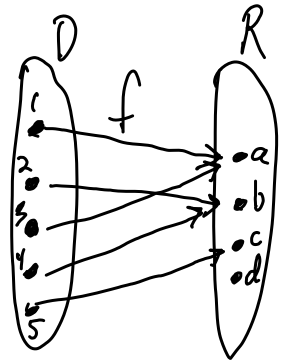
# Chapter 4 - Functions

**Def:** Given a set $D$ called a *domain* and a set $R$ called a *range*, a *function* $f: D \to R$ is an assignment of one element of $R$ to each element of $D$.

If $x \in D$, then the element of $R$ assigned to $x$ is called $f(x)$.

*example*




This diagram represents a function $f: D \to R$ in which $D = \{1, 2, 3, 4, 5\}$ and $R = \{a, b, c, d\}$ and

$f(1) = a$, $f(2) = b$, $f(3) = a$

$f(4) = b$, $f(5) = C$, and

for no $x \in D$ is $f(x) = d$.

In general, with "bubble diagrams" of functions every $x \in D$ has exactly one out-arrow.

There is no restriction on the number of in-arrows for elements $y \in R$.

example A predicate $P(x)$ with domain $D$ defines a function $P: D \to \{T, F\}$

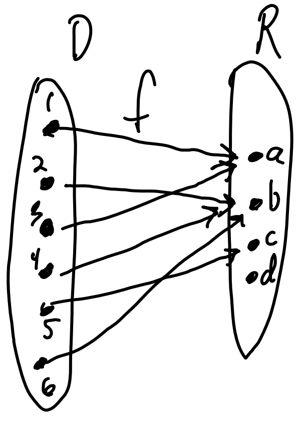
example In calculus the term function usually means an algebraic expression composed of powers of $x$, roots of $x$, $e^x$, $kx$, $\sin(x)$, $\cos(x)$, and maybe some other special functions. Their domains and ranges are both subsets of $\mathbb{R}$.

Def Given a function $f: D \to R$ and $A \subseteq D$, the image of $A$ under $f$ is defined as

$$f(A) = \{ f(x) \mid x \in A \}$$

Remark If $x \in D$, then $f(x) \in R$. If $A \subseteq D$, then $f(A) \subseteq R$.

example




If $A = \{1, 3, 5\}$, then $f(A) = \{a, c\}$.

If $A = \{2, 4, 6\}$, then $f(A) = \{b\}$

If $A = D$, then $f(D) = \{a, b, c\}$

**Remark** Sometimes for $f: D \to R$, the image of $D$ under $f$ (written $f(D)$) is just called <u>the image of $f$</u> and written as $Im(f)$.

In the example above $Im(f) = \{a, b, c\}$

**Example** Let $f: D \to R$. Prove that... If $A \subseteq B \subseteq D$, then $f(A) \subseteq f(B)$.

*proof*

Assume $A \subseteq B \subseteq D$. In order to show that $f(A) \subseteq f(B)$, consider some $y \in f(A)$. By the definition of image, there is $x \in A$ such that $y = f(x)$. Since $A \subseteq B$ we get $x \in B$, as well. Now by the definition of image, $y = f(x) \in f(B)$, as required. $\blacksquare$

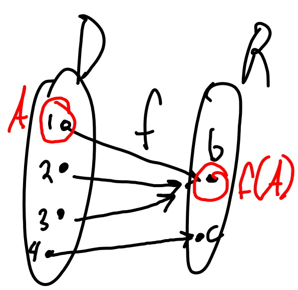
**example** Prove that the following is true or give a counterexample.

Given $f: D \to R$ and $A \subseteq D$, if $f(x) \in f(A)$, then $x \in A$.

False, here is a counterexample.

Def

$f(A) = \{f(x) \mid x \in A\}$




Let $A = \{1\}$. Thus $f(A) = \{b\}$

But $f(2) = b \in f(A)$ and $2 \notin A$.

**Example** Here is an incorrect proof of "If $A \subseteq B \subseteq D$, then $f(A) \subseteq f(B)$."

proof

Assume $A \subseteq B \subseteq D$. In order to show that $f(A) \subseteq f(B)$, consider some $f(x) \in f(A)$. By the definition of image, $x \in A$. (doesn't follow) Since $A \subseteq B$ we get $x \in B$ as well. Now $f(x) \in f(B)$, as required. ∎

**example** Let $f : D \to R$ and let $A, B \subseteq D$. This means $A \subseteq D$ and $B \subseteq D$.

Prove that
$$f(A \cap B) \subseteq f(A) \cap f(B)$$

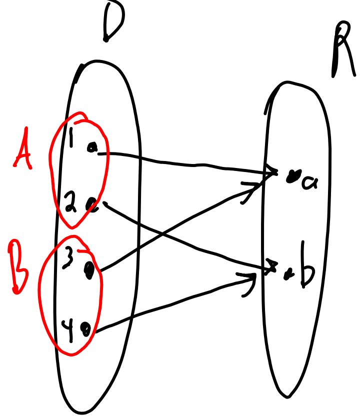
proof

Let $y \in f(A \cap B)$. By the definition of image, there is $x \in A \cap B$ such that $y = f(x)$. By the definition of intersection $x \in A$ and $x \in B$. By the definition of image $y = f(x) \in f(A)$ and $y = f(x) \in f(B)$. Therefore $y \in f(A) \cap f(B)$. ∎

**example** Give a counterexample to the following statement,

$$f(A) \cap f(B) \subseteq f(A \cap B)$$

**counterexample**




$A = \{1, 2\}$ $\quad B = \{3, 4\}$

So $f(A \cap B) = f(\emptyset) = \emptyset$

But $f(A) \cap f(B) = \{a, b\} \cap \{a, b\} = \{a, b\}$

So $f(A) \cap f(B) \not\subseteq f(A \cap B)$.

**example** For next time prove that

$$f(A) \cup f(B) = f(A \cup B)$$

**proof**

$\underline{f(A \cup B) \subseteq f(A) \cup f(B)}$

Let $y \in f(A \cup B)$. By definition of image, there is $x \in A \cup B$ such that $y = f(x)$. By definition of union $x \in A$ or $x \in B$. By definition of image, $f(x) \in f(A)$ or $f(x) \in B$. By definition of union $f(x) \in f(A) \cup f(B)$.

Thus $y = f(x) \in f(A) \cup f(B)$.

$f(A) \cup f(B) \subseteq f(A \cup B)$

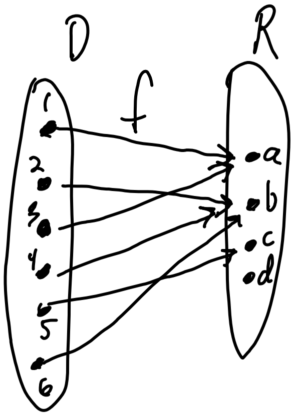
Let $y \in f(A) \cup f(B)$. By definition of union $y \in f(A)$ or $y \in f(B)$.
By definition of image there is $x \in A$ such that $y = f(x)$ or there is $x \in B$ such that $y = f(x)$. Therefore, by the definition of union, there is $x \in A \cup B$ such that $y = f(x)$. So by definition of image $y = f(x) \in f(A \cup B)$. $\blacksquare$

**Def** Given a function $f: D \to R$ and $A \subseteq R$,

The preimage of $A$ is

$$f^{-1}(A) = \{x \in D \mid f(x) \in A\}$$

Example




If $A = \{a, b\}$, then $f^{-1}(A) = \{1, 2, 3, 4, 6\}$

If $A = \{a, b, d\}$, then $f^{-1}(A) = \{1, 2, 3, 4, 6\}$

If $A = \{d\}$, then $f^{-1}(A) = \emptyset$

Also, $f^{-1}(\emptyset) =$

Remark If $x \in f^{-1}(A)$, then $f(x) \in A$.

If $f(x) \in A$, then $x \in f^{-1}(A)$.

Thus $x \in f^{-1}(A)$ if and only if $f(x) \in A$

This is different than <u>images</u>

If $B \subseteq D$. Then $x \in B$ implies $f(x) \in f(B)$

But $f(x) \in B$ does not imply $x \in B$.

<u>example</u> Let $f: D \to R$ and $A, B \subseteq R$, prove that

$$f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B).$$

<u>proof</u>

$x \in f^{-1}(A) \cap f^{-1}(B)$ if and only if (def of intersection)

$x \in f^{-1}(A)$ and $x \in f^{-1}(B)$ if and only if (def of preimage)

$f(x) \in A$ and $f(x) \in B$ if and only if (def of intersection)

$f(x) \in A \cap B$ if and only if (def of preimage)

$x \in f^{-1}(A \cap B)$ $\blacksquare$

<u>example</u> Let $f: D \to R$ and $A, B \subseteq R$, prove that

$$f^{-1}(A) - f^{-1}(B) = f^{-1}(A - B)$$

*proof*

$x \in f^{-1}(A) - f^{-1}(B)$ if and only if (definition of subtraction)

$x \in f^{-1}(A)$ and $x \notin f^{-1}(B)$ if and only if (definition of preimage)

$f(x) \in A$ and $f(x) \notin B$ if and only if (definition of subtraction)

$f(x) \in A - B$ if and only if (definition of preimage)

$x \in f^{-1}(A - B)$. $\blacksquare$

<u>example</u> Consider $f: D \to R$ and $A \subseteq R$.

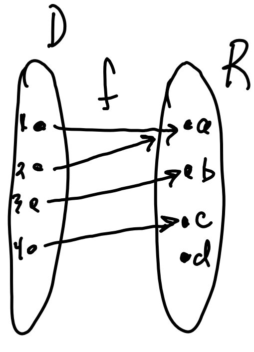
Prove that $f(f^{-1}(A)) \subseteq A$.

*proof*

Let $y \in f(f^{-1}(A))$. By definition of image, there is $x \in f^{-1}(A)$ such that $y = f(x)$. By definition of preimage $y = f(x) \in A$. $\blacksquare$

<u>example</u> Is it true that $A \subseteq f(f^{-1}(A))$ ?? No!

Here is a counterexample.




Let $A = \{d\}$

Then $f^{-1}(A) = \emptyset$.

Then $f(f^{-1}(A)) = \emptyset$.

So $A \not\subseteq f(f^{-1}(A))$.

**Def** A function $f: D \to R$ is called <u>surjective</u> (or <u>onto</u>) when for every $y \in R$, there is $x \in D$ such that $y = f(x)$.
Equivalently, for every $y \in R$, $f^{-1}(\{y\}) \neq \phi$.

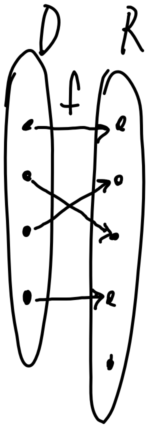
**example** Consider a surjective function $f: D \to R$ and $A \subseteq R$.
Prove that $f(f^{-1}(A)) = A$.

<u>$f(f^{-1}(A)) \subseteq A$</u> This was proved in the previous example.

<u>$A \subseteq f(f^{-1}(A))$</u> Let $y \in A$. Because $f$ is surjective, there is $x \in D$ such that $y = f(x)$. So by the definition of preimage $x \in f^{-1}(A)$. So now by the definition of image $y = f(x) \in f(f^{-1}(A))$, as required. ∎

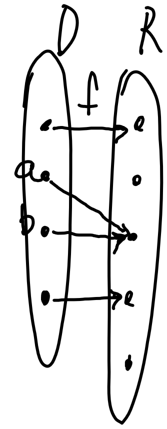
**Def** A function $f: D \to R$ is called <u>injective</u> (or <u>one-to-one</u>) when for all $a, b \in D$, $f(a) = f(b)$ implies $a = b$.
Equivalently, for all $a, b \in D$, $a \neq b$ implies $f(a) \neq f(b)$.

__example__




$f$ is injective

Every range element has 0 or 1 in-arrows




$f$ is __not__ injective

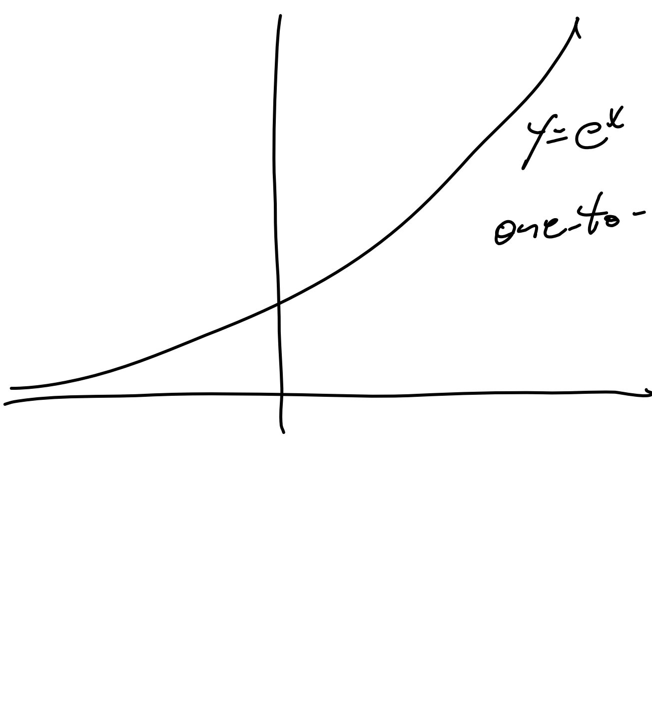
$a \neq b$ but $f(a) = f(b)$.

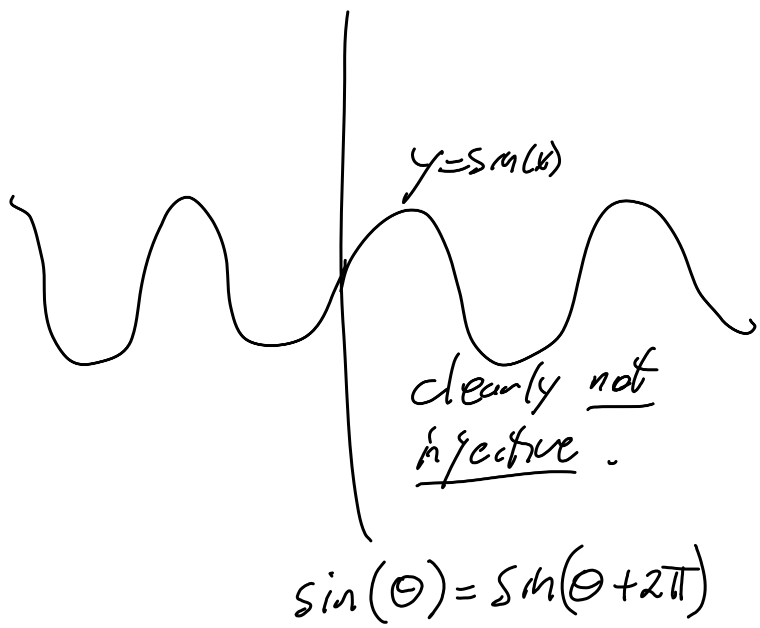
__example__ If $f : \mathbb{R} \to \mathbb{R}$ is a function, then

$f$ is injective

if and only

The graph of $y = f(x)$ satisfies the horizontal line test.




clearly __not__ __injective__.

$\sin(\theta) = \sin(\theta + 2\pi)$

where $\theta \neq \theta + 2\pi$

<u>example</u> Define $f: \mathbb{N}\times\mathbb{N} \longrightarrow \mathbb{N}$ (Remember $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$)

by $f((a,b)) = a^2+b^2$

an element of $\mathbb{N}\times\mathbb{N}$

an element of $\mathbb{N}$ because $a^2+b^2 \geq 0$.

ⓐ Let $A = \{(1,0),(0,1)\}$

$$f(A) = \{f(a,b) \mid (a,b) \in A\} = \{a^2+b^2 \mid (a,b) \in A\} = 1$$

ⓑ Let $B = \{0, 2, 5, 25\}$

$$f^{-1}(B) = \{(a,b) \in \mathbb{N}\times\mathbb{N} \mid f(a,b) = 0, 2, 5, \text{ or } 25\}$$

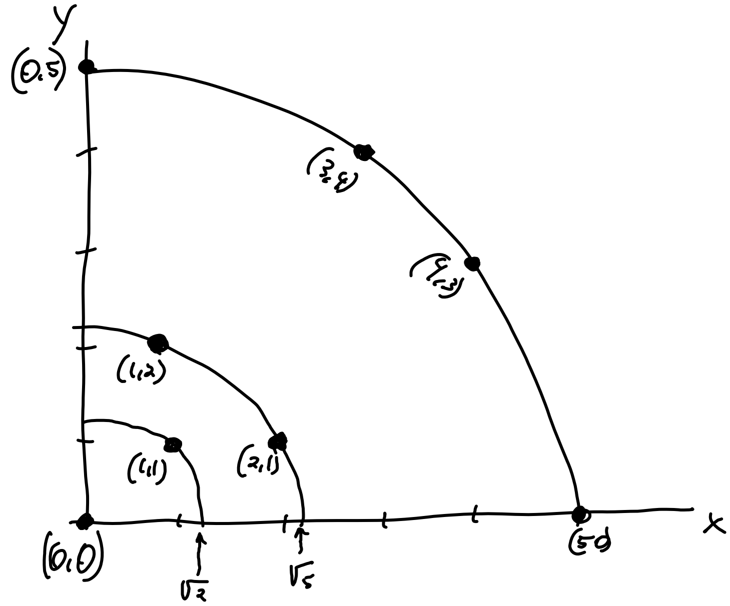
$$= \{(a,b) \in \mathbb{N}\times\mathbb{N} \mid a^2+b^2 = 0, 2, 5, \text{ or } 25$$

| $a^2+b^2=0$ | $a^2+b^2=2$ | $a^2+b^2=5$ | $a^2+b^2=25$ |
|---|---|---|---|
| $(0,0)$ | $(1,1)$ | $(2,1)$ and $(1,2)$ | $(3,4)$, $(4,3)$, $(5,0)$, $(0,5)$ |

ⓒ If we graph these points on an $xy$-plane ($1^{st}$ quadrant only)

Then we see that $f^{-1}(\{2\})$ $f^{-1}(\{25\})$ are all contained on $\frac{1}{4}$ circles of radii $0, \sqrt{2}, \sqrt{5}, 5$

Here they are.




(d)

Is $f: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by $f((a,b)) = a^2 + b^2$

Surjective? For any $y \in \mathbb{N}$ is there $(a,b) \in \mathbb{N} \times \mathbb{N}$ such that $f((a,b)) = a^2 + b^2 = y$ ??

No, for example $a^2 + b^2 \neq 3$ for any $(a,b) \in \mathbb{N} \times \mathbb{N}$.

Injective? For any $(a,b), (x,y) \in \mathbb{N} \times \mathbb{N}$ is it true that $f((a,b)) = f((x,y))$ implies $(a,b) = (x,y)$ ??

that is, $a^2 + b^2 = x^2 + y^2$ implies $a = x$ and $b = y$ ??

No. For example $f((3,4)) = f((5,0))$ but $(3,4) \neq (5,0)$.

example Consider $f: (\mathbb{R}-\{1\}) \to (\mathbb{R}-\{6\})$

defined by $f(x) = \dfrac{6x+3}{x-1}$

Is f surjective?

For any $y \in \mathbb{R}-\{6\}$ is there $x \in \mathbb{R}-\{1\}$ such that

$f(x) = y$ ?? That is,

$$\frac{6x+3}{x-1} = y \quad ??$$

Since is $y$ is a known element of $\mathbb{R}-\{6\}$, if can solve for $x$ in terms of $y$, then we will have found $x$.

$$\frac{6x+3}{x-1} = y$$

$$6x+3 = y(x-1)$$

$$6x+3 = xy - y$$

$$y+3 = xy - 6x$$

$$y+3 = x(y-6)$$

$$\frac{y+3}{y-6} = x$$ and since $y \neq 6$, we now have found $x$.

Thus f is surjective.

Is f injective? Does $f(a)=f(b)$ imply $a=b$??

Assume $f(a)=f(b)$

$$\frac{6a+3}{a-1}=\frac{6b+3}{b-1}$$

$$(6a+3)(b-1)=(6b+3)(a-1)$$

$$\cancel{6ab}+3b-6a-\cancel{3}=\cancel{6ab}+3a-6b-\cancel{3}$$

$$9b=9a$$

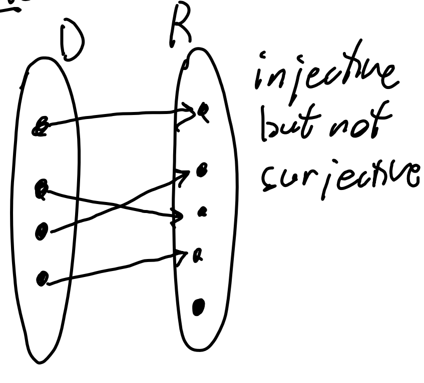
$$b=a.$$

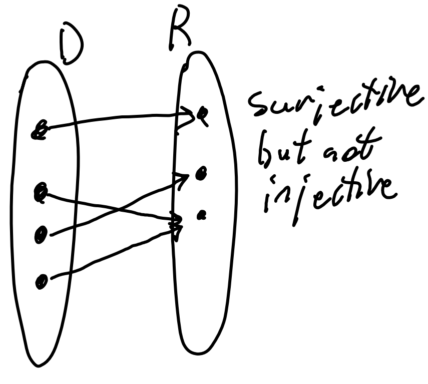
So yes, f is injective.

<u>Def:</u> A function $f: D \to R$ is called a <u>bijection</u> when it is both injective and surjective; That is, f is both one-to-one and onto.

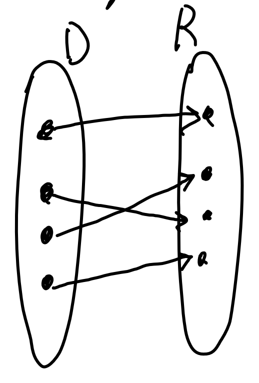
$\boxed{\text{Sometimes a bijection is called a one-to-one correspondence.}}$

<u>example</u>




$\boxed{\text{A bijection}}$

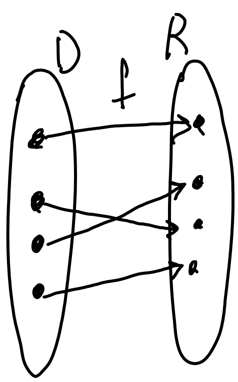


**Def:** If $f: D \to R$ is a bijection, the <u>inverse function</u> is a function $f^{-1}: R \to D$ defined by

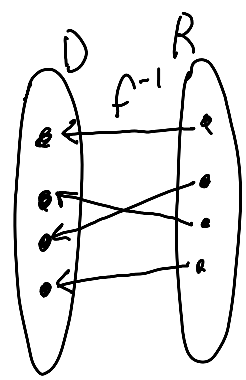
$$f^{-1}(y) = x \quad \text{when} \quad f(x) = y$$

<u>example</u>

Given $f$



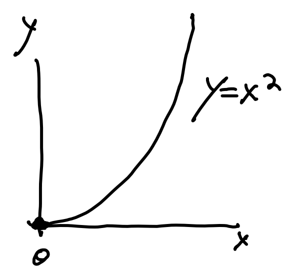

, $f^{-1}$ is




<u>example</u>

$f: [0, +\infty) \longrightarrow [0, +\infty)$ defined by $f(x) = x^2$

is a bijection




Here $f^{-1}(x) = \sqrt{x}$

*example*

Given $f: (\mathbb{R} - \{1\}) \to (\mathbb{R} - \{6\})$

defined by $f(x) = \dfrac{6x+3}{x-1}$,

we already showed that it's a bijection.

The inverse function $f^{-1}: (\mathbb{R} - \{6\}) \to (\mathbb{R} - \{1\})$

is defined by $f^{-1}(x) = \dfrac{x+3}{x-6}$

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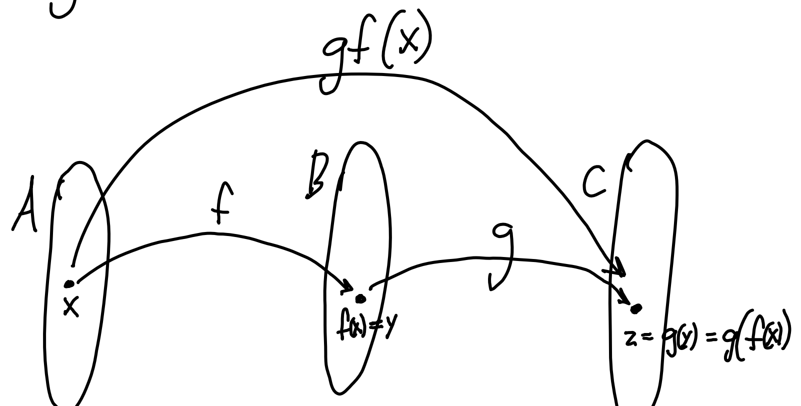
## Compositions of functions

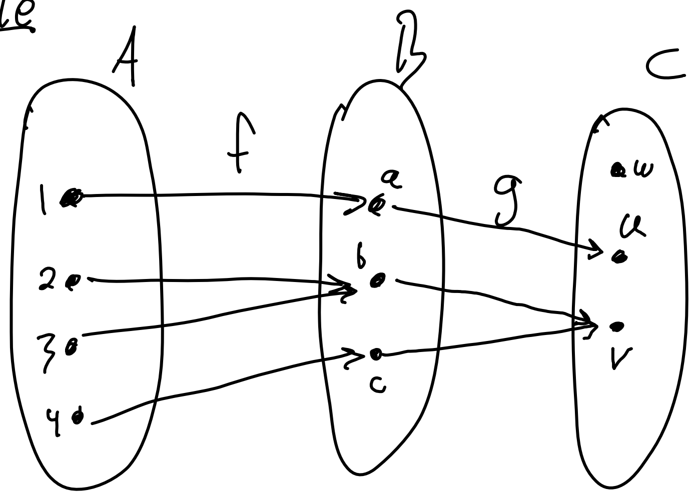
Given functions $f: A \to B$ and $g: B \to C$

The <u>composition</u> $gf: A \to C$ is called

"g following f" and is defined as $gf(x) = g(f(x))$



example




$gf(1) = g(f(1)) = g(a) = u$

$gf(2) = g(f(2)) = g(b) = v$

$gf(3) = g(f(3)) = g(b) = v$

$gf(4) = g(f(4)) = g(c) = v$

example Consider $f: A \to B$ and $g: B \to C$.

(1) Prove That, $f$ and $g$ injective implies $gf$ injective.

proof

Assume $f$ and $g$ are injective. To prove that $gf$ is injective consider $gf(a) = gf(b)$.

By the definition of composition $g(f(a)) = g(f(b))$

Because $g$ is injective $f(a) = f(b)$

Because $f$ is injective $a = b$, as required.

$\blacksquare$

② Prove that, $gf$ injective implies $f$ injective.

proof

Assume $gf$ is injective. To show that $f$ is injective consider $f(a) = f(b)$.

Apply $g$ to both sides to get $g(f(a)) = g(f(b))$.

By definition of composition $gf(a) = gf(b)$.

Because $gf$ is injective $a = b$, as required. ■

proof by contrapositive

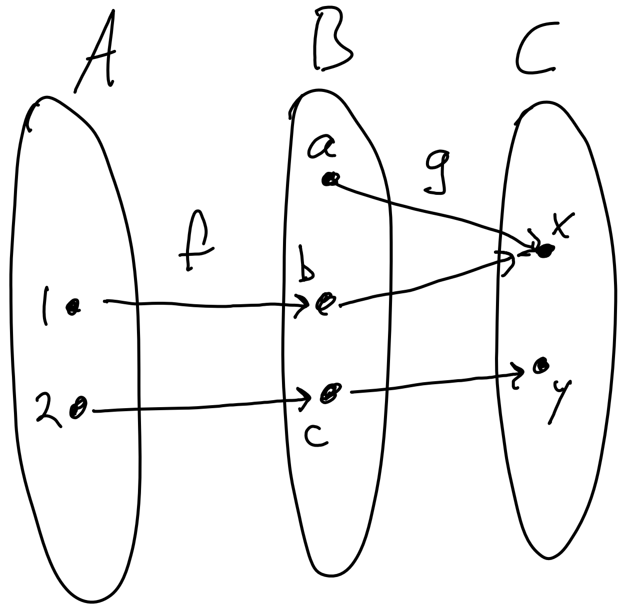
Assume $f$ is not injective. By definition there must exist $a, b \in A$ such that $a \neq b$ and $f(a) = f(b)$.

Apply $g$ to both sides to get $g(f(a)) = g(f(b))$.

By definition of composition $gf(a) = gf(b)$

where $a \neq b$. This means $gf$ is not injective. ■

example When $gf: A \to C$ is injective it does not necessarily imply that $g$ is injective.




gf is injective and g is not injective.

③ Prove that, f and g surjective implies gf surjective.

proof

Assume f and g are surjective. In order to show that gf is surjective, consider an arbitrary $z \in C$. Because g is surjective there is $y \in B$ such that $g(y) = z$. Because

$f$ is surjective, there is $x \in A$ such that

$f(x) = y$. So now $f(x) = y$ becomes

$g(f(x)) = g(y)$ which becomes

$gf(x) = z$, which shows

$gf$ is surjective. ■

(4) Prove that, $gf$ surjective implies $g$ surjective

proof

Assume $gf$ is surjective. In order to show

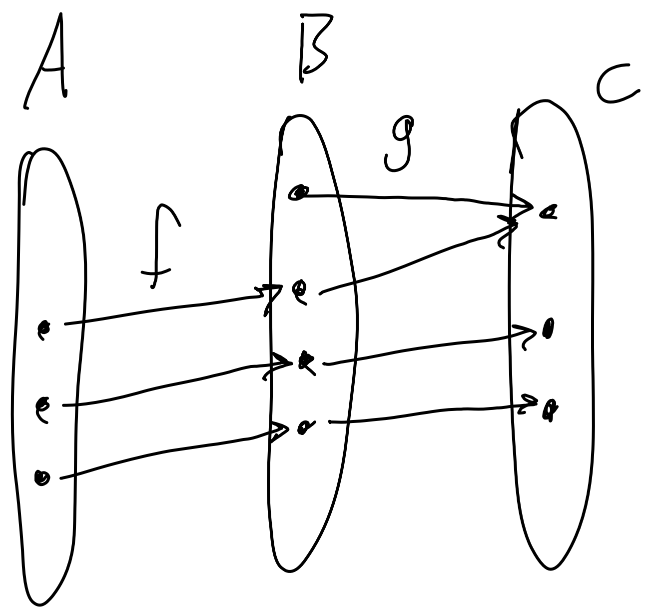
that $g$ is surjective, consider some

$z \in C$. Because $gf$ is surjective, there is

$x \in A$ such that $gf(x) = z$. By the definition of composition $g(f(x)) = z$ where $f(x) \in B$.

Thus $g$ is surjective. ∎

<u>example</u> When $gf$ is surjective, it doesn't necessarily imply that $f$ is surjective.




$f$ is not surjective but $gf$ is surjective.

<u>Here are two corollaries</u>

① If $gf$ is a bijection, then $f$ is an injection and $g$ is a surjection.

② If $g$ and $f$ are bijections, then $gf$ is a bijection.

"The following are equivalent"

Some Theorems in mathematics are stated as follows.

Theorem: Blah, Blah, Blah, - - - - - . Then The following are equivalent.

1. A
2. B
3. C
4. D

$\vdots$

Which means

$$\begin{tikzcd}
A \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] \arrow[dr, leftrightarrow] & B \arrow[d, leftrightarrow] \arrow[dl, leftrightarrow] \\
C \arrow[r, leftrightarrow] & D
\end{tikzcd}$$

This is 6 logical equivalences which includes 12 implications.

However, to show 12 implications one

need only show some subset of the 12 which includes a directed path between any two. This uses the <u>hypothetical syllogism</u> tautology.

$$(A \to B) \wedge (B \to C) \longrightarrow (A \to C)$$

<u>example</u>

proving these 4 implications

$$\begin{tikzcd} A \arrow[r] & B \arrow[d] \\ D \arrow[u] & C \arrow[l] \end{tikzcd}$$

proves all 12 implications.

<u>example</u>

proving these 5 implications

$$\begin{tikzcd} & A \arrow[dr] & & \\ C \arrow[ur] & & B \arrow[ll] \arrow[r, bend left] & D \arrow[l, bend left] \end{tikzcd}$$

proves all 12 implications

**example** Let $D$ and $R$ be finite sets such that $|D|=|R|$. Let $f: D \to R$ be some function. The following are equivalent.

1. $f$ is a bijection
2. $f$ is an injection
3. $f$ is a surjection

**proof** We will prove

$$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[dl] \\ 3 \arrow[u] & \end{tikzcd}$$

$(1 \to 2)$ Assuming $f$ is a bijection, $f$ is then an injection by definition.

$(2 \to 3)$ Assume $f$ is an injection. Write $D=\{a_1, a_2, \ldots, a_n\}$. Therefore $n=|D|=|R|$. Because $f$ is injective $\{f(a_1), f(a_2), \ldots, f(a_n)\}$ is a subset of $R$ having order $n$. But $|R| = n$ so $R = \{f(a_1), \ldots, f(a_n)\}$ which means that $f$ is surjective.

$(3 \to 1)$ Assume $f$ is surjective. To show that $f$ is a bijection we only need to show that $f$ is injective. Write $R = \{b_1, \ldots, b_n\}$. Therefore $n = |R| = |D|$. Because $f$ is surjective for each $b_i$ there is $a_i \in D$ such that $f(a_i) = b_i$. Therefore $f(a_1) = b_1$, $f(a_2) = b_2$, $\ldots$, $f(a_n) = b_n$ where $D = \{a_1, \ldots, a_n\}$. Thus $f$ is one-to-one (i.e., injective).

$\blacksquare$