6.3 Fundamental Theorem of calculus for line integrals.

Recall the ordinary Fundamental Theorem of Calculus.

$$\int_a^b f'(t)\,dt = f(b) - f(a)$$

Consider a gradient vector field $\nabla f(x,y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$

or

$\nabla f(x,y,z) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$

along a continuous curve $\vec{C}$ given by

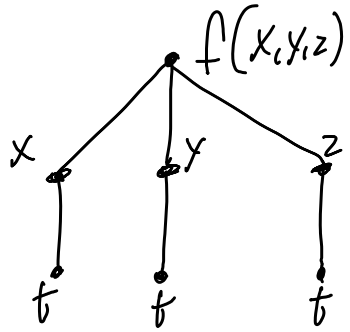
$\vec{r}(t) = \langle x(t), y(t) \rangle$ or $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$.

In this case,

$$\int_C \nabla f \cdot d\vec{r} = \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt = \int_a^b \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle dt$$

$$= \int_a^b \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\,dt = f(\underbrace{\vec{r}(b)}_{\text{endpoint for } \vec{C}}) - f(\underbrace{\vec{r}(a)}_{\text{start point for } \vec{C}})$$

via the chain Rule and ordinary fundamental Theorem of calculus.

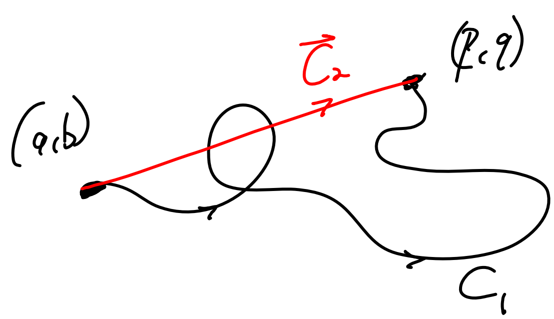


So $\dfrac{df}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial f}{\partial z}\dfrac{dz}{dt}$

__Consequence__

$$\int_{\vec{C}_1} \nabla f \cdot d\vec{r} = \int_{\vec{C}_2} \nabla f \cdot d\vec{r}$$

when $\vec{C}_1$ and $\vec{C}_2$ start and end in the same place.



__example__ Calculate $\displaystyle\int_{\vec{C}} \langle 2xy, x^2+y \rangle \cdot d\vec{r}$ where $\vec{C}$ is any continuous curve starting at $(1,1)$ and ending at $(2,3)$.

We calculated in 6.1 that $\langle 2xy, x^2+y\rangle = \nabla f$

for $f(x,y) = x^2y + \frac{1}{2}y^2$

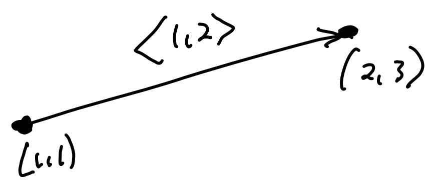
Therefore

$$\int_C \langle 2xy, x^2+y\rangle \cdot d\vec{r} = x^2y + \frac{1}{2}y^2 \Big]_{(1,1)}^{(2,3)}$$

$$= 12 + \frac{9}{2} - \left(1 + \frac{1}{2}\right) = 11 + 4 = 15$$

Let's calculate again

let $\vec{r}(t) = \langle 1+t, 1+2t\rangle$, $0 \le t \le 1$



$$\int_C \langle 2xy, x^2+y\rangle \cdot d\vec{r}$$

$\vec{r}(t) = \langle 1+t, 1+2t\rangle$ $0 \le t \le 1$ $\qquad \vec{F} = \langle 2xy, x^2+y\rangle$

$\vec{r}\,'(t) = \langle 1, 2\rangle$ $\qquad \vec{F}(\vec{r}(t)) = \langle 2(1+t)(1+2t), (1+t)^2 + (1+2t)\rangle$

$$= \langle 4t^2+6t+2, t^2+4t+2\rangle$$

So now

$$\int_C \langle 2xy, x^2+y\rangle \cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt =$$

$$\int_0^1 \langle 4t^2+6t+2, t^2+4t+2\rangle \cdot \langle 1,2\rangle\,dt = \int_0^1 4t^2+6t+2+2(t^2+4t+2)\,dt$$

$$= \int_0^1 6t^2+14t+6\,dt = 2t^3+7t^2+6t\Big]_{t=0}^{t=1} = 2+7+6 = 15$$

Same answer.