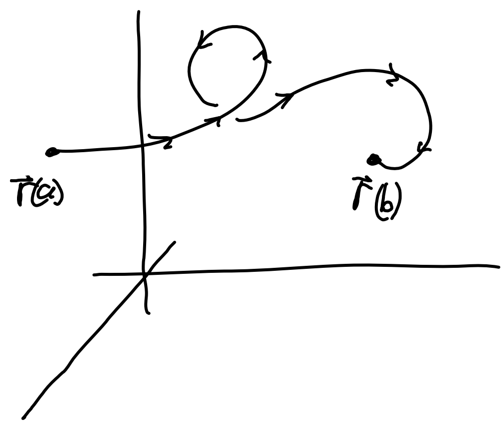
6.2 Two types of line integrals.

(I) Scalar line integral (Density/mass line integral).

Consider a curve $\vec{r}(t) = \langle x(t), y(t) \rangle$ in 2D or

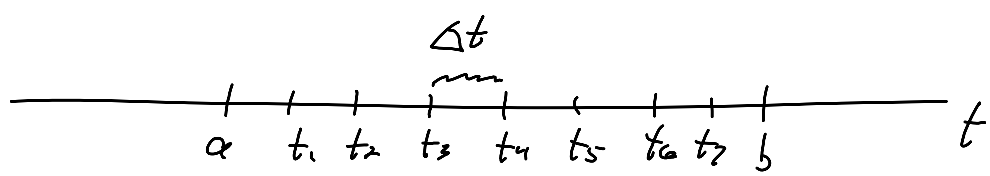
$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ in 3D for $a \le t \le b$.



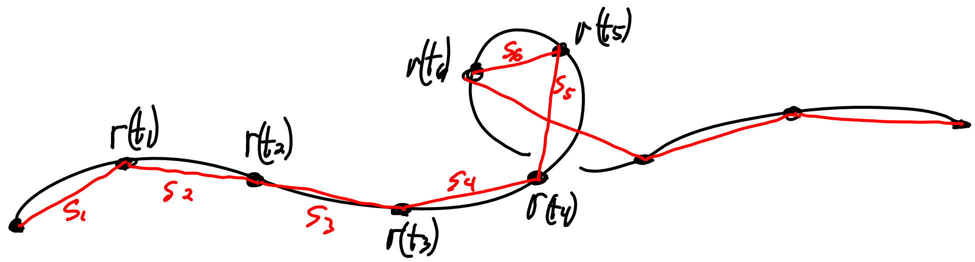
Let $f(x,y)$ (or $f(x,y,z)$) be a function measuring the density of the curve at any point (x,y) in terms of $\frac{\text{units mass}}{\text{unit length}}$.

How do we measure the total mass of the curve??

Take $a \le t \le b$ and subdivide into subintervals of length $\Delta t = \frac{b-a}{n}$.

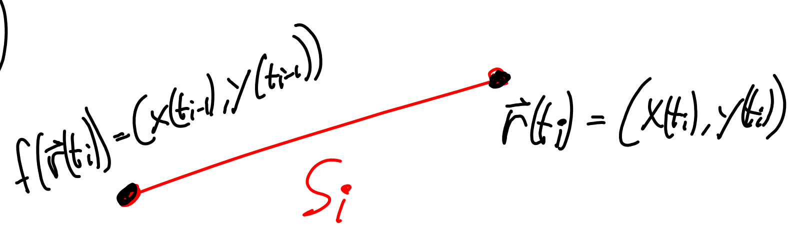


Look at the corresponding points on $\vec{r}(t)$ and connect with straight line segments.



Not that as $n \to \infty$ This approximation of $\vec{r}(t)$ gets better and better.

Now when Δt is really small segment S_i is as well and has approximately constant density $f(\vec{r}(t_i))$



So the mass of segment S_i is approximately

$$f(\vec{r}(t_i))\, \text{length}(S_i) =$$

$$f(\vec{r}(t_i))\sqrt{(\Delta x)^2 + (\Delta y)^2} =$$

So the mass of the entire curve is approximately

$$\sum_{i=1}^{n} f(\vec{r}(t_i))\sqrt{(\Delta x)^2 + (\Delta y)^2}$$

So the exact mass of the curve is

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(\vec{r}(t_i))\sqrt{(\Delta x)^2 + (\Delta y)^2} =$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(\vec{r}(t_i))\sqrt{(\Delta x)^2 + (\Delta y)^2}\ \frac{\Delta t}{\Delta t} =$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(\vec{r}(t_i))\sqrt{\frac{(\Delta x)^2}{(\Delta t)^2} + \frac{(\Delta y)^2}{(\Delta t)^2}}\ \Delta t =$$

$$\lim_{\substack{n\to\infty \\ \Delta t\to 0}} \sum_{i=1}^{n} f(\vec{r}(t_i))\sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}\ \Delta t =$$

$$\int_a^b f(\vec{r}(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt =$$

$$\boxed{\int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt}$$

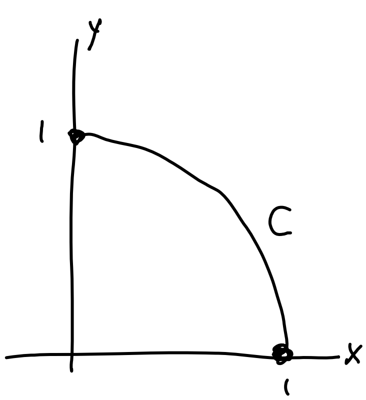
If we wish to refer to a scalar line integral over a curve C without actually specifying $\vec{r}(t)$ (because there are different possibilities for $\vec{r}(t)$ given any curve C)

we just write $\int_C f(x,y)\,ds$ or $\int_C f(x,y,z)\,ds$

Special consequence If $f(x,y)=1$, then mass(C) = length (C)

So length (C) $= \int_a^b |\vec{r}\,'(t)|\,dt$.

example Calculate $\int_C y\,ds$ where C is the quarter of the circle of radius 1 in quadrant 1.



Two ways to pick $\vec{r}(t)$

① $\vec{r}(t) = \langle \cos(t), \sin(t) \rangle \quad 0 \le t \le \frac{\pi}{2}$

② $\vec{r}(t) = \langle t, \sqrt{1-t^2} \rangle \quad 0 \le t \le 1$

Calculation ①

$\vec{r}(t) = \langle \cos(t), \sin(t) \rangle \quad 0 \le t \le \frac{\pi}{2}$

$f(x,y) = y$

$\vec{r}\,'(t) = \langle -\sin(t), \cos(t) \rangle$

$f(\vec{r}(t)) = \sin(t)$

$|\vec{r}\,'(t)| = \sqrt{\sin^2(t) + \cos^2(t)} = 1$

So

$$\int_C y\, ds = \int_0^{\frac{\pi}{2}} f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt = \int_0^{\frac{\pi}{2}} \sin(t)\, dt = -\cos(t)\Big]_0^{\frac{\pi}{2}} = 0 + 1 = 1$$

Calculation ②

$\vec{r}(t) = \langle t, \sqrt{1-t^2} \rangle, \quad 0 \le t \le 1$

$f(x,y) = y$

$\vec{r}\,'(t) = \left\langle 1, \frac{-t}{\sqrt{1-t^2}} \right\rangle$

$f(\vec{r}(t)) = \sqrt{1-t^2}$

$$|\vec{r}\,'(t)| = \sqrt{1^2 + \left(\frac{-t}{\sqrt{1-t^2}}\right)^2}$$

$$= \sqrt{1 + \frac{t^2}{1-t^2}}$$

$$= \sqrt{\frac{1-t^2}{1-t^2} + \frac{t^2}{1-t^2}}$$

$$= \frac{1}{\sqrt{1-t^2}}$$

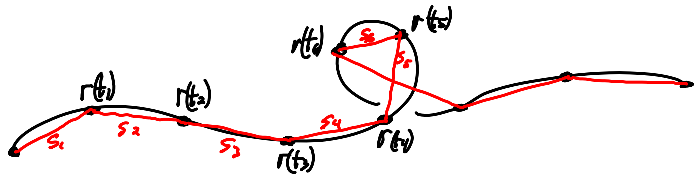
So now

$$\int_C y\, ds = \int_0^1 f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt$$

$$= \int_0^1 \cancel{\sqrt{1-t^2}}\, \frac{1}{\cancel{\sqrt{1-t^2}}}\, dt = \int_0^1 dt = 1$$

Final note about $\int_C f(x,y)\,ds$.

Calculating $\int_C f(x,y)\,ds$ we approximated C by using line segments

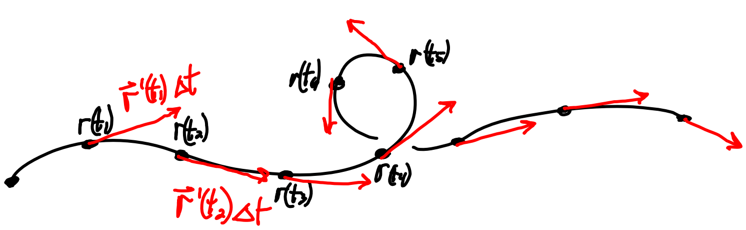


Doing this we obtained

$$\int_C f(x,y)\,ds = \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt = \lim_{n\to\infty} \sum_{i=1}^{n} f(\vec{r}(t_i))\,\underbrace{|\vec{r}\,'(t_i)|\Delta t}_{\text{tangent vectors to } \vec{r}(t).}$$

going back to a different Riemann Sum

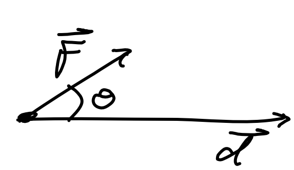
So C can also be approximated by tangent vectors multiplied by Δt.



Not as intuitively clear as the segments, but Riemann-Sum/Integral tells us it works in approximating C.

(II) Vector line integral (Force/work line integral)

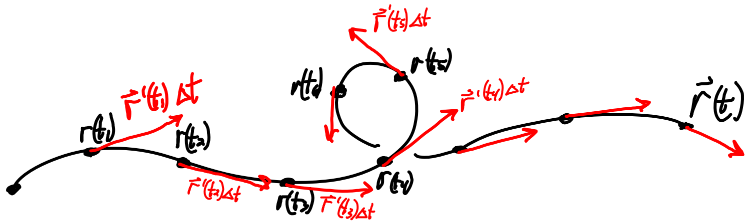
Given a curve $\vec{C}$ represented by $\vec{r}(t) = \langle x(t), y(t) \rangle$ (or $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ in 3D) for $a \le t \le b$ we now consider t as a time parameter and $\vec{r}(t)$ as the path of a particle through time. Now given a vector field $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$ (or $\vec{F}(x,y,z) = \langle P(xyz), Q(xyz), R(xyz) \rangle$ in 3D) which represents a field of force vectors acting on that moving particle, we now ask "what is the total amount of work done by the vector field $\vec{F}$ on the moving particle given by $\vec{r}(t)$??"

Remember, for constant vectors 

$\text{work} = \vec{F} \cdot \vec{d}$

$= |F||d| \cos(\theta)$

We'll approximate the curve $\vec{C}$ by tangent vectors



Now the work done by $\vec{F}$ along the curve $\vec{r}(t)$ is approximately

$$\sum_{i=1}^{n} \vec{F}(\vec{r}(t_i)) \cdot \vec{r}\,'(t)\Delta t$$

So the exact amount for total work is

$$\lim_{n\to\infty} \sum_{i=1}^{n} \vec{F}(\vec{r}(t_i)) \cdot \vec{r}\,'(t)\Delta t = \boxed{\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt}$$

by definition of integrals

So work done by $\vec{F}$ along $\vec{r}(t)$ $= \displaystyle\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$

which we write as $\displaystyle\int_C \vec{F}\cdot d\vec{r}$ if we don't want to mention $\vec{r}$ specifically.

Another notational shorthand

Given $\vec{F} = \langle P, Q \rangle$ and $\vec{r}(t) = \langle x(t), y(t) \rangle$

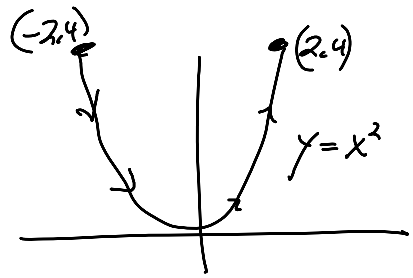
$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt = \int_a^b \langle P, Q \rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle dt = \int_a^b P \frac{dx}{dt} dt + Q \frac{dy}{dt} dt$$

which is abbreviated $\displaystyle\int_C P\,dx + Q\,dy.$

<u>example</u>

Say $\vec{F}(x,y) = \langle -y, x \rangle$

and $\vec{C}$ is the parabola



Let's pick $\vec{r}(t) = \langle t, t^2 \rangle$ for $-2 \le t \le 2$

Let's calculate $\displaystyle\int_C \vec{F} \cdot d\vec{r}$

$\vec{r}(t) = \langle t, t^2 \rangle \qquad \vec{F}(x,y) = \langle -y, x \rangle$

$\vec{r}\,'(t) = \langle 1, 2t \rangle \qquad \vec{F}(\vec{r}(t)) = \langle -t^2, t \rangle$

So

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt = \int_{-2}^{2} \langle -t^2, t\rangle \cdot \langle 1, 2t\rangle\, dt$$

$$= \int_{-2}^{2} -t^2+2t^2\,dt = \int_{-2}^{2} t^2\,dt = \frac{1}{3}t^3\Big]_{t=-2}^{t=2} = \frac{1}{3}(8-(-8)) = \frac{16}{3}$$