Math 2800 – Fall 2022 Exam 1 Name

(1) Let $A$, $B$, and $C$ be propositions. Prove the following tautology using a string of logical equivalences with the appropriate named tautology underneath.

$$(A \wedge B) \to C \quad \longleftrightarrow \quad A \to (B \to C)$$

$$(A \wedge B) \to C \xleftrightarrow[\text{implication}]{} (A \wedge B)' \vee C \xleftrightarrow[\text{DeMorgan}]{} (A' \vee B') \vee C \xleftrightarrow[\text{association}]{}$$

$$A' \vee (B' \vee C) \xleftrightarrow[\text{implication} \times 2]{} A \to (B \to C)$$

(2) For each of the following predicate-logic statements, determine the truth value (TRUE or FALSE) over the three different domains specified: the set of all integers (i.e., whole numbers), the set of natural numbers (i.e., $\{0, 1, 2, 3, \ldots\}$), and $\{0\}$.

| Statement | Integers $\{0, \pm 1, \pm 2, \ldots\}$ | Natural Numbers $\{0, 1, 2, 3, \ldots\}$ | $\{0\}$ |
|---|---|---|---|
| $(\forall x)(\exists y)(x + y = x)$ | T | T | T |
| $(\exists x)(\forall y)(x + y = 0)$ | F | F | T |
| $(\forall x)(\exists y)(x + y = 0)$ | T | F | T |
| $(\forall x)(\exists y)(y < x)$ | T | F | F |
| $(\forall x)[(x > 0) \vee (\exists y)(y < x)]$ | T | F | F |
| $(\forall x)[(x < 0) \to (\exists y)(x + y = 0)]$ | T | T | T |
| $(\forall x)(\forall y)[(x = y) \vee (x < y) \vee (y < x)]$ | T | T | T |

(3) Translate the following statements from English to propositional logic.

$F$ =flowers will bloom $\quad$ $R$ =there is plentiful rain $\quad$ $S$ =the soil is good

(a) If the soil is good and there is plentiful rain, then the flowers will bloom.

$S \wedge R \rightarrow F$

(b) The soil is good only if there is plentiful rain.

$S \rightarrow R$

(c) The soil is good or there is plentiful rain, if the flowers bloom.

$F \rightarrow S \vee R$

(d) Good soil is a sufficient condition for flowers to bloom.

$S \rightarrow F$

(e) Flowers will bloom if and only if there is plentiful rain and good soil.

$F \leftrightarrow R \wedge S$

(4) Let $A$, $B$, and $C$ be sets. Prove that

$$A \times (B - C) = (A \times B) - (A \times C)$$

by proving that $A \times (B - C) \subseteq (A \times B) - (A \times C)$ and $(A \times B) - (A \times C) \subseteq A \times (B - C)$.

$\underline{A \times (B - C) \subseteq (A \times B) - (A \times C)}$

Let $(p,q) \in A \times (B-C)$. By the definition of cartesian products, $p \in A$ and $q \in B - C$. By the definition of set subtraction, $q \in B$ and $q \notin C$. So by the definition of Cartesian products $(p,q) \in A \times B$ but $(p,q) \notin A \times C$. Therefore $(p,q) \in (A \times B) - (A \times C)$.

$\underline{(A \times B) - (A \times C) \subseteq A \times (B - C)}$

Let $(p,q) \in (A \times B) - (A \times C)$. By the definition of set subtraction, $(p,q) \in (A \times B)$ but $(p,q) \notin A \times C$. By definition of Cartesian products $p \in A$ and $q \in B$. This also yields $q \notin C$ because $(p,q) \notin A \times C$. Thus $q \in B - C$ by the definition of set subtraction. Thus $(p,q) \in A \times (B - C)$.

(5) Let $A$, $B$, and $C$ be sets. Give a direct proof of the following statement.

$$\text{If } A \subseteq B, \text{ then } A - C \subseteq B - C.$$

Assume that $A \subseteq B$. 4 Now consider some $x \in A - C$. 4 By definition of set subtraction $x \in A$ and $x \notin C$. 4 Because $A \subseteq B$, $x \in B$ 4 as well. Thus by the definition of set subtraction $x \in B - C$, 4 as required.