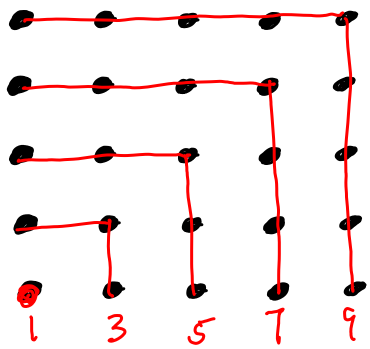
Chapter 6 Mathematical induction.

example



n^2 dots

$$1+3+5+\cdots+2n-1=n^2$$

example

Let $S=1+2+3+4+\cdots+n$

So $2S = 1+2+3+\cdots+n+$
$n+(n-1)+(n-2)\cdots+1=$

$(n+1)+(n+1)+(n+1)+\cdots+(n+1)=n(n+1)$

So $S=\dfrac{n(n+1)}{2}.$

$$1+2+3+\cdots+n=\frac{n(n+1)}{2}.$$

example

The sum of the measures of the interior angles of an n-sided polygon is $(n-2)180^\circ$ ($n\geq 3$ of course).

1^{st} Principle of Mathematical Induction (Simple Induction)

To show that $(\forall n)P(n)$ is true over domain $D = \{1, 2, 3, 4, \ldots\}$

Show that

① Show that $P(1)$ is true ← Base case

and

② Show that $(\forall n)(P(n) \to P(n+1))$ is true ← Inductive step

<u>example</u> Prove that $\underbrace{1+3+5+\cdots+2n-1 = n^2}_{P(n)}$ for all $n \geq 1$.

<u>Base Case</u> $(n=1)$

When $n=1$, $1+3+5+\cdots+2n-1 = 1$ and $n^2 = 1$

same

<u>Inductive Step</u> $(\forall n)(P(n) \to P(n+1))$

Assume that $1+3+5+\cdots+2n-1 = n^2$ for some $n \geq 1$. ← "Induction hypothesis"

We need to show that $1+3+5+\cdots+2n+1 = (n+1)^2$ is also true.

Well $1+3+5+\cdots+2n+1 = (1+3+5+\cdots+2n-1)+2n+1 \underset{\substack{\uparrow \\ \text{by the} \\ \text{induction} \\ \text{hypothesis}}}{=} n^2+2n+1 = (n+1)^2$, as required. ∎

Example Prove that $\underbrace{1^2+2^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}}_{P(n)}$ for all $n \geq 1$.

Base Case ($n=1$)

When $n=1$, $1^2+2^2+3^2+\cdots+n^2 = 1$ and $\frac{n(n+1)(2n+1)}{6} = \frac{1(2)(3)}{6} = 1$

same

Induction Step ($P(n) \to P(n+1)$)

Assume that $1^2+2^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$ for some $n \geq 1$. ← Induction hypothesis

Show that $1^2+2^2+\cdots+(n+1)^2 = \frac{(n+1)(n+2)(2n+3)}{6}$ is also true. ← Target Conclusion.

Well $1^2+2^2+\cdots+(n+1)^2 = \left[1^2+2^2+\cdots+n^2\right]+(n+1)^2 \underset{\text{by the induction hypothesis}}{=} \frac{n(n+1)(2n+1)}{6} + (n+1)^2$

$$= (n+1)\left(\frac{n(2n+1)}{6} + (n+1)\right) = (n+1)\left(\frac{2n^2+n}{6} + \frac{6n+6}{6}\right) = (n+1)\,\frac{2n^2+7n+6}{6} = (n+1)\,\frac{(2n+3)(n+2)}{6}$$

$$= \frac{(n+1)(n+2)(2n+3)}{6},$$ as required. ■

Working With Inequalities

Given $A = B \le C \le D = F < G = H$ we obtain $A < H$.

Given $A = B \le C \le D = F \le G = H$ we obtain $A \le H$.

Also, (1) $A \le B$ and $C \le D$ imply $A + C \le B + D$

(2) $A \le B$ and $C \ge 0$ imply $AC \le BC$

(3) $A \le B$ and $C < 0$ imply $AC \ge BC$

Factorials

Def $n! = n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1$

e.g. $1! = 1$

$2! = 2 \cdot 1 = 2$

$3! = 3 \cdot 2 \cdot 1 = 6$

$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$

$5! = 5 \cdot 4! = 120$

$6! = 6 \cdot 5! = 720$

$\vdots$

In general $(n+1)! = (n+1)(n!)$

example Prove that $\underbrace{n! > \left(\frac{5}{2}\right)^{n-1}}_{P(n)}$ for all $n \geq 4$.

$\boxed{\text{Side Note}}$ $P(1) =$ "$1! > \left(\frac{5}{2}\right)^0$" $=$ "$1 > 1$" which is false

$P(2) =$ "$2! > \left(\frac{5}{2}\right)^1$" $=$ "$2 > \frac{5}{2}$" which is false

$P(3)$ is also false.

Base Case ($n=4$)

When $n = 4$, $n! = 4! = 24$ and $\left(\frac{5}{2}\right)^{n-1} = \left(\frac{5}{2}\right)^3 = \frac{125}{8} = 15 + \frac{5}{8}$ and

$24 > 15 + \frac{5}{8}$.

Induction Step ($P(n) \to P(n+1)$)

Assume that $n! > \left(\frac{5}{2}\right)^{n-1}$ for some $n \geq 4$.

Show that $(n+1)! > \left(\frac{5}{2}\right)^n$ is also true.

Well $(n+1)! = (n+1)(n!) \underset{\text{by the induction hypothesis}}{>} (n+1)\left(\frac{5}{2}\right)^{n-1} \underset{\text{because } n \geq 4}{\geq} 5\left(\frac{5}{2}\right)^{n-1} > \left(\frac{5}{2}\right)\left(\frac{5}{2}\right)^{n-1} = \left(\frac{5}{2}\right)^n$.

Thus $(n+1)! > \left(\frac{5}{2}\right)^n$, as required. ∎

example Prove $\boxed{n^n > n!}$ for all $n \geq 2$.

$P(n)$

proof

Base Case ($n=2$)

When $n=2$, $n^n = 2^2 = 4$ and $n! = 2! = 2$, and $4 > 2$.

Induction Step ($P(n) \to P(n+1)$)

Assume that $n^n > n!$ for some arbitrary $n \geq 2$. Induction hypothesis

Show that, $(n+1)^{n+1} > (n+1)!$ is true as well. Target conclusion.

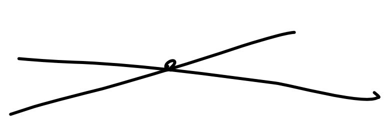
Well, $(n+1)^{n+1} = (n+1)(n+1)^n \geq (n+1)n^n > (n+1)(n!) = (n+1)!$. Therefore $(n+1)^{n+1} > (n+1)!$, as required. ■

because $n \geq 2$

by the induction hypothesis

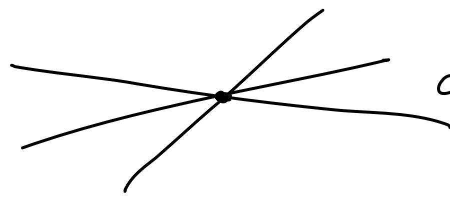
<u>example</u> A collection of lines $\mathcal{L}$ in the 2-dimensional plane is said to be in <u>general position</u> when

① Every pair of lines intersect.



and

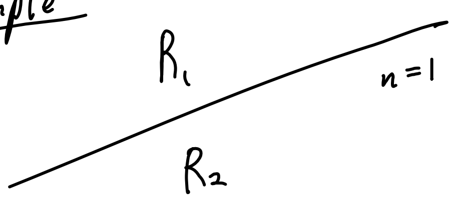
② No three lines intersect at the same point.



does <u>not</u> occur.

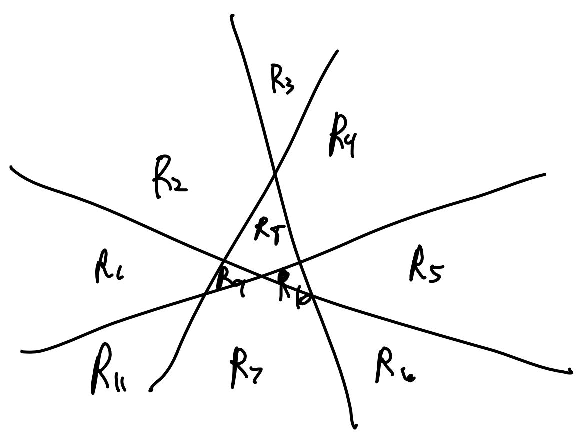
Let's prove that any collection $\mathcal{L}$ of <u>n lines in general position divides the plane into $\frac{n^2+n+2}{2}$ regions.</u> $P(n)$

<u>example</u>



$n=1$ line

$\frac{1^2+1+2}{2} = 2$ regions



$n=4$ lines

$\frac{4^2+4+2}{2} = \frac{22}{2}$
$= 11$ regions.

Proof

<u>Base Case ($n=1$)</u>

One line defines 2 regions and $\frac{1^2+1+2}{2} = 2$.

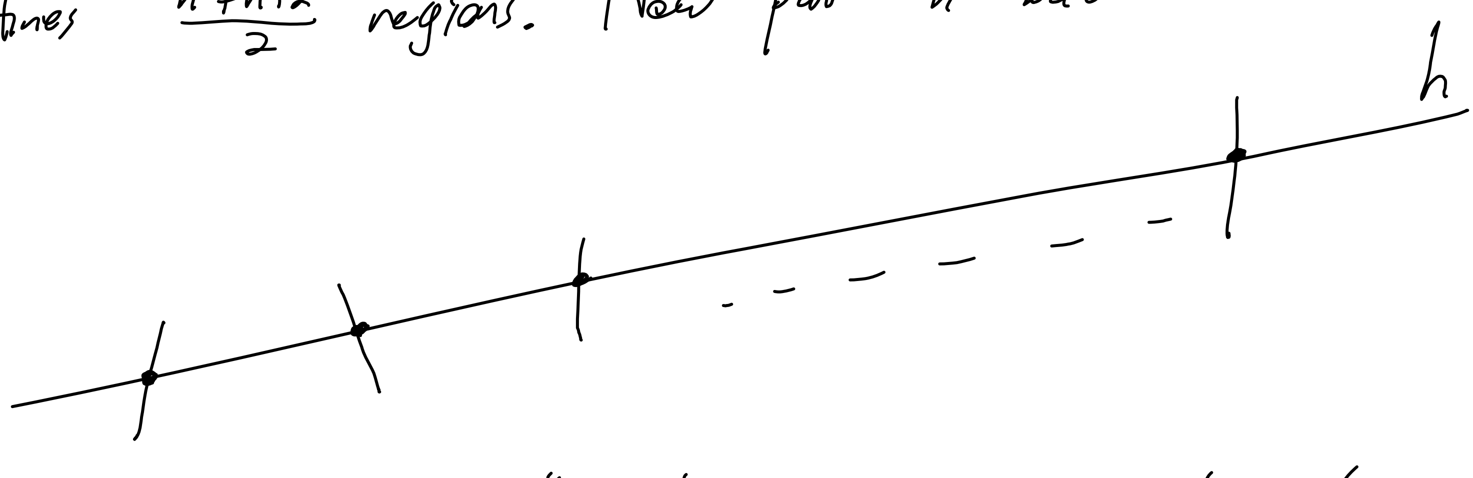
Induction Step

Assume that any n lines in general position define $\frac{n^2+n+2}{2}$ regions.

Show that $n+1$ lines in general position define $\frac{(n+1)^2+(n+1)+2}{2}$ regions.

First, note $\frac{(n+1)^2+(n+1)+2}{2} = \frac{n^2+3n+4}{2}$.

Let $\mathcal{L}$ be a collection of $n+1$ lines in general position. Let $h \in \mathcal{L}$. Now $\mathcal{L}-\{h\}$ is a collection of n lines in general position. By the induction hypothesis, $\mathcal{L}-\{h\}$ defines $\frac{n^2+n+2}{2}$ regions. Now put h back in.



h intersects the n other lines in one point each for a total of n intersection points which divides

h into $n+1$ segments. Each segment separates a region defined by $\mathcal{L}-\{h\}$ into two new regions.

This yields a total gain of $n+1$ to the number of regions.

So the total number of regions defined by $\mathcal{L}$

number of regions given by $\mathcal{L}-\{h\}$

$$\frac{n^2+n+2}{2} + (n+1) =$$

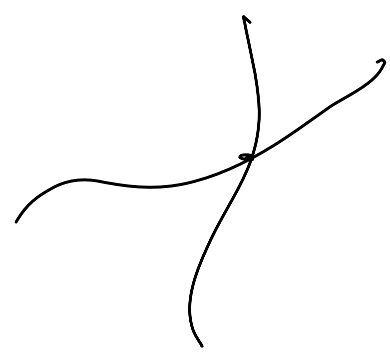
total gain by putting h back in

$$\frac{n^2+n+2}{2} + \frac{2n+2}{2} = \frac{n^2+3n+4}{2}, \quad \text{as required} \quad \blacksquare$$

<u>example</u> A pseudoline is just a line with some waviness.

A collection of pseudolines are in general position when

1. every pair of pseudolines intersect once and do so transversely



2. No three pseudolines intersect at the same point.

A collection of n pseudolines defines $\frac{n^2+n+2}{2}$ regions.

Same proof.

Strong Induction (also called the 2^{nd} principle of induction)

To show that $(\forall n)\, P(n)$ is true over domain $D=\{1,2,3,\ldots\}$

we can show that

① $P(1)$ is true

and

② $(\forall n)(P(1) \wedge P(2) \wedge \cdots \wedge P(n) \longrightarrow P(n+1))$.

example Prove that every integer $n \geq 2$ is either prime or a product of primes.

Base Case ($n=2$)

$n=2$ is prime.

Induction Step $P(2) \wedge P(3) \wedge \cdots \wedge P(n) \longrightarrow P(n+1)$

Assume that every $k \in \{2,\ldots,n\}$ is either prime or a product of primes

Show that $n+1$ is either prime or a product of primes.

If $n+1$ is prime, then we are done. If $n+1$ is not prime, then $n+1 = ab$ where $a, b \in \{2, \ldots, n\}$.

By the induction hypothesis

$$a = p_1 \cdot p_2 \cdot \cdots \cdot p_s \quad \text{where each } p_i \text{ is prime}$$

and

$$b = q_1 \cdot q_2 \cdot \cdots \cdot q_t \quad \text{where each } q_j \text{ is prime.}$$

Therefore $n+1 = ab = p_1 \cdot p_2 \cdot \cdots \cdot p_s \cdot q_1 \cdot q_2 \cdot \cdots \cdot q_t$ which is a product of primes. ∎

<u>example</u> Prove that every integer $n \geq 0$ can be written as

$$n = a_0 2^0 + a_1 2^1 + a_2 2^2 + \cdots a_\ell 2^\ell \quad \text{where each } a_i \in \{0, 1\}.$$

~~proof~~

<u>Base case</u> $n = 0 = 0 \cdot 2^0$.

<u>Induction Step</u> $P(0) \wedge P(1) \wedge \cdots \wedge P(n) \longrightarrow P(n+1)$

Assume that every $k \in \{0, \ldots, n\}$ can be written as

$k = a_0 2^0 + a_1 2^1 + a_2 2^2 + \cdots a_\ell 2^\ell$ where each $a_i \in \{0, 1\}$.

Show that $n+1$ can also be written as a sum of consecutive powers of 2 with coefficients in $\{0, 1\}$.

Well $n+1$ is either even or odd. So

$n+1 = 2k$ or $n+1 = 2k+1$ where $k \in \{0, ..., n\}$.

By the induction hypothesis $k = a_0 2^0 + a_1 2^1 + a_2 2^2 + \cdots a_\ell 2^\ell$

where each $a_i \in \{0, 1\}$.

If $n+1 = 2k$, then, $n+1 = a_0 2^1 + a_1 2^2 + a_2 2^3 + \cdots a_\ell 2^{\ell+1}$ so

$$n+1 = 0 \cdot 2^0 + a_0 2^1 + a_1 2^2 + a_2 2^3 + \cdots a_\ell 2^{\ell+1}, \text{ as required.}$$

If $n+1 = 2k+1$, then $n+1 = 1 + a_0 2^1 + a_1 2^2 + a_2 2^3 + \cdots a_\ell 2^{\ell+1}$ so

$$n+1 = 1 \cdot 2^0 + a_0 2^1 + a_1 2^2 + a_2 2^3 + \cdots a_\ell 2^{\ell+1}, \text{ as required.}$$

$\blacksquare$

example

Base-10	Base 2	typically written
0	$0 \cdot 2^0$	000
1	$1 \cdot 2^0$	001
2	$0 \cdot 2^0 + 1 \cdot 2^1$	010
3	$1 \cdot 2^0 + 1 \cdot 2^1$	011
4	$0 \cdot 2^0 + 0 \cdot 2^1 + 1 \cdot 2^2$	100
5	$1 \cdot 2^0 + 0 \cdot 2^1 + 1 \cdot 2^2$	101
6	$0 \cdot 2^0 + 1 \cdot 2^1 + 1 \cdot 2^2$	110
7	$1 \cdot 2^0 + 1 \cdot 2^1 + 1 \cdot 2^2$	111

Now

$$10 = 2 \cdot 5 = 2(1 \cdot 2^0 + 0 \cdot 2^1 + 1 \cdot 2^2) = 0 \cdot 2^0 + 1 \cdot 2^1 + 0 \cdot 2^2 + 1 \cdot 2^3$$

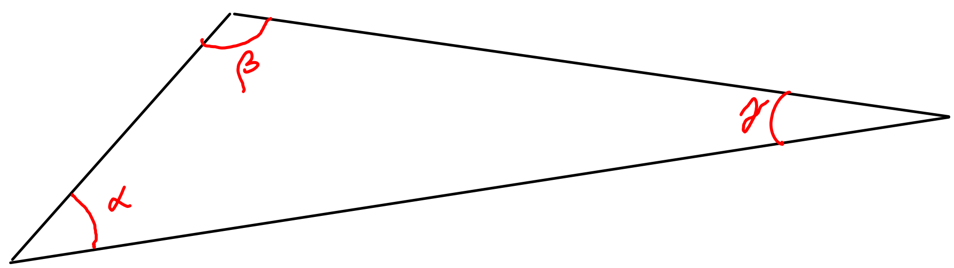
$$11 = 2 \cdot 5 + 1 = 2(1 \cdot 2^0 + 0 \cdot 2^1 + 1 \cdot 2^2) + 1 = 1 \cdot 2^0 + 1 \cdot 2^1 + 0 \cdot 2^2 + 1 \cdot 2^3$$

example Prove that the sum of the measures of the interior angles of an n-sided polygon ($n \geq 3$) is $(n-2)180°$.

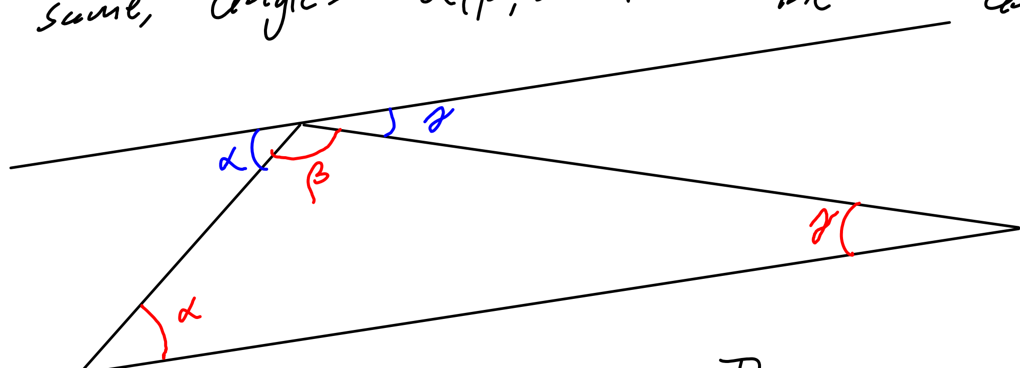
Base Case ($n=3$)

We want to show that the sum of the interior angles of a triangle is $(3-2)180° = 180°$.

Consider an arbitrary triangle with angles as indicated



Take one side of the triangle and place a parallel copy on the opposite vertex. Because opposite interior angles of transversal cutting two parallel lines are the same, angles α, β, γ form the angle of a straight line



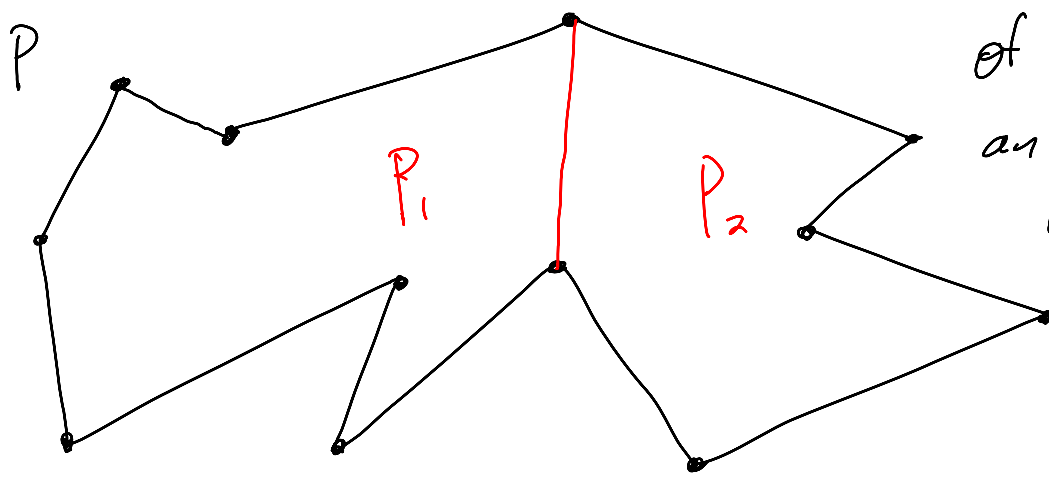
Thus $\alpha + \beta + \gamma = 180°$.

Induction Step $P(3) \wedge \cdots \wedge P(n) \longrightarrow P(n+1)$

Assume that for every $k \in \{3, \ldots, n\}$ that the sum of the interior angles of a k-sided polygon is $(k-2)180°$.

Show that the sum of the interior angles of an $(n+1)$-sided polygon is $(n-1)180°$.

Let P be an $(n+1)$-sided polygon. Use a line to connect two vertices of P by an interior line segment which cuts P into two smaller polygons, P_1 and P_2

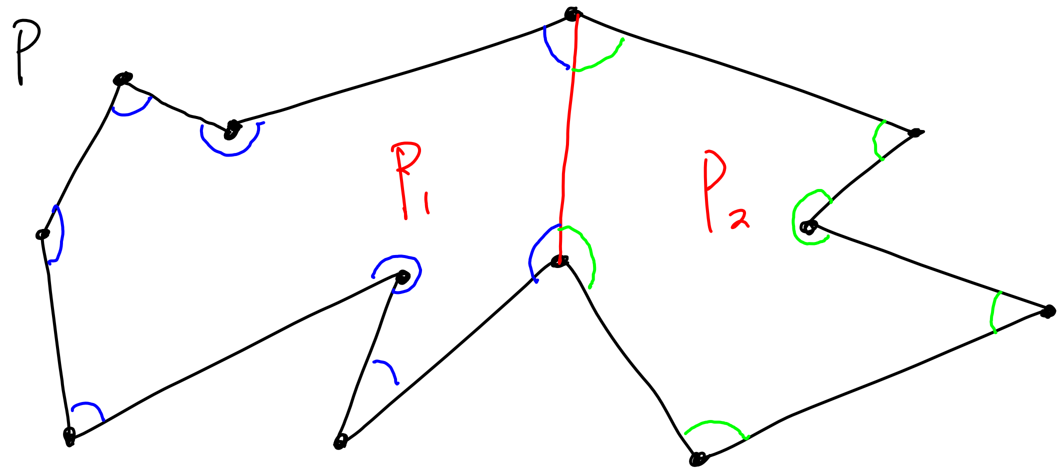


Say that P_1 has a sides and P_2 has b sides.

Therefore $a + b = n+3$ and $a, b \in \{3, \ldots, n\}$.

By the induction hypothesis

Sum of angles of $P_1 = (a-2)180°$ Sum of angles of $P_2 = (b-2)180°$



Now

$$\text{Sum of the angles of } P = \text{Sum of the angles of } P_1 + \text{Sum of the angles of } P_2$$

$$= (a-2)180^\circ + (b-2)180^\circ$$

$$= (a+b-4)180^\circ$$

$$= (n+3-4)180^\circ$$

$$= (n-1)180^\circ \quad \blacksquare$$

<u>Example</u> Show that every propositional-logic statement can be rewritten equivalently using no connectors other than $\rightarrow$ and $'$.

<u>proof</u>
The proof will be by induction on the number of connectors in a statement with base case $n=0$ connectors.

<u>Base Case</u>
If P has $n=0$ connectors, then P uses no connectors other than $\rightarrow$ and $'$.

<u>Induction Step</u>
Assume that any propositional-logic statement with $k\in\{0,...,n\}$ connectors can be rewritten equivalently using no connectors other than $\rightarrow$ and $'$.

Show that any propositional-logic statement with $n+1$ connectors can be rewritten equivalently using no connectors other than $\rightarrow$ and $'$.

If P is a statement with $n+1$ connectors. Then either $P = A \vee B$, $A \wedge B$, $A \to B$, $A \leftrightarrow B$, or A' where A has s connectors, B has t connectors and $s+t=n$. This implies that $s,t \in \{0,\dots,n\}$ so by the induction hypothesis, $A \leftrightarrow C$ and $B \leftrightarrow D$ where C and D use no connectors other than $\to$ and $'$.

So now either,

$$P = A \wedge B \leftrightarrow C \wedge D \leftrightarrow (C' \vee D')' \leftrightarrow (C \to D')' \quad \text{or}$$

$$P = A \vee B \leftrightarrow C \vee D \leftrightarrow C' \to D \quad \text{or}$$

$$P = A \to B \leftrightarrow C \to D \quad \text{or}$$

$$P = (A \leftrightarrow B) \leftrightarrow (C \leftrightarrow D) \leftrightarrow (C \to D) \wedge (D \to C) \leftrightarrow ((C \to D) \to (D \to C)')' \quad \text{or}$$

$$P = A' \leftrightarrow C'.$$

In each case, the final statement uses only the connectors $\to$ and $'$.

$\blacksquare$