by $G = [\, I_{11} \,\vdots\, A \,]$ where A is 11×4 matrix whose rows are all length-4 binary words of weight at least 2.

$$\left[\begin{array}{ccccccccccc:cccc}
1&0&0&0&0&0&0&0&0&0&0&1&1&0&0\\
0&1&0&0&0&0&0&0&0&0&0&1&0&1&0\\
0&0&1&0&0&0&0&0&0&0&0&1&0&0&1\\
0&0&0&1&0&0&0&0&0&0&0&0&1&1&0\\
0&0&0&0&1&0&0&0&0&0&0&0&1&0&1\\
0&0&0&0&0&1&0&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&1&0&0&0&0&1&1&1&0\\
0&0&0&0&0&0&0&1&0&0&0&1&1&0&1\\
0&0&0&0&0&0&0&0&1&0&0&1&0&1&1\\
0&0&0&0&0&0&0&0&0&1&0&0&1&1&1\\
0&0&0&0&0&0&0&0&0&0&1&1&1&1&1
\end{array}\right] = G$$

generator matrix

$$H = \left[\begin{array}{ccccccccccc:cccc}
1&1&1&0&0&0&1&1&1&0&1&1&0&0&0\\
1&0&0&1&1&0&1&1&0&1&1&0&1&0&0\\
0&1&0&1&0&1&1&0&1&1&1&0&0&1&0\\
0&0&1&0&1&1&0&1&1&1&1&0&0&0&1
\end{array}\right]$$

parity-check matrix.

$\boxed{A}$ Write down a generator matrix and a parity-check matrix for the ternary Hamming code of length $\frac{3^2-1}{3-1}=4$.

The q-ary Hamming code of length $\dfrac{q^n-1}{q-1}$ is a

$$\left[\frac{q^n-1}{q-1},\ \frac{q^n-1}{q-1}-n,\ 3\right]_q\text{-linear code.}$$

In our example we are constructing a $[4,2,3]_3$-linear code.

$$G=\left[\, I_k \;\vdots\; A \,\right]$$ where A has all possible q-ary words of length n and weight ≥ 2 up to scalar multiplication.

$$G=\left[\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{array}\right]$$ generates $[4,2,3]_3$-linear code

$$H=\left[-A^T \,\middle|\, I_2\right]=\begin{bmatrix} 2 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}$$

Here's the generator matrix for the ternary Hamming code of length $\frac{3^3-1}{3-1} = 13$ dimension $\frac{3^3-1}{3-1} - 3 = 10$

$$\left[\begin{array}{c|ccc}
 & 1 & 1 & 0 \\
 & 1 & 2 & 0 \\
 & 1 & 0 & 1 \\
 & 1 & 0 & 2 \\
I_{10} & 0 & 1 & 1 \\
 & 0 & 1 & 2 \\
 & 1 & 1 & 1 \\
 & 1 & 2 & 1 \\
 & 1 & 1 & 2 \\
 & 1 & 2 & 2
\end{array}\right]$$

$\boxed{c}$ Calculate the spherical bounds (both of them), the Plotkin Bound, and the singleton bound for $A_2(5,3)$. Use these to construct an optimal binary code of length $=5$, $d=3$.

<u>Sphere Covering</u> $A_2(5,3) \geq \frac{2^5}{V_2^5(3-1)} = \frac{32}{\binom{5}{0}+\binom{5}{1}+\binom{5}{2}} = \frac{32}{1+5+10} = 2$

<u>Sphere packing</u> $A_2(5,3) \leq \frac{2^5}{V_2^5\left(\lfloor\frac{3-1}{2}\rfloor\right)} = \frac{32}{\binom{5}{0}+\binom{5}{1}} = \frac{32}{1+5} = 5.333\ldots$

<u>Singleton</u> $A_2(5,3) \leq 2^{5-3+1} = 2^3 = 8$

Plotkin $A_2(5,3) \leq 2\left\lfloor \frac{d+1}{2d+1-n} \right\rfloor = 2\left\lfloor \frac{4}{7-5} \right\rfloor = \lfloor 4 \rfloor = 4$

Now $G = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$ generates a $[5,2,d]_2$-linear code

with words $0\ 0\ 0\ 0\ 0,$
$1\ 1\ 1\ 0\ 0,$
$0\ 0\ 1\ 1\ 1,$
$1\ 1\ 0\ 1\ 1,$

Which has minimum weight = minimum distance = 3

So This is an optimal code.