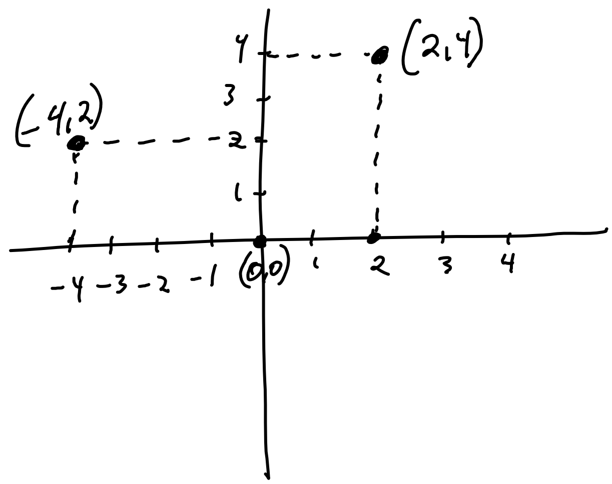
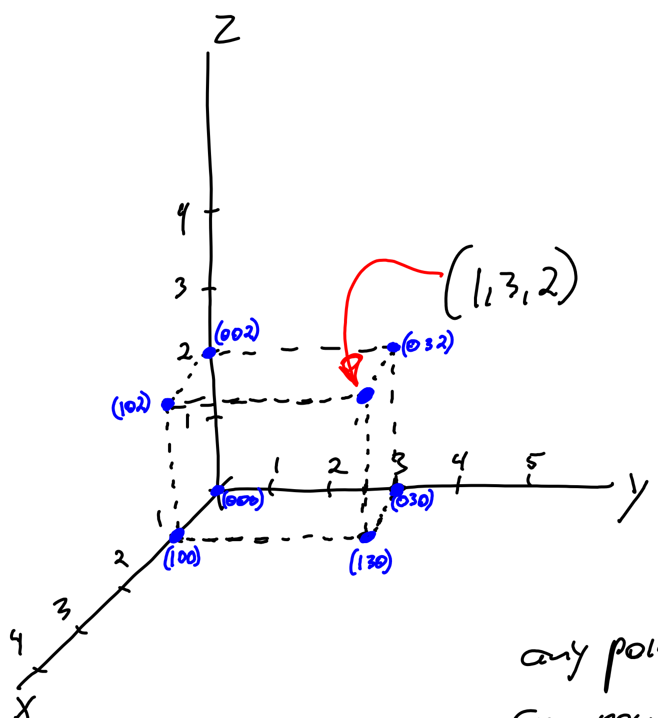
Sections 2.1 and 2.3

Recall the 2-dimensional xy-plane and its "rectangular" or "Cartesian" coordinate system.

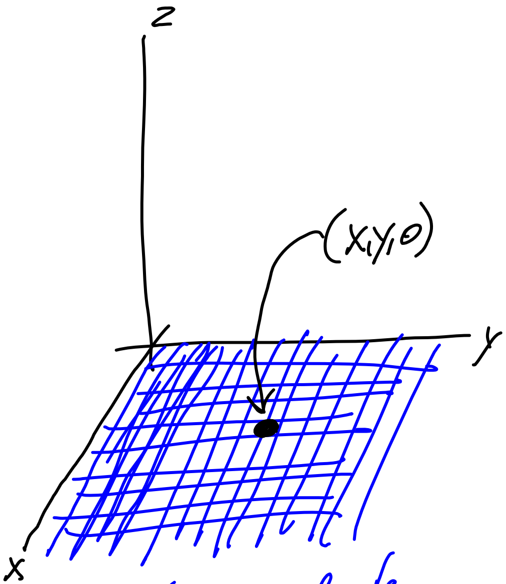


This can be generalized to 3 dimensions with 3 mutually perpendicular coordinate axes.

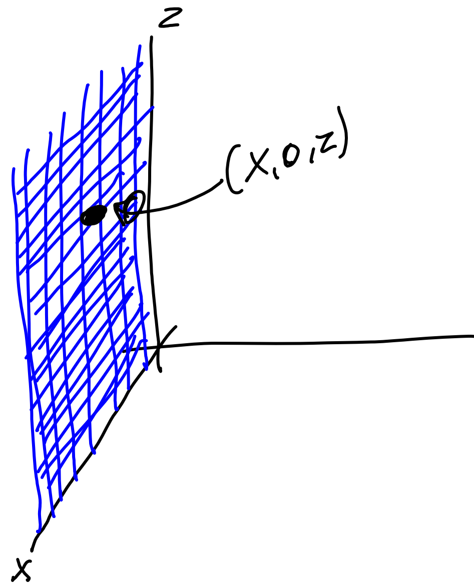


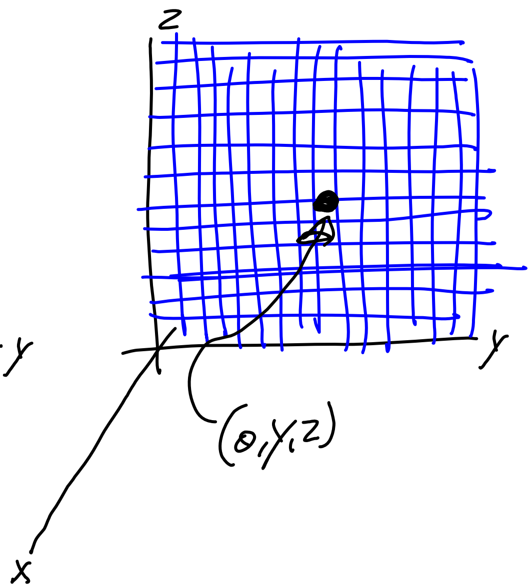
any point in 3D can now be identified by an ordered triple (x, y, z)

Each pair of these 3 coordinate axes defines what's known as a <u>coordinate plane</u>.

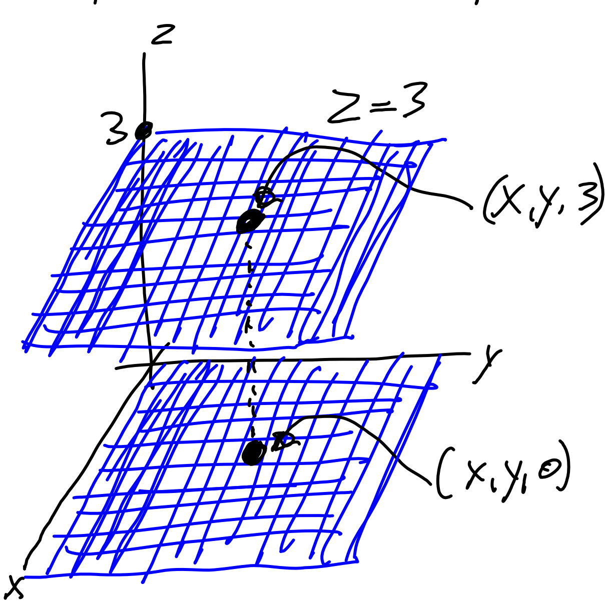


XY-coordinate plane.

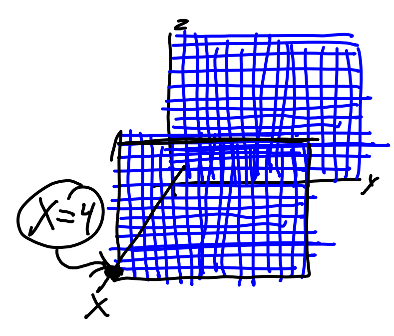




To specify a plane parallel to the xy-plane we write that $z=k$. This is the plane parallel to the xy-plane which passes through the z-axis at $z=k$.

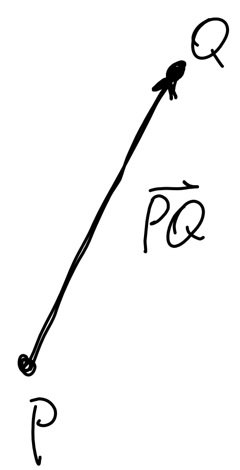


Similarly, a plane parallel to the yz-plane is specified by the equation $x=k$.



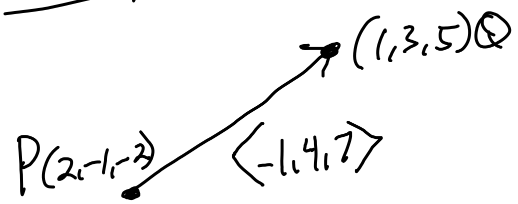
Vectors

Given two points P and Q in either 2D or 3D
The vector $\overrightarrow{PQ}$ is the directed line segment whose head is at Q and tail is at P.



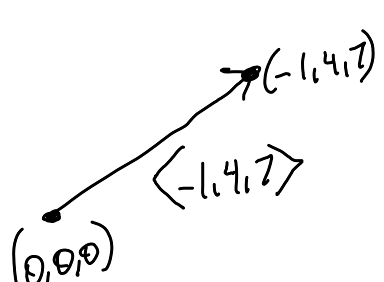
A vector can be represented by a pair in 2D or a triple in 3D which is obtained by the difference head-tail in each coordinate.

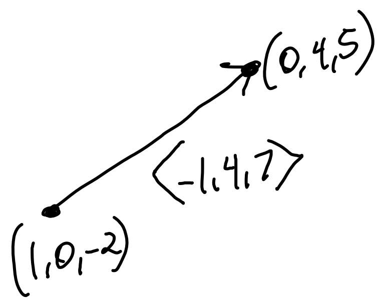
example



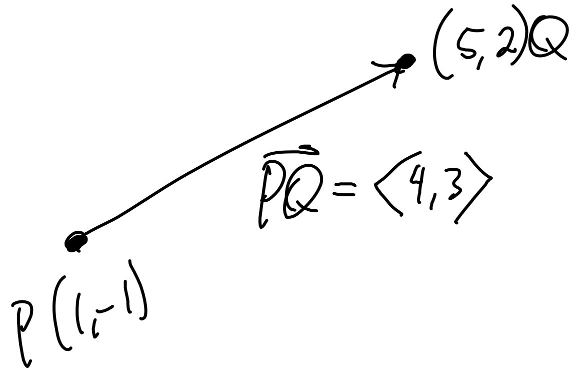
$$\overrightarrow{PQ} = \langle 1-2,\ 3-(-1),\ 5-(-2) \rangle = \langle -1,4,7 \rangle$$

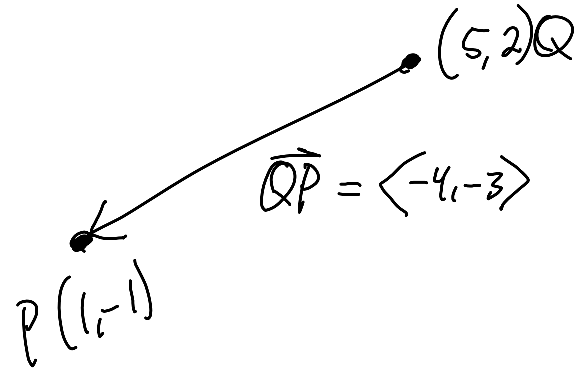
Note that this same vector has infinitely many head, tail points.





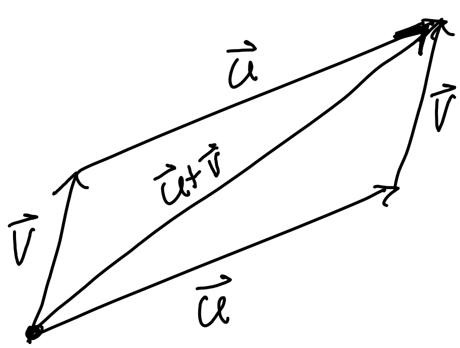
example Note that the reverse of $\overrightarrow{PQ}$ namely $\overrightarrow{QP}$ is obtained by negating the coordinates of the vector.





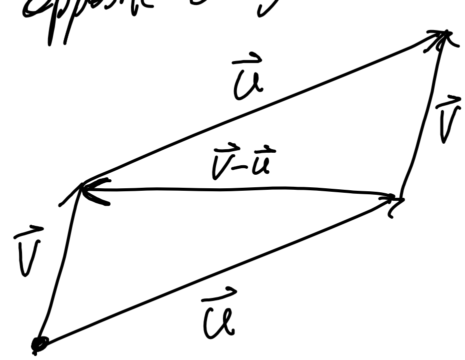
Vector Addition

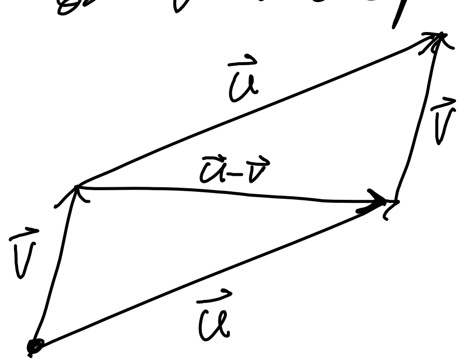
Given two vectors $\vec{u}$ and $\vec{v}$ they can be placed head to tail to form a parallelogram as shown.



The sum $\vec{u}+\vec{v}$ is the diagonal indicated.

The opposite diagonal is $\vec{u}-\vec{v}$ or $\vec{v}-\vec{u}$ depending on direction.





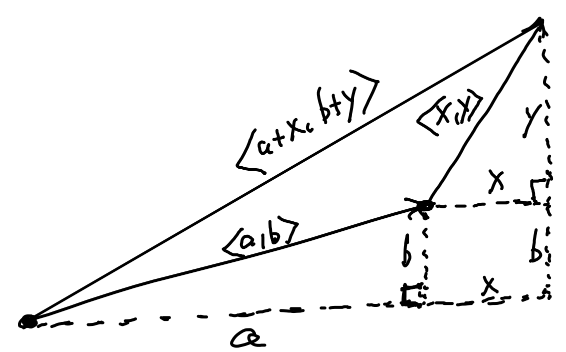
This geometric definition of $\vec{u}+\vec{v}$ also yields the following arithmetic property.

Given $\vec{u}=\langle a,b\rangle$ and $\vec{v}=\langle x,y\rangle$, $\vec{u}+\vec{v}=\langle a+x, b+y\rangle$

and

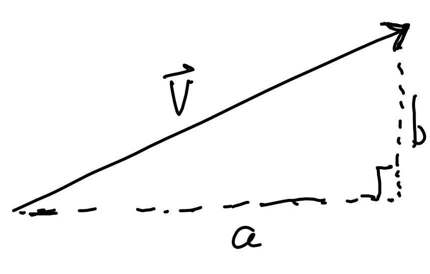
Given $\vec{u}=\langle a,b,c\rangle$ and $\vec{v}=\langle x,y,z\rangle$, $\vec{u}+\vec{v}=\langle a+x, b+y, c+z\rangle$

Why? The following figure proves this property in 2 dimensions



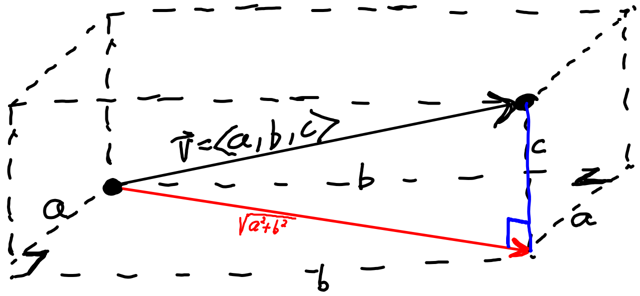
Length

The length of vector $\vec{v}=\langle a,b\rangle$ is $|\vec{v}|=\sqrt{a^2+b^2}$



The length of vector $\vec{v}=\langle a,b,c\rangle$ is $|\vec{v}|=\sqrt{a^2+b^2+c^2}$

This is again the geometric length of the line segment represented by $\vec{v}$



The black/red/blue right triangle has hypotenuse $|\vec{v}|$.
So now by the Pythagorean Theorem

$$|\vec{v}| = \sqrt{\left(\sqrt{a^2+b^2}\right)^2 + c^2}$$
$$= \sqrt{a^2+b^2+c^2}$$

which is the result we want.

example

$$|\langle 1,3,5 \rangle| = \sqrt{1^2+3^2+5^2} = \sqrt{37}$$

$$|\langle 3,4 \rangle| = \sqrt{3^2+4^2} = \sqrt{25} = 5$$

$$|\langle 2,2,1 \rangle| = \sqrt{2^2+2^2+1} = \sqrt{9} = 3$$

example A unit vector is a vector whose length is 1.

$$\left|\left\langle \frac{2}{3}, \frac{-3}{3}, \frac{1}{3} \right\rangle\right| = \sqrt{\frac{4}{9}+\frac{4}{9}+\frac{1}{9}} = \sqrt{\frac{9}{9}} = 1.$$

$$\left|\left\langle \frac{-3}{5}, \frac{4}{5} \right\rangle\right| = \sqrt{\frac{9}{25}+\frac{16}{25}} = \sqrt{\frac{25}{25}} = 1.$$

Scalar multiplication

Given a vector $\vec{v} = \langle a, b \rangle$ or $\vec{v} = \langle a, b, c \rangle$ and a real number k which is called a scalar, The scalar product $k\vec{v}$ is defined as

$$k\vec{v} = k\langle a, b \rangle = \langle ka, kb \rangle \quad \text{or}$$

$$k\vec{v} = k\langle a, b, c \rangle = \langle ka, kb, kc \rangle.$$

Geometrically The vector $k\vec{v}$ is parallel to $\vec{v}$ and has length

$$|k\vec{v}| = |k\langle a, b, c \rangle| = |\langle ka, kb, kc \rangle| =$$

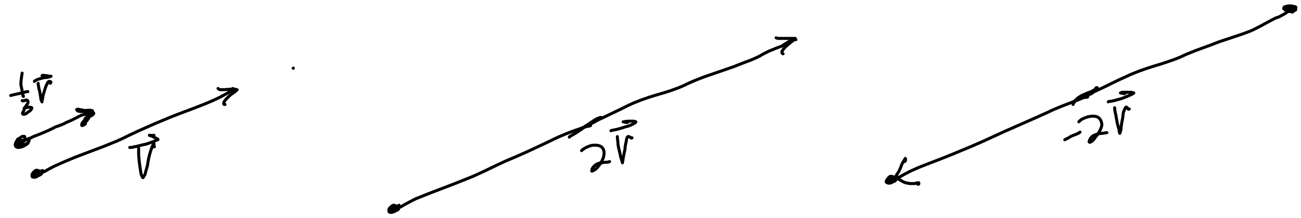
$$\sqrt{(ka)^2 + (kb)^2 + (kc)^2} = \sqrt{k^2a^2 + k^2b^2 + k^2c^2} =$$

$$\sqrt{k^2(a^2 + b^2 + c^2)} = \sqrt{k^2}\sqrt{a^2 + b^2 + c^2}$$

$$= |k|\sqrt{a^2 + b^2 + c^2}$$

$$= |k||\langle a, b, c \rangle| = |k||\vec{v}|$$

With the reverse direction if $k < 0$ same direction if $k > 0$.



Basic algebraic properties

① $\vec{u}+\vec{v}=\vec{v}+\vec{u}$

② $0\vec{v}=\vec{0}$ where $\vec{0}$ is the zero-vector which is $\langle 0,0\rangle$ or $\langle 0,0,0\rangle$ depending on which dimension we are working in.

③ $\vec{u}+(\vec{v}+\vec{w})=(\vec{u}+\vec{v})+\vec{w}$

④ $k(\vec{u}+\vec{v})=k\vec{u}+k\vec{v}$

⑤ $(k+l)\vec{u}=k\vec{u}+l\vec{u}$

⑥ $k(l\vec{u})=(kl)\vec{u}$

Some special unit vectors

In 2D $\hat{\imath}=\langle 1\ 0\rangle$ $\hat{\jmath}=\langle 0\ 1\rangle$

In 3D $\hat{\imath}=\langle 1\ 0\ 0\rangle$ $\hat{\jmath}=\langle 0\ 1\ 0\rangle$ $\hat{k}=\langle 0\ 0\ 1\rangle$

Now any other vector in 2D or 3D can be written using $\hat{\imath},\hat{\jmath},\hat{k}$ along with scalar multiplication and vector addition.

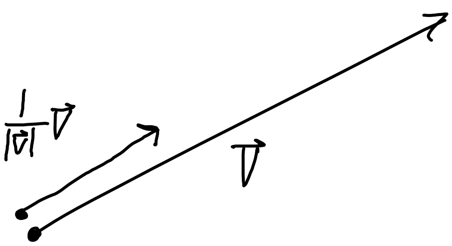
$$\langle a,b\rangle=\langle a,0\rangle+\langle 0,b\rangle=a\langle 1,0\rangle+b\langle 0,1\rangle=a\hat{\imath}+b\hat{\jmath}$$

$$\langle a,b,c\rangle=\langle a\,0\,0\rangle+\langle 0\,b\,0\rangle+\langle 0\,0\,c\rangle=a\langle 1\,0\,0\rangle+b\langle 0\,1\,0\rangle+c\langle 0\,0\,1\rangle$$

$$=a\hat{\imath}+b\hat{\jmath}+c\hat{k}$$

example $\langle 3,0,-2\rangle=3\hat{\imath}-2\hat{k}=-2\hat{k}+3\hat{\imath}$ $\langle 0\,1\,2\rangle=\hat{\jmath}+2\hat{k}$

Question Given a vector $\vec{v} = \langle a, b, c \rangle$ how scalar multiplication to obtain a unit vector which is parallel and in the same direction as $\vec{v}$??



example $\langle 1, 1, 1 \rangle$ which has length $\sqrt{1+1+1} = \sqrt{3}$

has parallel unit vector $\frac{1}{\sqrt{3}}\langle 1, 1, 1 \rangle = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$

also pointing in the same direction

Given two points (a, b) and (x, y) we know that their distance in 2D is $\sqrt{(x-a)^2 + (y-a)^2}$

and

Given two points (a, b, c) and (x, y, z) in 3D we now know their distance apart is

$$\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$$

Now given a fixed point (a,b,c) in 3D

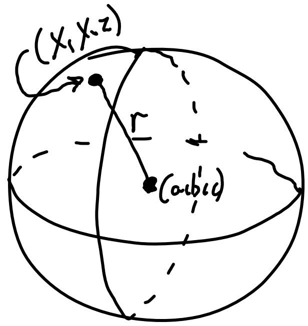
The collection of all points (x,y,z) whose distance from (a,b,c) is a constant r form a sphere of radius r centered at (a,b,c).

Its equation is therefore

$$r = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$$

or

$$r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$$



example

Find the equation of the sphere centered at $(1,-2,2)$ which contains the point $(1,0,1)$ on its surface.

Since $(1,0,1)$ is on the sphere $r^2 = (1-1)^2 + (0-(-2))^2 + (1-2)^2$
$= 0+4+1=5$

The equation of the sphere is

$$\boxed{(x-1)^2 + (y+2)^2 + (z-2)^2 = 5}$$

example $x^2+y^2+z^2-6x-8y+10z+25=0$ represents a sphere in 3D. What is the center and radius ??

$$x^2+y^2+z^2-6x-8y+10z+25=0$$

$$(x^2-6x)+(y^2-8y)+(z^2+10z) = -25$$

$$(x^2-6x+9)+(y^2-8y+16)+(z^2+10z+25) = 9+16+25-25$$

$$(x-3)^2+(y-4)^2+(z+5)^2 = 25$$

center $=(3,4,-5)$ radius $=5$.

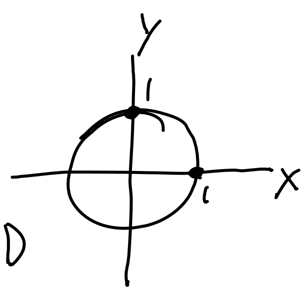
Some other simple surfaces given by the "<u>cylinder construction</u>".

Any equation in 2 variables defines a curve in 2 dimensions. If we look at the same equation in 3 dimensions then gets extended parallel to the axis of the missing variable.

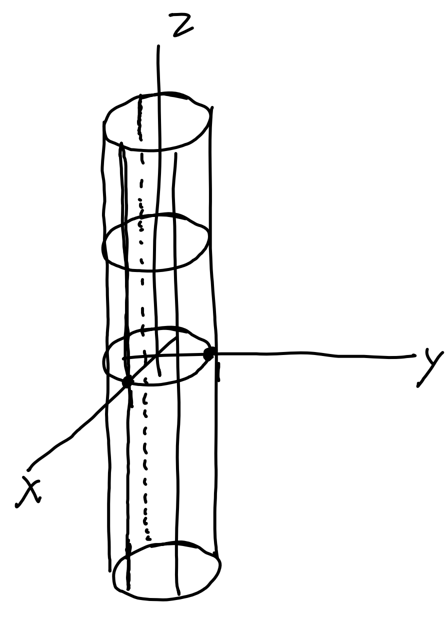
<u>example</u>

$x^2+y^2=1$

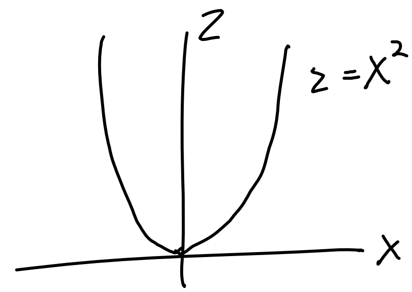
in 2D



$x^2+y^2=1$ in 3D



<u>example</u> $z = x^2$ in 2D



$z = x^2$ in 3D