6.1 Vector Fields

A 2-dimensional vector field is a 2-variable vector function

$$\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$$

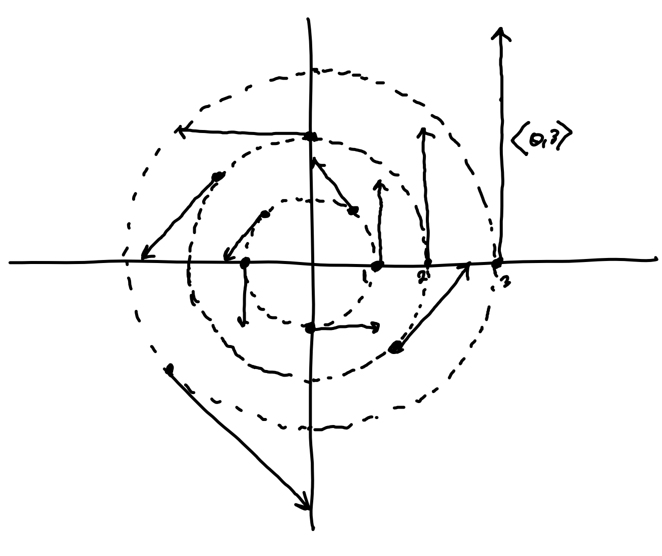
A 3-dimensional vector field is a 2-variable vector function

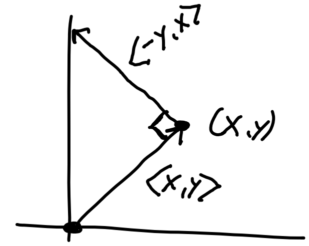
$$\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$$

A vector field associates to each point (x,y) in 2D (or (x,y,z) in 3D) a vector $\vec{F}(x,y)$ (or $\vec{F}(x,y,z)$).

Typically we consider $\vec{F}$ to be a force or flow value at that point.

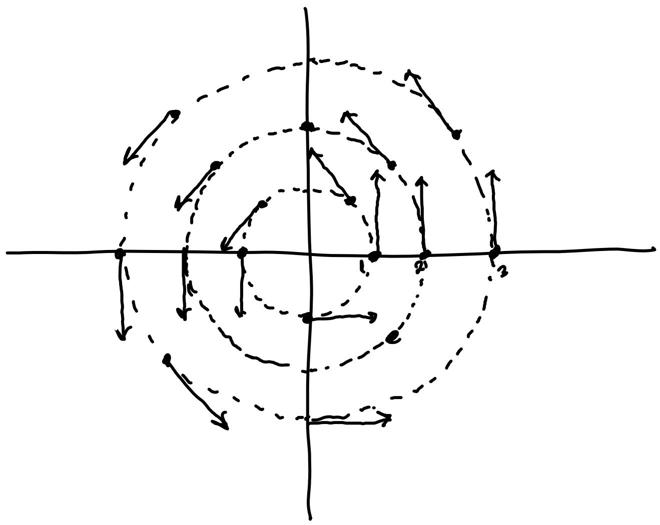
example $\vec{F}(x,y) = \langle -y, x \rangle$ looks like the following.





example $\vec{F}(x,y) = \left\langle \frac{-y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}} \right\rangle = \frac{1}{\sqrt{x^2+y^2}} \langle -y, x \rangle = \frac{1}{|\langle -y, x\rangle|} \langle -y, x \rangle$

So this vector field looks like the previous example but each vector now has length 1.



Def: Given a 2-variable function $f(x,y)$ or 3-variable function $f(x,y,z)$,

The gradient field (or conservative field) associated with $f(x,y)$ or $f(x,y,z)$ is

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle \quad \text{or}$$

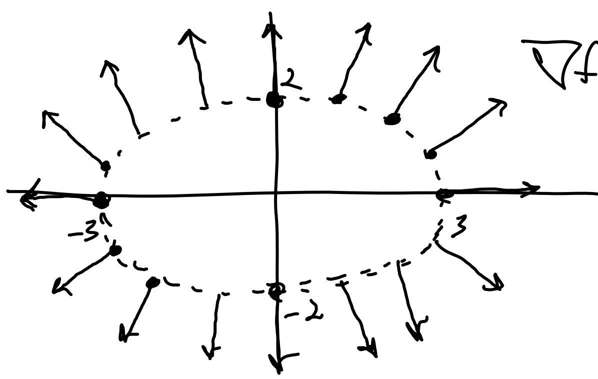
$$\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z) \rangle$$

In section 2.5 we showed that $\nabla f(x,y)$ is perpendicular to the level curves of $f(x,y)$.

Also $\nabla f(x,y,z)$ is perpendicular to the level surfaces of $f(x,y,z)$.

This gives use another way of helping to visualize <u>gradient vector fields</u>

<u>example</u> Given $f(x,y) = \frac{x^2}{9} + \frac{y^2}{4}$ The level curves are ellipses. $\frac{x^2}{9} + \frac{y^2}{4} = 1$ is the ellipse with x-radius 3, y-radius 2.



$\nabla f(x,y) = \langle \frac{2}{9}x, \frac{1}{2}y \rangle$

<u>Remember Clairaut's Theorem</u>

Given $f(x,y)$ with continuous 1st partial derivatives,

$$f_{xy} = f_{yx}$$

Given $f(x,y)$ with continuous 1^{st} partial derivatives,

$f_{xy} = f_{yx}$, $f_{xz} = f_{zx}$, and $f_{yz} = f_{zy}$

So when considering a vector field

$\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$

and we ask, is there $f(x,y)$ such that

$\nabla f(x,y) = \vec{F}(x,y)$; that is $P(x,y) = f_x(x,y)$ and $Q(x,y) = f_y(x,y)$??

Clairaut's Theorem says that,

① If $P_y \neq Q_x$, then no $\vec{F}(x,y) \neq \nabla f(x,y)$ at all.

② If $P_y = Q_x$, then maybe $\vec{F}(x,y) = \nabla f(x,y)$ for some $f(x,y)$.

<u>example</u> Consider $\vec{F}(x,y) = \langle -y, x \rangle$

Here $P = -y$ and $Q = x$

So $P_y = -1$ and $Q_x = 1$

So $\langle -y, x \rangle$ is <u>not</u> conservative, that is, <u>not</u> a gradient field.

Example $\vec{F}(x,y) = \langle 2xy, x^2+y \rangle$

Here $P = 2xy$ and $Q = x^2+y$

so $P_y = 2x$ and $Q_x = 2x$

So now we search for $f(x,y)$ such that $\nabla f = \langle 2xy, x^2+y \rangle$

such a function $f(x,y)$ is called a potential function for $\vec{F}(x,y) = \langle 2xy, x^2+y \rangle$.

If such a function $f(x,y)$ exists, then

$$f_x = P \quad \text{and} \quad f_y = Q$$

$$f_x = 2xy \quad \text{and} \quad f_y = x^2+y$$

$$f(x,y) = x^2y + p(y) \qquad f(x,y) = x^2y + \tfrac{1}{2}y^2 + q(x)$$

letting $p(y) = \frac{1}{2}y^2$ and $q(x) = 0$ we get

$$\boxed{f(x,y) = x^2y + \tfrac{1}{2}y^2}$$

which satisfies $\nabla f = \langle 2xy, x^2+y \rangle$.