E5 Triple Integrals in Cyclindrical and spherical coordinates

Cyclindrical coordinates

Given a 3-D region described as follows

$$R = \boxed{\begin{array}{c} a \le r \le b \\ \alpha \le \theta \le \beta \\ p(x,y) \le z \le q(x,y) \end{array}}$$

We can transform $\iiint\limits_R f(x,y,z)\,dV$ into an integral in cylindrical coordinates just by calculating dz in the first iteration and then transforming the remaining double integral into polar coordinates.

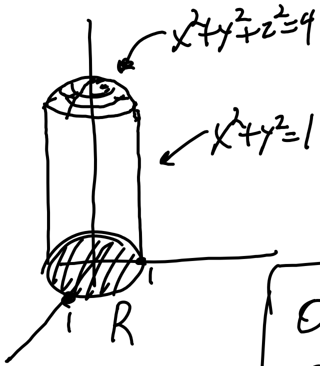
Thus

$$\iiint f(x,y,z)\,dV = \int_\alpha^\beta \int_a^b \int_{p(x,y)}^{q(x,y)} f(x,y,z)\,dz\,r\,dr\,d\theta$$

Calculate z in rectangular coordinates then the remaining x's and y's get transformed to polar coordinates.

<u>example</u> Take the part of the sphere $x^2+y^2+z^2=4$ inside the cylinder $x^2+y^2=1$ and above the xy-plane.

This is R. Calculate $\iiint_R z\,dV$



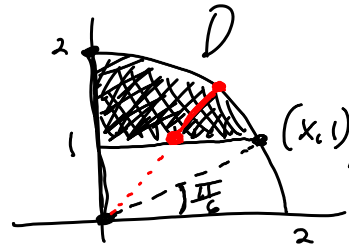
$$\boxed{\begin{array}{l} 0 \le r \le 1 \\ 0 \le \theta \le 2\pi \\ 0 \le z \le \sqrt{4-x^2-y^2} \end{array}}$$

$$\iiint_R z\,dV = \int_0^{2\pi}\int_0^1\int_0^{\sqrt{4-x^2-y^2}} z\,dz\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 \frac{1}{2}z^2\Big]_{z=0}^{z=\sqrt{4-x^2-y^2}} r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 \frac{1}{2}(4-(x^2+y^2))\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 \frac{1}{2}(4-r^2)\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 2r-\frac{1}{2}r^3\,dr\,d\theta$$

$$= \int_0^{2\pi} r^2-\frac{1}{8}r^4\Big]_{r=0}^{r=1} d\theta = \int_0^{2\pi} 1-\frac{1}{8}\,d\theta = \int_0^{2\pi} \frac{7}{8}\,d\theta = \boxed{\frac{7\pi}{4}}$$

(210) $\iint_D \left(\int_0^{4x^2+4y^2} y\, dz \right) dA$

$D =$ 

$x^2+1^2=4$
$x^2=3$
$x=\sqrt{3}$

$(\sqrt{3},1)$ on the circle of radius $=1$ This corresponds to $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

$\frac{\pi}{6} \le \theta \le \frac{\pi}{2}$

The line $y=1$ in terms of r and θ is $r\sin(\theta)=1$

$$r = \frac{1}{\sin(\theta)} = \csc(\theta)$$

$$D = \boxed{\begin{array}{c} \frac{\pi}{6} \le \theta \le \frac{\pi}{2} \\ \csc(\theta) \le r \le 2 \end{array}}$$

$$\iint_D \left(\int_0^{4x^2+4y^2} y\, dz \right) dA = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \int_{\csc(\theta)}^{2} \int_0^{4x^2+4y^2} y\, dz\, r\, dr\, d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \int_{\csc(\theta)}^{2} y(4x^2+4y^2-0)\, r\, dr\, d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \int_{\csc(\theta)}^{2} r\sin(\theta)\, 4r^2 r\, dr\, d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \int_{\csc(\theta)}^{2} 4r^4 \sin(\theta)\, dr\, d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left[\frac{4}{5} r^5 \sin(\theta) \right]_{r=\csc(\theta)}^{r=2} d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{128}{5}\sin(\theta) - \frac{4}{5}\csc^{\cancel{5}\,4}(\theta)\cancel{\sin(\theta)}\, d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{128}{5}\sin(\theta) - \frac{4}{5}\csc^4(\theta)\, d\theta$$

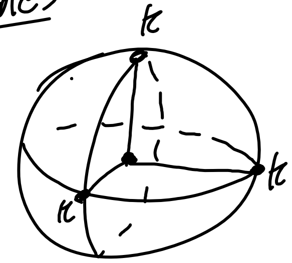
$$\left.-\frac{128}{5}\cos(\theta) + \frac{4}{15}\cot(\theta)\left(\csc^2(\theta)+2\right)\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} = 0 + 0 - \left(-\frac{128}{5}\frac{\sqrt{3}}{2} + \sqrt{3}(6)\right)$$

$$= -\left(-\frac{64\sqrt{3}}{5} + 6\sqrt{3}\right) = \boxed{\frac{34\sqrt{3}}{5}}$$

Spherical Coordinates

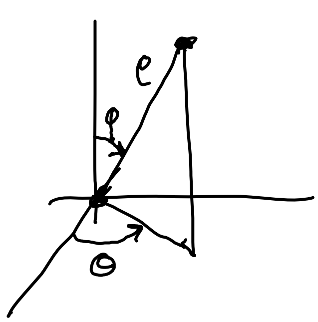
Some regions are naturally described w/ spherical coordinates

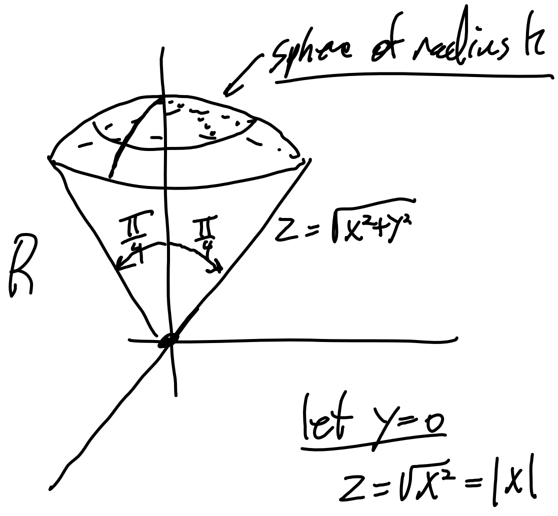
examples

① 

sphere of radius R.

$$\boxed{\begin{array}{l} 0 \le \rho \le R \\ 0 \le \theta \le 2\pi \\ 0 \le \varphi \le \pi \end{array}}$$

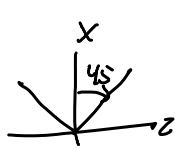


② 

$$\boxed{\begin{array}{l} 0 \le \rho \le R \\ 0 \le \theta \le 2\pi \\ 0 \le \varphi \le \frac{\pi}{4} \end{array}}$$

let $y=0$

$z = \sqrt{x^2} = |x|$



Given
$$R = \boxed{\begin{array}{l} a \le \rho \le b \\ \alpha \le \theta \le \beta \\ p \le \varphi \le q \end{array}}$$

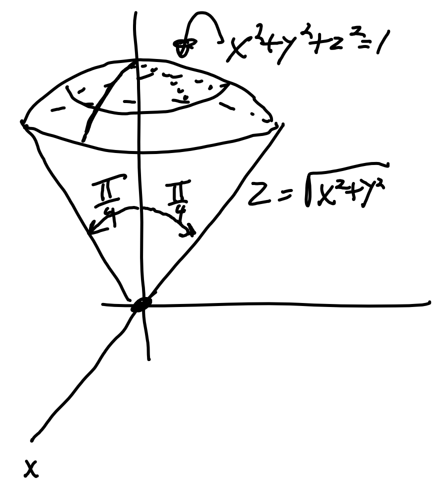
$$\boxed{\begin{array}{l} x = \rho\cos(\theta)\sin(\varphi) \\ y = \rho\sin(\theta)\sin(\varphi) \\ z = \rho\cos(\varphi) \end{array}} \quad \boxed{x^2+y^2+z^2=\rho^2}$$

$$\iiint_R f(x,y,z)\,dV = \int_a^b\int_\alpha^\beta\int_p^q f(\rho\cos(\theta)\sin(\varphi), \rho\sin(\theta)\sin(\varphi), \rho\cos(\varphi))\,\rho^2\sin(\varphi)\,d\varphi\,d\theta\,d\rho$$

Correction factor for converting from xyz to $\rho\theta\varphi$

example

Calculate the volume of $R = \boxed{\begin{array}{l} 0 \le \rho \le 1 \\ 0 \le \theta \le 2\pi \\ 0 \le \varphi \le \frac{\pi}{4} \end{array}}$



$$\text{Volume}(R) = \iiint_R 1\,dV = \int_0^1\int_0^{2\pi}\int_0^{\frac{\pi}{4}} 1\,\rho^2\sin(\varphi)\,d\varphi\,d\theta\,d\rho$$

$$= \int_0^1\int_0^{2\pi} \left[-\rho^2\cos(\varphi)\right]_{\varphi=0}^{\varphi=\frac{\pi}{4}} d\theta\,d\rho = \int_0^1\int_0^{2\pi} -\rho^2\left(\frac{\sqrt{2}}{2}-1\right)d\theta\,d\rho = \frac{2-\sqrt{2}}{2}\int_0^1\int_0^{2\pi}\rho^2\,d\theta\,d\rho$$

$$= \pi(2-\sqrt{2})\int_0^1 \rho^2\,d\rho = \pi(2-\sqrt{2})\,\frac{1}{3}\rho^3\Big]_{\rho=0}^{\rho=1} = \frac{\pi(2-\sqrt{2})}{3}$$