Math 2800 – Fall 2022 – Exam 2 Name ____________________________

(1) Consider a function $f: D \to R$ and subsets $A, B \subseteq R$.

(a) Prove that $f^{-1}(A) \cap f^{-1}(B) \neq \emptyset$ implies that $A \cap B \neq \emptyset$.

(b) Using $D = \{1, 2, 3\}$ and $R = \{a, b, c\}$, give an example in which $A \cap B \neq \emptyset$ but $f^{-1}(A) \cap f^{-1}(B) = \emptyset$.

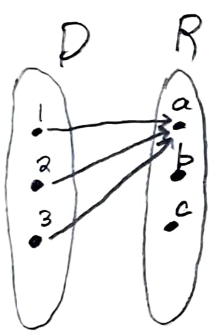
(a)

proof

Assume $f^{-1}(A) \cap f^{-1}(B) \neq \emptyset$. That is, there is $x \in f^{-1}(A) \cap f^{-1}(B)$. By the definition of intersection, $x \in f^{-1}(A)$ and $x \in f^{-1}(B)$. By the definition of preimage $f(x) \in A$ and $f(x) \in B$. Thus $f(x) \in A \cap B$.

Thus $A \cap B \neq \emptyset$. $\blacksquare$

(b)



$A = \{b, c\}$, $B = \{b, c\}$, $A \cap B \neq \emptyset$

But $f^{-1}(A) = \emptyset$, $f^{-1}(B) = \emptyset$, $f^{-1}(A) \cap f^{-1}(B) = \emptyset$.

(2) Consider a function $f : D \to R$ and subsets $A, B \subseteq D$.

(a) Prove that $A \subseteq B$ implies that $f(A) \subseteq f(B)$.

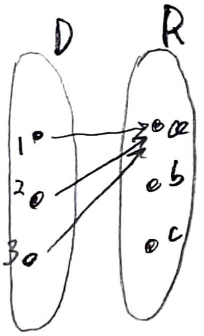
(b) Using $D = \{1, 2, 3\}$ and $R = \{a, b, c\}$, give an example in which $f(A) \subseteq f(B)$ but $A \not\subseteq B$.

(a)

proof

Assume $A \subseteq B$. In order to show That $f(A) \subseteq f(B)$ consider $y \in f(A)$. By The definition of image, There is $x \in A$ such That $y = f(x)$. By The definition of subset $x \in B$. By The definition of image $f(x) \in f(B)$ so $y \in f(B)$, as required. ■

(b)



$A = \{1, 2\}$, $B = \{1, 3\}$, $A \nsubseteq B$.

But $f(A) = \{a\} = f(B)$ so $f(A) \subseteq f(B)$.

(3) Define a relation $\sim$ on $\mathbb{N} \times \mathbb{N}$ by $(x, y) \sim (a, b)$ when $x + y = a + b$.

(a) Prove that $\sim$ is an equivalence relation.

(b) Find all of the elements of the equivalence classes $[(1, 0)]$, $[(2, 0)]$, and $[(3, 0)]$.

(c) Draw all of the elements of $[(1, 0)]$, $[(2, 0)]$, and $[(3, 0)]$ in 2-dimensional plane along with the familiar geometric objects which contain each one them.

ⓐ reflexive Let $(a,b) \in \mathbb{N} \times \mathbb{N}$. Because $a+b = a+b$, $(a,b) \sim (a,b)$.

Symmetric Let $(a,b), (x,y) \in \mathbb{N} \times \mathbb{N}$ and assume $(a,b) \sim (x,y)$.

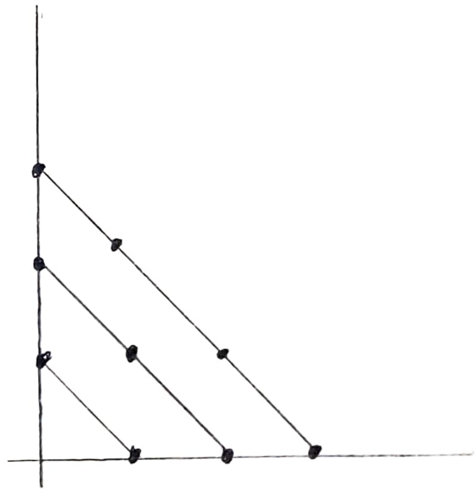
Thus $a+b = x+y$ so $x+y = a+b$ so $(x,y) \sim (a,b)$.

transitive Let $(a,b), (x,y), (p,q) \in \mathbb{N} \times \mathbb{N}$ and assume that $(a,b) \sim (x,y)$ and $(x,y) \sim (p,q)$. Thus $a+b = x+y$ and $x+y = p+q$ which imply that $a+b = p+q$. Thus $(a,b) \sim (p,q)$.

ⓑ $[(1,0)] = \{(1,0), (0,1)\}$, $[(2,0)] = \{(2,0), (1,1), (0,2)\}$

$[(3,0)] = \{(3,0), (2,1), (1,2), (0,3)\}$

ⓒ



(4) In this problem, let $\leq$ be the standard ordering on $\mathbb{Z}$. Define a relation $\leq$ on $\mathbb{Z} \times \mathbb{Z}$ by

$$(a, b) \leq (x, y) \text{ when either } \quad a < x \text{ and } b < y \quad \text{or} \quad a = x \text{ and } b = y.$$

Prove that $\leq$ is a partial ordering on $\mathbb{Z} \times \mathbb{Z}$.

Reflexive Let $(a,b) \in \mathbb{Z}\times\mathbb{Z}$. Because $a=a$ and $b=b$, $(a,b) \leq (a,b)$.

Anti-symmetric Let $(a,b),(x,y) \in \mathbb{Z}\times\mathbb{Z}$ and assume that $(a,b) \leq (x,y)$ and $(x,y) \leq (a,b)$. By the definition of $\leq$ on $\mathbb{Z}\times\mathbb{Z}$,

$a=x$ and $b=y$ OR $a<x$ and $b<y$

AND

$x=a$ and $y=b$ OR $x<a$ and $y<b$.

If $a=x$ and $b=y$ or $x=a$ and $y=b$, then $(a,b)=(x,y)$, as required.

Otherwise, $a<x$ and $x<a$; however, this is impossible.

Thus $(a,b)=(x,y)$.

Transitive Let $(a,b),(x,y),(p,q) \in \mathbb{Z}\times\mathbb{Z}$ and assume that $(a,b) \leq (x,y)$ and $(x,y) \leq (p,q)$.

By the definition of $\leq$,

$a=x$ and $b=y$ OR $a<x$ and $b<y$

AND

$x=p$ and $y=q$ OR $x<p$ and $y<q$.

This yields four possibilities: $a=x=p$ and $b=y=q$, $a=x<p$ and $b=y<q$, $a<x=p$ and $b<y=q$, or $a<x<p$ and $b<y<q$. In the first case, $a=p$ and $b=q$ and in cases 2 through 4 $a<p$ and $b<q$.

Therefore $(a,b) \leq (p,q)$.