Assignment #6 due Friday 10/28 by 5pm

Exam 2 will cover chapters 4 and 5.
Scheduled for Monday 10/31.

For assignment #6 problem 2

You may use the fact that odd + odd = even
odd + even = odd
even + even = even.

Proposition If $a, b \in \mathbb{Z}$, then

$a+b$ is even if and only if a and b have the same parity.

proof
Assume $a, b \in \mathbb{Z}$.
($\Leftarrow$) Assume a and b have the same parity.

That is, either $a = 2k$ and $b = 2l$ for some $k, l \in \mathbb{Z}$

OR

$a = 2k+1$ and $b = 2l+1$ for some $k, l \in \mathbb{Z}$.

In the former case $a+b = 2k+2l = 2(k+l)$ which is even.

In the latter case $a+b=2k+1+2\ell+1=2(k+\ell+1)$ which is even.

$(\Leftarrow)$ Assume a and b do not have the same parity.

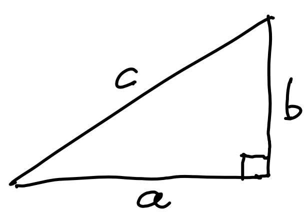
Therefore, without a loss of generality,

$$a=2k \text{ and } b=2\ell+1 \text{ for some } k,\ell\in\mathbb{Z}.$$

So now $a+b=2k+2\ell+1=2(k+\ell)+1$ which is not even. $\blacksquare$

<u>Extra topic</u> <u>Pythagorean Theorem</u>

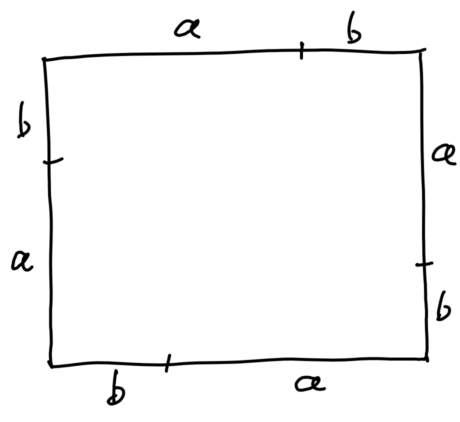
<u>Theorem</u> For the right triangle shown with side lengths as indicated,



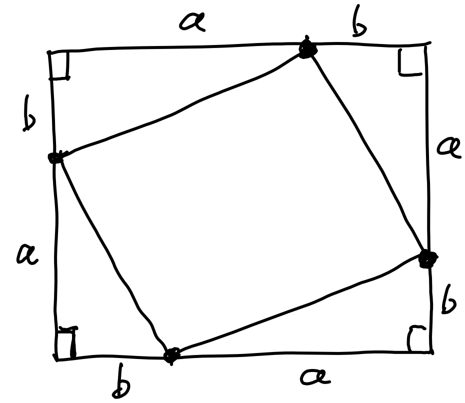
$$a^2+b^2=c^2$$

<u>proof</u>

Consider the square shown whose sides have length $a+b$ each.

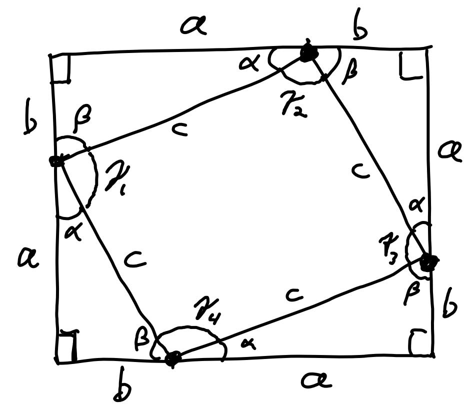


Connect the dividing marks with lines as shown.



Since the four triangles created in this picture are right triangles whose legs have lengths a and b, the third side must therefore have length c.

Labeling the angles of the four triangles and the inner rhombus. We now use the fact that a straight-line angle is 180° to obtain the fact that $\alpha+\beta+\gamma_i=180^\circ$, that is $\gamma_i=180^\circ-\alpha-\beta$ for each $i\in\{1,2,3,4\}$. Now assuming that the sum of the angles of a triangle is 180° we obtain



$$\alpha+\beta+90^\circ=180^\circ \quad \text{so}$$

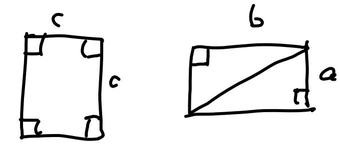
$$\alpha+\beta=90^\circ \quad \text{so}$$

$$\gamma_i=180^\circ-90^\circ=90.$$

Thus the inner rhombus is a square.

Now the area of the large square is $(a+b)^2$.

Also, the area of the large square is $c^2 + 2ab$



Therefore $(a+b)^2 = c^2 + 2ab$ so

$$a^2 + b^2 + \cancel{2ab} = c^2 + \cancel{2ab} \quad \text{so}$$

$$a^2 + b^2 = c^2.$$

$\blacksquare$

Pythagorean Triples

A Pythagorean Triple is 3 positive integers $a < b < c$ such that $a^2 + b^2 = c^2$.

example $3^2 + 4^2 = 5^2$
$5^2 + 12^2 = 13^2$

example $a < b < c$ is a pythagorean triple if and only if $ka < kb < kc$ is a pythagorean triple for any positive integer k.

$$a^2 + b^2 = c^2 \quad \text{iff}$$

$$k^2(a^2 + b^2) = k^2 c^2 \quad \text{iff}$$

$$k^2a^2+k^2b^2=k^2c^2 \text{ iff}$$
$$(ka)^2+(kb)^2=(kc)^2.$$

A pythagorean triple a.b.c is called <u>primitive</u> when a, b, c have no common factor.

Proposition Ⓐ If $m<n$ are positive integers. Then

$$n^2-m^2,\ 2mn,\ n^2+m^2$$

is a pythagorean triple.

proof

$$(n^2-m^2)^2+(2mn)^2=n^4-2n^2m^2+m^4+4n^2m^2=n^4+2n^2m^2+m^4=(m^2+n^2)^2. \blacksquare$$

Proposition Ⓑ If $m<n$ are positive integers without a common fact and with different parity. Then

$$m^2-n^2,\ 2mn,\ m^2+n^2$$

is a primitive pythagorean triple.

proof

Proposition A already tells us that m^2-n^2, $2mn$, m^2+n^2 is a pythagorean triple. We need to assume the following facts. ① a prime number p is a factor of an integer n if and only if p is a factor of n^2. Also integer k is a factor of n if and only if there is a prime number p which is a common factor of both k and n.

By way of contradiction, assume that m and n have no common factor and different parities but

$$n^2-m^2,\ 2mn,\ n^2+m^2$$

is not primitive, that is, there is a positive integer $k \geq 2$ which is a common factor of $n^2-m^2, 2mn, n^2+m^2$.

Therefore there is a prime number p which is a common factor of n^2-m^2, $2mn$, n^2+m^2.

Suppose $p=2$. That is n^2-m^2, $2mn$, n^2+m^2 are all even. Well n^2+m^2 is even if and only if n^2 and m^2 have the same parity if and only if n and m both have the same parity, a contradiction of our assumption.

Suppose $p \geq 3$. Assuming that $p \mid mn$ if and only if $p \mid m$ or $p \mid n$ we now get without loss of generality that $p \mid m$. So p is a factor of m^2. So since p is a factor of m^2+n^2 we must also have that p is a factor of n^2, as well. Thus p is a factor of n as well, a contradiction. $\blacksquare$

<u>Theorem:</u> $a<b<c$ is a primitive Pythagorean triple

if and only if

a, b, c is of the form $n^2-m^2, 2nm, n^2+m^2$ for some positive integers $m<n$ which have no common factor and have different parities.