$\boxed{A}$ Let $W = \text{Row}(A)$ be a subspace of $\mathbb{F}_5^6$ where

$$A = \begin{bmatrix} 1 & 4 & 3 & 0 & 0 & 1 \\ 2 & 1 & 1 & 0 & 3 & 4 \\ 3 & 0 & 4 & 0 & 3 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 1 & 1 & 0 \end{bmatrix}$$

* Find A basis for W.
* What is $\dim(W)$?
* What is $\dim(W^\perp)$?
* Find a basis for $W^\perp$.

$$\begin{matrix} 1 & 4 & 3 & 0 & 0 & 1 \\ 2 & 1 & 1 & 0 & 3 & 4 \\ 3 & 0 & 4 & 0 & 3 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 1 & 1 & 0 \end{matrix} \xrightarrow[\substack{R_2-2R_1 \\ R_3-3R_1 \\ R_4-R_1}]{} \begin{matrix} 1 & 4 & 3 & 0 & 0 & 1 \\ 0 & 3 & 0 & 0 & 3 & 2 \\ 0 & 3 & 0 & 0 & 3 & 2 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 1 & 0 \end{matrix} \xrightarrow[R_3-R_2]{} \begin{matrix} 1 & 4 & 3 & 0 & 0 & 1 \\ 0 & 3 & 0 & 0 & 3 & 2 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 1 & 0 \end{matrix} \longrightarrow$$

$$\begin{matrix} 1 & 4 & 3 & 0 & 0 & 1 \\ 0 & 3 & 0 & 0 & 3 & 2 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 1 & 0 \end{matrix} \xrightarrow[R_2 \leftrightarrow R_3]{} \begin{matrix} 1 & 4 & 3 & 0 & 0 & 1 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 3 & 0 & 0 & 3 & 2 \\ 0 & 2 & 0 & 1 & 1 & 0 \end{matrix} \xrightarrow[\substack{R_3-3R_2 \\ R_4-2R_2}]{} \begin{matrix} 1 & 4 & 3 & 0 & 0 & 1 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 4 & 1 & 4 & 0 \end{matrix} \xrightarrow[R_4-4R_3]{}$$

$$\begin{matrix} 1 & 4 & 3 & 0 & 0 & 1 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 4 & 2 \end{matrix}$$

Row echelon form

So rows are linearly independent

A basis for W is The rows of This matrix

$\dim(W) = 4$

$-9 = -4$

$$\begin{matrix} 1 & 4 & 3 & 0 & 0 & 1 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 4 & 2 \end{matrix} \xrightarrow[\substack{R_2-3R_3 \\ R_1-3R_3}]{} \begin{matrix} 1 & 4 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 4 & 2 \end{matrix} \xrightarrow[R_1-4R_2]{} \begin{matrix} 1 & 0 & 0 & 0 & 1 & 4 \\ 0 & 1 & 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 4 & 2 \end{matrix}$$

Reduced row-echelon form.

$x_5, x_6 =$ free variables

$$x_4 + 4x_5 + 2x_6 = 0$$
$$x_3 \qquad + 2x_6 = 0$$
$$x_2 + x_5 + 4x_6 = 0$$
$$x_1 + x_5 + 4x_6 = 0$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 4x_5 + x_6 \\ 4x_5 + x_6 \\ 3x_6 \\ x_5 + 3x_6 \\ x_5 \\ x_6 \end{bmatrix} = x_5 \begin{bmatrix} 4 \\ 4 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} 1 \\ 1 \\ 3 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

Basis for $W^{\perp}$ 440110, 113301

$$\dim(W^{\perp}) = 2$$

$\boxed{\text{notice } \dim(W^{\perp}) = 6 - \dim(W)}$

number of columns of matrix A.

3.20 Is too much calculation for an exam. Here is a similar but easier calculation.

(i) Calculate the cyclotomic cosets of 2 mod 16.

(i) x^4+x+1 is a primitive polynomial in $\mathbb{F}_2[x]$.

Thus $\alpha = x$ is a primitive element of $\mathbb{F}_{16} = \frac{\mathbb{F}_2[x]}{(x^4+x+1)}$.

Find the minimal polynomials for $\alpha, \alpha^2, \alpha^3$, and α^5.

To aid with this calculation, here are the powers of α.

$\alpha^4 = \alpha+1$ $\qquad \alpha^9 = \alpha^3+\alpha$

$\alpha^5 = \alpha^2+\alpha$ $\qquad \alpha^{10} = \alpha^2+\alpha+1$

$\alpha^6 = \alpha^3+\alpha^2$ $\qquad \alpha^{11} = \alpha^3+\alpha^2+\alpha$

$\alpha^7 = \alpha^3+\alpha+1$ $\qquad \alpha^{12} = \alpha^3+\alpha^2+\alpha+1$

$\alpha^8 = \alpha^2+1$ $\qquad \alpha^{13} = \alpha^3+\alpha^2+1$ $\qquad \alpha^{14} = \alpha^3+1$

(i) Cyclotomic cosets of 2 mod 15.

$C_0 = \{0\}$ $\quad C_1 = \{1,2,4,8\}$ $\quad C_3 = \{3,6,12,9\}$

$C_5 = \{5,10\}$ $\quad C_7 = \{7,14,13,11\}$

(ii) Because $C_1=\{1,2,4,8\}$, $M^{(1)}(x)=M^{(2)}(x)=x^4+x+1$

Because $C_5=\{5,10\}$ $M^{(5)}(x)=(x-\alpha^5)(x-\alpha^{10})$

$$= x^2+(\alpha^5+\alpha^{10})+\alpha^{15}$$

$$\boxed{= x^2+x+1}$$

$$\begin{array}{rl} \alpha^5 = & \alpha^2+\alpha \\ +\ \alpha^{10} = & \alpha^2+\alpha+1 \\ \hline & 1 \end{array}$$

Because $C_3=\{3,6,12,9\}$,

$$M^{(3)}(x)=(x-\alpha^3)(x-\alpha^6)(x-\alpha^9)(x-\alpha^{12})$$

$$= x^4+(\alpha^3+\alpha^6+\alpha^9+\alpha^{12})x^3$$

$$+(\alpha^9+\alpha^{12}+\cancel{\alpha^{15}}+\cancel{\alpha^{15}}+\alpha^{18}+\alpha^{21})x^2$$

$$+(\alpha^{27}+\alpha^{24}+\alpha^{21}+\alpha^{18})x$$

$$+\alpha^{30}$$

$$\boxed{= x^4+x^3+x^2+x+1}$$

$$\begin{array}{rl} \alpha^3 = & \alpha^3 \\ \alpha^6 = & \alpha^3+\alpha^2 \\ \alpha^9 = & \alpha^3 \quad +\alpha \\ +\ \alpha^{12} = & \alpha^3+\alpha^2+\alpha+1 \\ \hline & 1 \end{array}$$

$$\begin{array}{rl} \alpha^9 = & \alpha^3 \quad +\alpha \\ \alpha^{12} = & \alpha^3+\alpha^2+\alpha+1 \\ \alpha^{18} = & \alpha^3 \\ +\ \alpha^{21} = \alpha^6 = & \alpha^3+\alpha^2 \\ \hline & 1 \end{array}$$

$$\begin{array}{rl} \alpha^{27} = \alpha^{12} = & \alpha^3+\alpha^2+\alpha+1 \\ \alpha^{24} = \alpha^9 = & \alpha^3 \quad +\alpha \\ \alpha^{21} = \alpha^6 = & \alpha^3+\alpha^2 \\ \alpha^{18} = & \alpha^3 \\ \hline & 1 \end{array}$$