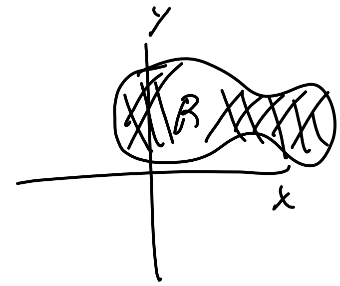
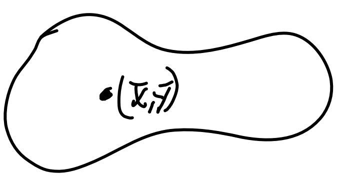
# 5.6 Centers of mass

Consider a region $R$ in the $xy$-plane and a function $f(x,y)$ on $R$. If we think of $f(x,y)$ as a density function in $\frac{\text{units mass}}{\text{unit area}}$. Then




$$\text{mass}(R) = \iint_R f(x,y)\,dA$$

what is the center of mass $(\bar{x},\bar{y})$ ??





balancing point.

We will show that

$$\bar{x} = \frac{1}{\text{mass}} \iint_R x\,f(x,y)\,dA$$

$$\bar{y} = \frac{1}{\text{mass}} \iint_R y\,f(x,y)\,dA.$$

For next time do 299 and 300 in the book for discussion.

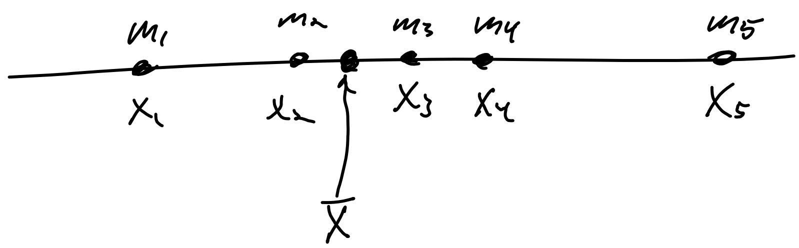
## Recall Archimedes' Lever law

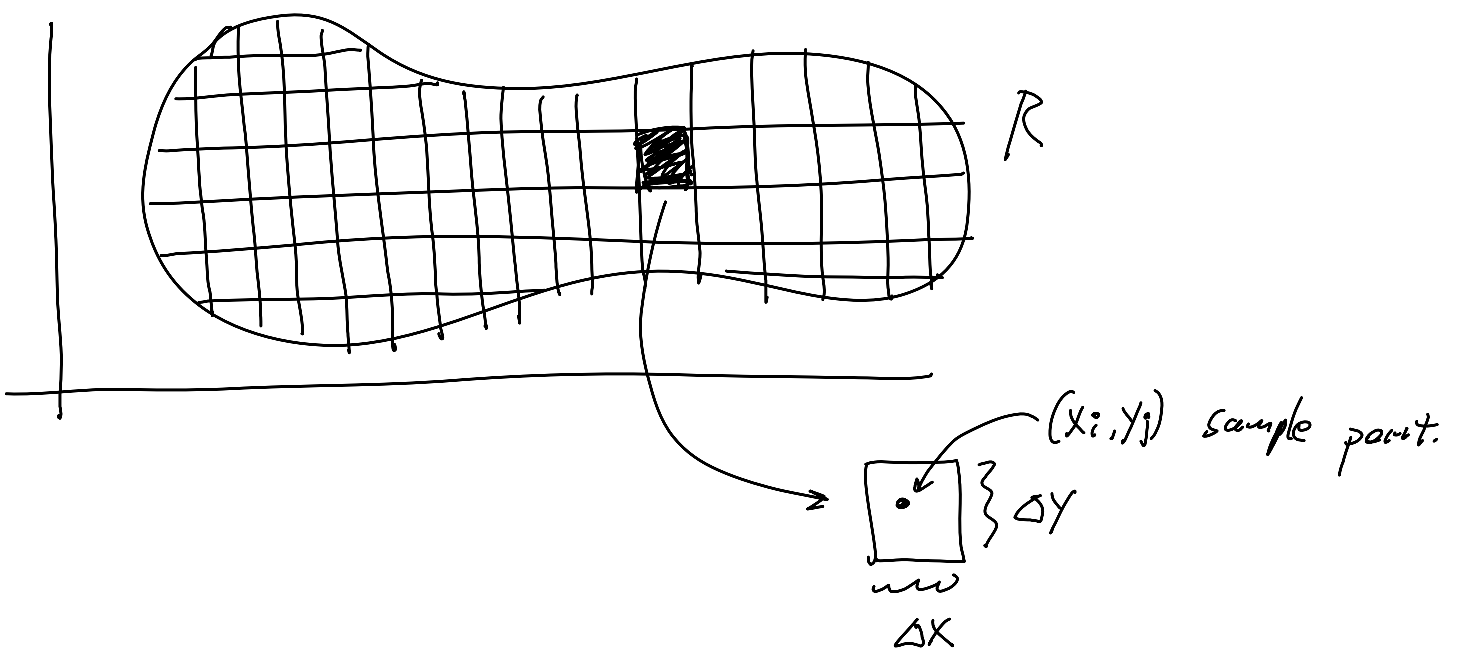
Given point masses $m_1, m_2, \ldots, m_n$

along a rod of negligible mass at coordinates $x_1, x_2, \ldots, x_n$

The center of mass is

$$\bar{x} = \frac{\text{total moments}}{\text{total mass}} = \frac{m_1 x_1 + m_2 x_2 + \cdots + m_n x_n}{m_1 + m_2 + \cdots + m_n}$$




Assuming that $f(x,y)$ is continuous and so "almost" constant

$f(x_i, y_j)$ $\frac{\text{units mass}}{\text{unit area}}$ on the rectangle of area $\Delta A = \Delta x \, \Delta y$.

The mass of the little rectangle is approximated by a point mass $f(x_i, y_j)\,\Delta A$ units mass at coordinates $(x_i, y_j)$.

So now to approximate $\bar{x}$ and $\bar{y}$ we have by Archimedes' lever law that

$$\bar{x} \approx \frac{\text{total } x\text{-moments}}{\text{total mass}} = \frac{\sum_i \sum_j \overbrace{x_i}^{x\text{-coordinate}} \overbrace{f(x_i, y_j)\,\Delta A}^{\text{mass}}}{\sum_i \sum_j f(x_i y_j)\,\Delta A}$$

and

$$\bar{y} \approx \frac{\text{total } y\text{-moments}}{\text{total mass}} = \frac{\sum_i \sum_j y_i f(x_i, y_j)\,\Delta A}{\sum_i \sum_j f(x_i y_j)\,\Delta A}$$

The approximation gets more accurate as $\Delta x, \Delta y \to 0$.

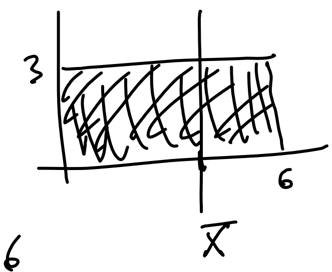
Therefore

$$\bar{x} = \lim_{\Delta x \to 0} \frac{\sum_i \sum_j x_i f(x_i, y_j)\,\Delta A}{\sum_i \sum_j f(x_i y_j)\,\Delta A} = \frac{\iint_R x f(x,y)\,dA}{\underbrace{\iint_R f(x,y)\,dA}_{\text{mass of } R.}}$$

$$= \frac{1}{\text{mass}(R)} \iint_R x f(x,y)\,dA.$$

Similarly $\quad \bar{y} = \frac{1}{mass(R)} \iint_R y f(x,y)\,dA.$

example (299) $\quad R = \boxed{\begin{matrix} 0 \le x \le 6 \\ 0 \le y \le 3 \end{matrix}} \quad$ density $\; f(x,y) = \sqrt{xy}$




$$\text{mass} = \iint_R \sqrt{xy}\,dA = \int_0^6\int_0^3 x^{\frac12} y^{\frac12}\,dy\,dx = \int_0^6 \frac{2}{3} y^{\frac32} x^{\frac12}\Big]_0^3 dx = \int_0^6 \frac{2}{3}\sqrt{27}\, x^{\frac12}\,dx$$

$$= \frac{4\sqrt3}{3} x^{\frac32}\Big]_0^6 = \frac{4\sqrt3\,\cancel{6}^{\,2}\sqrt6}{\cancel{3}} = 8\sqrt{18} = 24\sqrt2$$

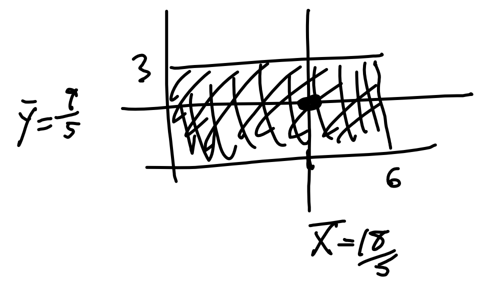
Let's also calculate the center of mass $(\bar{x}, \bar{y})$

$$\bar{x} = \frac{1}{mass}\iint_R x f(x,y)\,dA = \frac{1}{24\sqrt2}\int_0^6\int_0^3 x^{\frac32} y^{\frac12}\,dy\,dx = \frac{1}{24\sqrt2}\int_0^6 \frac23 y^{\frac32} x^{\frac32}\Big]_{y=0}^{y=3} dx$$

$$= \frac{2\sqrt3}{24\sqrt2}\int_0^6 x^{\frac32}\,dx = \frac{\sqrt3}{12\sqrt2}\,\frac25 x^{\frac52}\Big]_0^6 = \frac{\sqrt3}{12\sqrt{\cancel{2}}}\,\frac25\, 36\sqrt{\cancel{6}}^{\,\sqrt3} = \frac{216}{60} = \frac{108}{30} = \frac{18}{5}$$

$$\bar{y} = \frac{1}{mass}\iint_R y f(x,y)\,dA = \frac{1}{24\sqrt{2}}\int_0^6\int_0^3 x^{\frac{1}{2}}y^{\frac{3}{2}}\,dy\,dx = \frac{1}{24\sqrt{2}}\,\frac{2}{5}\int_0^6 \left[x^{\frac{1}{2}}y^{\frac{5}{2}}\right]_0^3 dx$$

$$= \frac{1}{\cancel{24}_{12}\sqrt{2}}\,\frac{\cancel{2}}{5}\,9\sqrt{3}\int_0^6 x^{\frac{1}{2}}\,dx = \frac{9\sqrt{3}}{60\sqrt{2}}\,\frac{2}{3}x^{\frac{3}{2}}\Big]_0^6 = \frac{\cancel{9}^{3}\cancel{\sqrt{3}}\,\cancel{6}\cancel{\sqrt{6}}}{\cancel{60}_{\cancel{30}\,\cancel{15}\,5}\cancel{\sqrt{2}}}\cdot\frac{\cancel{2}}{\cancel{3}} = \frac{9}{5}$$




Do #300 calculate mass and center of mass $(\bar{x},\bar{y})$.

$$R = \boxed{\begin{matrix} 0 \le x \le 3 \\ 1 \le y \le 3 \end{matrix}} \qquad \text{density } f(x,y) = x^2 y$$

$$mass = \iint_R f(x,y)\,dA = \int_0^3\int_1^3 x^2 y\,dy\,dx = \int_0^3 \left[\tfrac{1}{2}x^2y^2\right]_{y=1}^{y=3} dx = \int_0^3 4x^2\,dx$$

$$= \frac{4}{3}x^3\Big]_0^3 = 4\cdot 9 = 36$$

$$\bar{x} = \frac{1}{mass}\iint_R x f(x,y)\,dA = \frac{1}{36}\int_0^3\int_1^3 x^3 y\,dy\,dx = \frac{1}{36}\int_0^3 \left[\tfrac{1}{2}x^3y^2\right]_{y=1}^{y=3} dx =$$

$$\frac{4}{36}\int_0^3 x^3\,dx = \frac{1}{36}x^4\Big]_0^3 = \frac{81}{36} = \frac{9}{4} = 2.25$$

$$\bar{y} = \frac{1}{mass}\iint_R y f(x,y)\,dA = \frac{1}{36}\int_0^3\int_1^3 x^2y^2\,dy\,dx = \frac{1}{36}\frac{1}{3}\int_0^3 x^2y^3\Big]_{y=1}^{y=3} dx$$

$$= \frac{26}{108}\int_0^3 x^2\,dx = \frac{13}{54}\frac{1}{3}x^3\Big]_0^3 = \frac{13}{54}9 = \frac{13}{6} \approx 2.1616\ldots$$