Chapter 1 - Propositional Logic

Def A proposition (or a statement) is a sentence (or a part of a sentence which is on its own a sentence) which has a definite truth value, either true or false.

example

1. "4 is even." is true
2. "9 is a prime number." is false
3. "Mark goes to Wright State." is true or false depending on who Mark is.

non examples

1. "Mark is tall." Has no definite truth value in all cases. Is more of a value judgement.
2. "Do you like chocolate?" This is a question.
3. "Trees leaves." Not a sentence.

Logical Connectors

Are used to take smaller statements and put them together to get larger statements.

1. **Conjunction** $\wedge$

Given two propositions A and B. The truth value of $A \wedge B$ is defined by the following table.

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

In English $A \wedge B$ can be written as

"A and B"

"A but B"

"A also B"

"A furthermore B"

"A moreover B"

"A however B"

Conjunction can also be used to join together three or more propositions.

$$A_1 \wedge A_2 \wedge \cdots \wedge A_n$$

is true if and only if each A_i is true on its own.

2. Disjunction - $\vee$

In English $A \vee B$ is the inclusive or

A or B meaning

A is true or B is true or both are true.

Other English language usages of disjunction.

"A or else B"

"Either A or B"

"If not A, then B."

Given two propositions A and B

The truth value of $A \vee B$ is defined by the following table.

A	B	$A \vee B$
T	T	T
T	F	T
F	T	T
F	F	F

Disjunction can also be used to join together three or more propositions.

$A_1 \vee A_2 \vee A_3 \vee \cdots \vee A_n$ is true

if and only if at least one A_i

is true.

3. <u>Implication</u> $\rightarrow$

In statement

$A \rightarrow B$

A is the <u>hypothesis</u>.
B is the <u>conclusion</u>.

In English $A \rightarrow B$ can be written in any the following ways.

Given two propositions A and B
The truth value of $A \rightarrow B$ is defined by the following table.

A	B	$A \rightarrow B$
T	T	T
T	F	F
F	T	T
F	F	T

<u>Notice</u>
An implication is false only when the hypothesis is true and the conclusion is false.
In particular, when the hypothesis is false the implication is true regardless of the conclusion.

In English $A \rightarrow B$
can be written in lots of ways.

"A implies B."

"If A, then B."

"B follows from A."

"B is necessary for A"

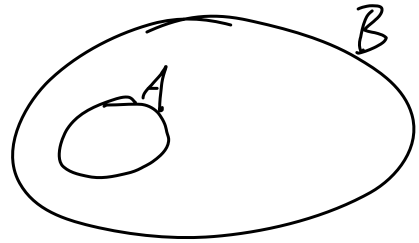
"A is sufficient for B."

"B if A"

"A only if B."

"B when A"

What do these all have in common? They all describe a containment relationship



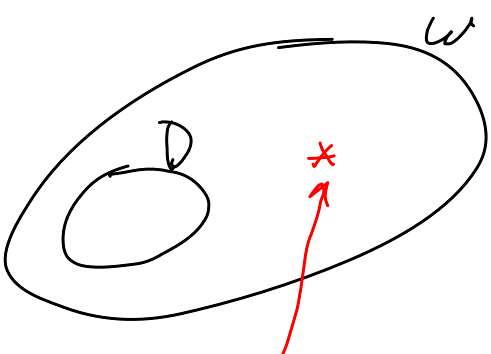
Instances where A is true are contained within instances where B is true.

What if we get confused. How can we tell which way the implication goes? Look at both possibilities and consider an instance in the gap between the two.

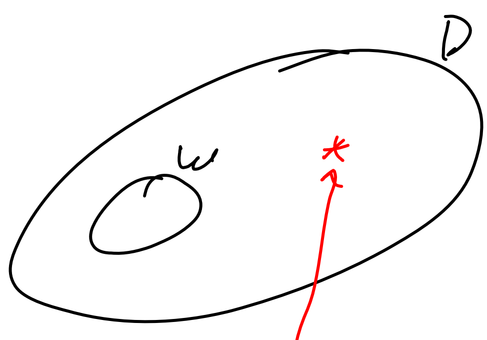
Being Dr. Slilaty's student is sufficient for being a student at WSU.

$D \to W$ correct

$\cancel{W \to D}$



You are a student at WSU and not Dr. Slilaty's student.



~~You are Dr. Slilaty's student and not a student at WSU.~~

Contradicts the original statement.

(4) Logical Equivalence $\longleftrightarrow$

Given two propositions A and B. The truth value of $A \leftrightarrow B$ is defined by the following table.

A	B	$A \leftrightarrow B$
T	T	T
T	F	F
F	T	F
F	F	T

In English

$A \leftrightarrow B$ is said as

"A is equivalent to B."

"A if and only if B."

"A iff B"

(5) Negation $'$

A' means "not A".

Given proposition A. The truth value of A' is given by

A	A'
T	F
F	T

Operator Precedence

1. $'$
2. $\vee, \wedge$ neither has precedence over the other.
3. $\rightarrow$
4. $\leftrightarrow$

example $A \wedge B \rightarrow C$ means $(A \wedge B) \rightarrow C$

Because conjunction has precedence over implication. So parentheses are not needed.

In contrast to this, parentheses are needed for $A \wedge (B \rightarrow C)$

example $A \wedge (B \vee C)$ is different than $(A \wedge B) \vee C$.

Furthermore $A \wedge B \vee C$ is ambiguous because neither conjunction nor disjunction has precedence over the other.

Truth tables

Given a compound propositional-logic statement, a <u>truth table</u> for that statement shows its truth value for every possible combination of truth values of its bare propositional variables.

<u>example</u> Construct the truth table for $A \wedge (A \to B) \longleftrightarrow B$

A	B	$A \to B$	$A \wedge (A \to B)$	$A \wedge (A \to B) \longleftrightarrow B$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	F
F	F	T	F	T

Notice that $A \wedge (A \to B) \leftrightarrow B$ is not always true.

So $A \wedge (A \to B) \leftrightarrow B$ is <u>not</u> a tautology.

A <u>tautology</u> is a propositional-logic statement which is always true.

<u>example</u> Create a truth table for $A \wedge (A \to B) \longrightarrow B$

A	B	$A \to B$	$A \wedge (A \to B)$	$A \wedge (A \to B) \to B$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

So $A \wedge (A \to B) \to B$ is a tautology.

<u>example</u> Construct a truth table for $A \wedge (B \vee C) \longleftrightarrow (A \wedge B) \vee C$

A	B	C	$B \vee C$	$A \wedge B$	$A \wedge (B \vee C)$	$(A \wedge B) \vee C$	$A \wedge (B \vee C) \longleftrightarrow (A \wedge B) \vee C$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T
T	F	T	T	F	T	T	T
T	F	F	F	F	F	F	T
F	T	T	T	F	F	T	F
F	T	F	T	F	F	F	T
F	F	T	T	F	F	T	F
F	F	F	F	F	F	F	T

So the statement is not a tautology.

Some Named Tautologies (These are all logical equivalences)

Implication

$A \to B \leftrightarrow A' \vee B$

Contrapositive

$A \to B \leftrightarrow B' \to A'$

Double Implication

$(A \leftrightarrow B) \leftrightarrow (A \to B) \wedge (B \to A)$

Associativity

$(A \wedge B) \wedge C \leftrightarrow A \wedge (B \wedge C)$

$(A \vee B) \vee C \leftrightarrow A \vee (B \vee C)$

Therefore it makes sense to write $A \wedge B \wedge C$ and $A \vee B \vee C$

Commutativity

$A \wedge B \leftrightarrow B \wedge A$

$A \vee B \leftrightarrow B \vee A$

Double Negation

$(A')' \leftrightarrow A$

Distribution

$A \wedge (B \vee C) \leftrightarrow (A \wedge B) \vee (A \wedge C)$

$A \vee (B \wedge C) \leftrightarrow (A \vee B) \wedge (A \vee C)$

DeMorgan's Laws

$(A \wedge B)' \leftrightarrow A' \vee B'$

$(A \vee B)' \leftrightarrow A' \wedge B'$

Complementation

$A \wedge A' \leftrightarrow 0$

$A \vee A' \leftrightarrow 1$

Identity

$A \wedge 1 \leftrightarrow A$

$A \vee 0 \leftrightarrow A$

Idempotent

$A \vee A \leftrightarrow A$

$A \wedge A \leftrightarrow A$

0 is a proposition which is always false

1 is a tautology, That is, a proposition which is always True

annihilation

$A \wedge 0 \leftrightarrow 0$

$A \vee 1 \leftrightarrow 1$

These named tautologies may be used to establish other larger tautologies.

<u>example</u> (Exportation) Prove that $A \to (B \to C) \longleftrightarrow A \wedge B \to C$ using a chain of logical equivalences with the name of the tautologies used underneath each one.

$$A \to (B \to C) \underset{\text{Implication}}{\longleftrightarrow} A \to (B' \vee C) \underset{\text{Implication}}{\longleftrightarrow} A' \vee (B' \vee C) \underset{\text{Associativity}}{\longleftrightarrow} (A' \vee B') \vee C \underset{\text{DeMorgan}}{\longleftrightarrow}$$

$$(A \wedge B)' \vee C \underset{\text{Implication}}{\longleftrightarrow} A \wedge B \to C$$

<u>example</u> Prove that the following is a tautology.

$$(A \to B) \vee (A \to C) \longleftrightarrow A \to B \vee C$$

$$(A \to B) \vee (A \to C) \underset{\substack{\text{Implication} \\ \times 2}}{\longleftrightarrow} (A' \vee B) \vee (A' \vee C) \underset{\substack{\text{Commutativity} \\ + \\ \text{associativity}}}{\longleftrightarrow} (A' \vee A') \vee (B \vee C) \underset{\text{Idempotent}}{\longleftrightarrow}$$

$$A' \vee (B \vee C) \underset{\text{Implication}}{\longleftrightarrow} A \to B \vee C$$

<u>example</u> Show that the following is a tautology.

$$(A \wedge (A \rightarrow B))' \vee B \leftrightarrow 1$$

$$(A \wedge (A \rightarrow B))' \vee B \underset{\text{DeMorgan}}{\longleftrightarrow} [A' \vee (A \rightarrow B)'] \vee B \underset{\text{Implication}}{\longleftrightarrow} [A' \vee (A' \vee B)'] \vee B \underset{\text{Demorgan + Double neg}}{\longleftrightarrow}$$

$$[A' \vee (A \wedge B')] \vee B \underset{\text{Distribution}}{\longleftrightarrow} [(A' \vee A) \wedge (A' \vee B')] \vee B \underset{\text{Complement}}{\longleftrightarrow} [1 \wedge (A' \vee B')] \vee B$$

$$\underset{\text{Identity}}{\longleftrightarrow} (A' \vee B') \vee B \underset{\text{Associativity}}{\longleftrightarrow} A' \vee (B' \vee B) \underset{\text{Complement}}{\longleftrightarrow} A' \vee 1 \underset{\text{annihilation}}{\longleftrightarrow} 1$$

Translations of English statements to propositional logic

Write a propositional-logic statement that is equivalent to an English-language statement.

<u>example</u>

H = "The horse is fresh" A = "The Armor is strong"

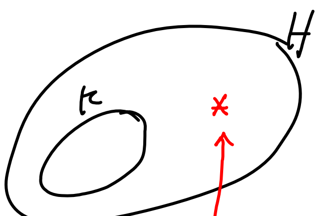
K = "The knight wins" O = "The opponent cheated."

ⓐ If the horse is fresh and the armor is strong, then the knight will win.

$$H \wedge A \to K$$

ⓑ The knight will win when the horse is fresh.

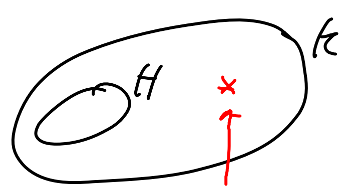
$\cancel{K \to H}$



Horse is fresh
and
The knight loses

This contradicts the meaning of the original statement

$\boxed{H \to K}$ correct implication.

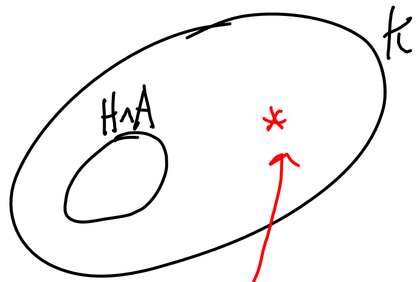


The knight wins
and
the horse is tired.

(c) A fresh horse and strong armor are necessary for the knight to win.

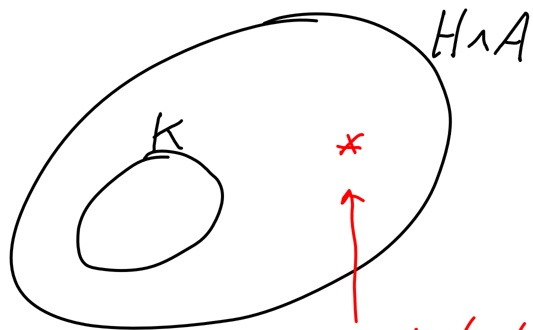
$\cancel{H \wedge A \to K}$

$\boxed{K \to H \wedge A}$ correct answer.



knight wins despite a deficiency in horse or armor

Contradicts the original statemen



knight lost even though he has a fresh horse and strong armor.

<u>Note</u> The <u>necessary</u> condition in an implication is the conclusion.
The sufficient condition in an implication is the hypothesis.

(d) A fresh horse and strong armor are sufficient for the knight to win.

$$H \wedge A \to K$$

(e) A fresh horse and strong armor are necessary and sufficient for the knight to win.

$$H \wedge A \longleftrightarrow K$$

(f) The knight will win unless the opponent cheats.

What is meant by K unless O is

$$K \vee (K' \wedge O)$$

However, logically this is just $K \vee O$.

proof

$$K \vee (K' \wedge O) \underset{\text{distribution}}{\longleftrightarrow} (K \vee K') \wedge (K \vee O) \underset{\text{complement}}{\longleftrightarrow} 1 \wedge (K \vee O) \underset{\text{identity}}{\longleftrightarrow} K \vee O$$

(g) The knight lost, the armor was strong and the horse was fresh, or the opponent cheated.

$$K \vee (A \wedge H) \vee O$$

Negating English-language statements

1. Given statement, translate to propositional logic.
2. Take the propositional statement and negate the whole thing (statement in here)'.
3. Use tautologies to distribute and simplify (——)'
4. Translate back to English.

example

Write the negation of the following statements.

1. If the restaurant is good, then the prices are high.

Translates to $G \to P$

Negate $(G \to P)' \underset{\text{Implication}}{\longleftrightarrow} (G' \vee P)' \underset{\substack{\text{DeMorgan} \\ + \\ \text{double negation}}}{\longleftrightarrow} G \wedge P'$

Translate back: The restaurant is good and prices are reasonable.

OR

The restaurant is good but the prices are reasonable.

<u>Note</u> It is worth remembering that $(A \to B)' \leftrightarrow A \wedge B'$

2. "If the food is good, then either the prices are high or the service is bad."

$$(G \to P \vee B)' \longleftrightarrow G \wedge (P \vee B)' \underset{\text{DeMorgan}}{\longleftrightarrow} G \wedge (P' \wedge B') \underset{\text{associativity}}{\longleftrightarrow} G \wedge P' \wedge B'$$

The food is good, the service is good, and the prices are reasonable.