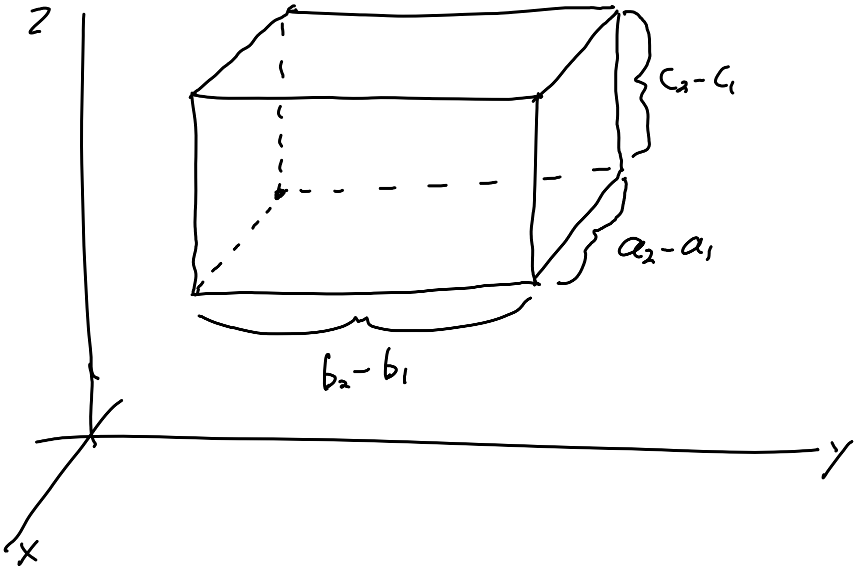
5.5 Triple integrals

Consider a rectangular box in 3D

$$\boxed{\begin{array}{c} a_1 \le x \le a_2 \\ b_1 \le y \le b_2 \\ c_1 \le z \le c_2 \end{array}} = R$$



Consider a 3-variable "density" function $f(x,y,z)$ on R in units mass/unit volume.

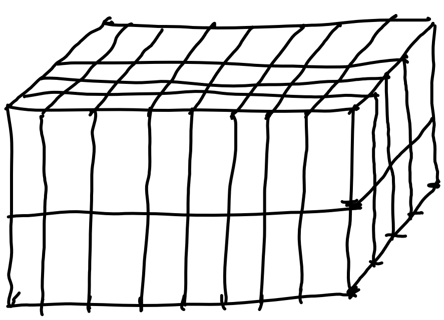
What is the total mass of the box??

We estimate the mass by using a Riemann Sum.

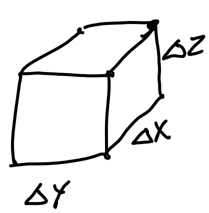
Subdivide x, y, z into subintervals of width $\Delta x = \frac{a_2 - a_1}{\ell}$

$$\Delta y = \frac{b_2 - b_1}{m}$$

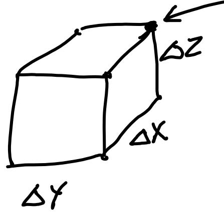
$$\Delta z = \frac{c_2 - c_1}{n}$$



each little box



has volume $\Delta V = \Delta x \Delta y \Delta z$.



let (x_i, y_j, z_k) be any point in this little box

an as long as the density function $f(x,y,z)$ is continuous we can assume the little box has approximately the constant density $f(x_i, y_j, z_k)$

Therefore the mass of the little box is approximately

$$f(x_i, y_j, z_k)\,\Delta V$$

constant density $\frac{\text{units mass}}{\text{unit volume}}$

volume $\Delta V = \Delta x \Delta y \Delta z$

So the total mass of the box is approximately.

$$\sum_i \sum_j \sum_k f(x_i, y_j, z_k)\,\Delta V$$

As $\Delta x, \Delta y, \Delta z \to 0$ the approximation continually improves.
(i.e., $l, m, n \to \infty$)

So the actual mass of the box is

$$\lim_{l,m,n\to\infty} \sum_i \sum_j \sum_k f(x_i, y_j z_k)\Delta V$$

we call this limit of a Riemann Sum the <u>triple integral</u>.

$$\lim_{l,m,n\to\infty} \sum_i \sum_j \sum_k f(x_i, y_j z_k)\Delta V = \iiint_R f(x,y,z)\,dV$$

Calculating this integral is again done using iteration where there are six choices for the ordering

$dx\,dy\,dz$
$dx\,dz\,dy$
$dy\,dx\,dz$
$dy\,dz\,dx$
$dz\,dx\,dy$
$dz\,dy\,dx$.

<u>example</u> Calculate $\iiint_R x^2+yz\,dV$ for $R = \boxed{\begin{matrix} -1\le x\le 1 \\ 0\le y\le 2 \\ 1\le z\le 2\end{matrix}}$ in 2 different ways.

$$\int_1^2\int_0^2\int_{-1}^1 x^2+yz\,dx\,dy\,dz = \int_1^2\int_0^2 \left.\frac{1}{3}x^3+yzx\right]_{x=-1}^{x=1} dy\,dz = \int_1^2\int_0^2 \frac{2}{3}+2yz\,dy\,dz$$

$$= \int_1^2 \left[\frac{2}{3}y + y^2 z \right]_{y=0}^{y=2} dz = \int_1^2 \frac{4}{3} + 4z \, dz = \left. \frac{4}{3}z + 2z^2 \right]_{z=1}^{z=2} = \frac{8}{3} + 8 - \left(\frac{4}{3} + 2\right)$$

$$= \frac{32}{3} - \frac{10}{3} = \frac{22}{3}$$

$$\int_{-1}^{1}\int_{1}^{2}\int_{0}^{2} x^2 + yz \, dy \, dz \, dx = \int_{-1}^{1}\int_{1}^{2} \left. x^2 y + \frac{1}{2}y^2 z \right]_{y=0}^{y=2} dz \, dx = \int_{-1}^{1}\int_{1}^{2} 2x^2 + 2z \, dz \, dx$$

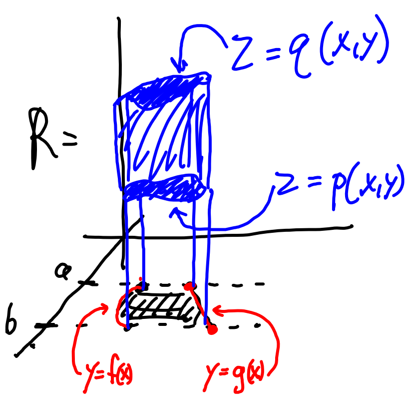
$$= \int_{-1}^{1} \left. 2x^2 z + z^2 \right]_{z=1}^{z=2} dx = \int_{-1}^{1} 4x^2 + 4 - (2x^2 + 1) \, dx = \int_{-1}^{1} 2x^2 + 3 \, dx$$

$$= \left. \frac{2}{3}x^3 + 3x \right]_{x=-1}^{x=1} = \frac{2}{3} + 3 - \left(-\frac{2}{3} - 3\right) = \left(\frac{2+9}{3}\right) 2 = \frac{22}{3}$$

Same answer

More General 3D regions

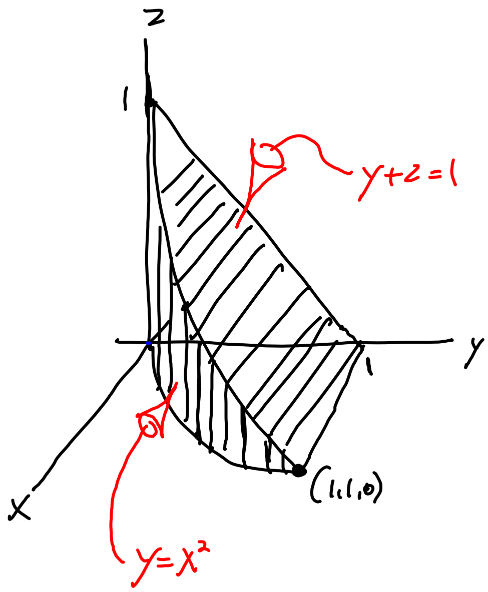
$R =$



Such a region has the following description

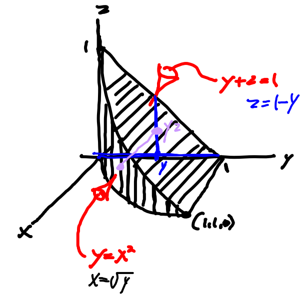
$$\boxed{\begin{array}{c} a \le x \le b \\ f(x) \le y \le g(x) \\ p(x,y) \le z \le q(x,y) \end{array}} = R$$

example

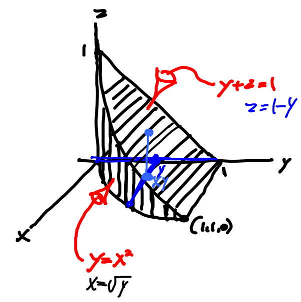


Three descriptions of R

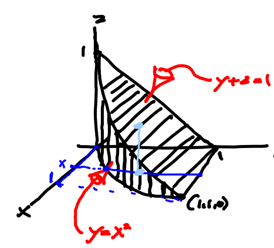
$$\boxed{\begin{array}{l} 0 \le y \le 1 \\ 0 \le z \le 1-y \\ 0 \le x \le \sqrt{y} \end{array}}$$



$$\boxed{\begin{array}{l} 0 \le y \le 1 \\ 0 \le x \le \sqrt{y} \\ 0 \le z \le 1-y \end{array}}$$



$$\boxed{\begin{array}{l} 0 \le x \le 1 \\ x^2 \le y \le 1 \\ 0 \le z \le 1-y \end{array}}$$



Can actually be described for any ordering of x, y, z.

Let's now calculate $\iiint_R z\,dV$ in 2 different ways.

Both yield $\frac{8}{105}$.

First Calculation

$$R = \boxed{\begin{array}{c} 0 \le y \le 1 \\ 0 \le z \le 1-y \\ 0 \le x \le \sqrt{y} \end{array}}$$

$$\iiint_R z\,dV = \int_0^1\int_0^{1-y}\int_0^{\sqrt{y}} z\,dx\,dz\,dy = \int_0^1\int_0^{1-y} z(\sqrt{y}-0)\,dz\,dy$$

$$= \int_0^1\int_0^{1-y} z y^{\frac{1}{2}}\,dz\,dy = \int_0^1 \left.\frac{1}{2}z^2 y^{\frac{1}{2}}\right]_{z=0}^{z=1-y} dy = \int_0^1 \frac{1}{2}(1-y)^2 y^{\frac{1}{2}}\,dy = \frac{1}{2}\int_0^1 y^{\frac{5}{2}} - 2y^{\frac{3}{2}} + y^{\frac{1}{2}}\,dy$$

$$= \frac{1}{2}\left(\left.\frac{2}{7}y^{\frac{7}{2}} - \frac{4}{5}y^{\frac{5}{2}} + \frac{2}{3}y^{\frac{3}{2}}\right]_{y=0}^{y=1}\right) = \frac{1}{2}\left(\frac{2}{7} - \frac{4}{5} + \frac{2}{3}\right) = \frac{1}{2}\,\frac{30-84+70}{105} = \frac{8}{105}$$

Second Calculation

$$R = \boxed{\begin{array}{c} 0 \le x \le 1 \\ x^2 \le y \le 1 \\ 0 \le z \le 1-y \end{array}}$$

$$\iiint_R z\,dV = \int_0^1\int_{x^2}^1\int_0^{1-y} z\,dz\,dy\,dx = \int_0^1\int_{x^2}^1 \left.\frac{1}{2}z^2\right]_{z=0}^{z=1-y} dy\,dx = \frac{1}{2}\int_0^1\int_{x^2}^1 (1-y)^2\,dy\,dx$$

let $u = 1-y$

$$= \frac{1}{2}\int_0^1 \left.-\frac{1}{3}(1-y)^3\right]_{y=x^2}^{y=1} dx = \frac{-1}{6}\int_0^1 0 - (1-x^2)^3\,dx = \frac{1}{6}\int_0^1 (1-x^2)^3\,dx = \frac{1}{6}\int_0^1 -x^6+3x^4-3x^2+1\,dx$$

$$= \frac{1}{6}\left(-\frac{1}{7}x^7 + \frac{3}{5}x^5 - x^3 + x\right)\Big]_{x=0}^{x=1} = \frac{1}{6}\left(-\frac{1}{7} + \frac{3}{5}\right) = \frac{1}{6}\,\frac{21-5}{35} = \frac{1}{3\cancel{6}}\,\frac{\cancel{16}\,8}{35} = \frac{8}{105}$$

(18)

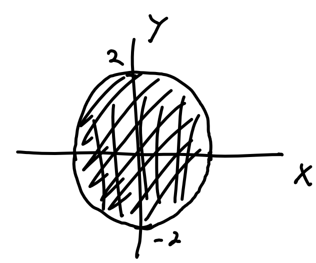
$$\int_1^2\int_{-2}^{-1}\int_0^1 \frac{x+y}{z}\,dx\,dy\,dz = \int_1^2\int_{-2}^{-1}\int_0^1 \frac{x}{z}+\frac{y}{z}\,dx\,dy\,dz = \int_1^2\int_{-2}^{-1} \left.\frac{x^2}{2z}+\frac{y}{z}x\right]_{x=0}^{x=1} dy\,dz$$

$$= \int_1^2\int_{-2}^{-1} \frac{1}{2}z^{-1}+yz^{-1}\,dy\,dz = \int_1^2 \left.\frac{1}{2}z^{-1}y+\frac{1}{2}y^2z^{-1}\right]_{y=-2}^{y=-1} dz$$

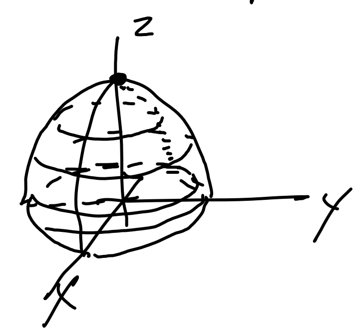
$$= \int_1^2 -\cancel{\tfrac{1}{2}}z^{-1}+\cancel{\tfrac{1}{2}}z^{-1}-\left(-\cancel{z^{-1}}+\cancel{2}z^{-1}\right)dz = \int_1^2 -\frac{1}{z}\,dz = \left.-\ln(2)\right]_{z=1}^{z=2} = -\ln(2)$$

(19)

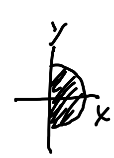
$$\iiint_R x\,dV = 0 \quad \text{where} \quad R = \boxed{\begin{array}{l} -2\le x\le 2 \\ -\sqrt{4-x^2}\le y\le\sqrt{4-x^2} \\ 0\le z\le 4-(x^2+y^2)\end{array}}$$



because
the region
is symmetric
around the yz-plane
and every positive x-value in R
has the corresponding neg x-value
in R.



The more interesting calculation is $\iiint_R z\,dV$ over $R = \boxed{\begin{array}{l} 0\le x\le 2 \\ -\sqrt{4-x^2}\le y\le\sqrt{4-x^2} \\ 0\le z\le 4-(x^2+y^2)\end{array}}$



we'll do this in the next section.

(202) $R = \boxed{\begin{array}{c} 2 \le z \le 3 \\ 0 \le y \le 1 \\ 2-2y \le x \le 2+\sqrt{y} \end{array}}$ calculate $\iiint_R z \, dV$

$\int_0^b k \, dt = k(b-a)$

$$\int_2^3 \int_0^1 \int_{2-2y}^{2+\sqrt{y}} z \, dx \, dy \, dz = \int_2^3 \int_0^1 z\left(\cancel{2}+\sqrt{y} - (\cancel{2} - 2y)\right) dy \, dz = \int_2^3 \int_0^1 z\left(y^{\frac{1}{2}} + 2y\right) dy \, dz$$

$$= \int_2^3 z\left(\frac{2}{3}y^{\frac{3}{2}} + y^2\right)\Big]_{y=0}^{y=1} dz = \frac{5}{3}\int_2^3 z \, dz = \frac{5}{6} z^2 \Big]_{z=2}^{z=3} = \frac{5}{6}(5) = \frac{25}{6}$$

Average value of a function

$f(x)$ on $a \le x \le b$	$f(x,y)$ on R	$f(x,y,z)$ on R
$\frac{1}{b-a}\int_a^b f(x)\,dx$	$\frac{1}{\text{Area}(R)}\iint_R f(x,y)\,dA$	$\frac{1}{\text{vol}(R)}\iiint_R f(x,y,z)\,dV$

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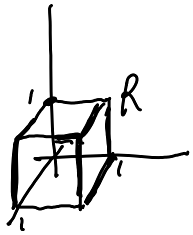

Average value of $f(x,y,z)=xyz$ $= \dfrac{1}{Vol(R)} \iiint_R xyz\, dV = \dfrac{1}{1}\int_0^1\int_0^1\int_0^1 xyz\, dz\, dy\, dx$

$$= \int_0^1\int_0^1 \frac{1}{2}xyz^2\Big]_{z=0}^{z=1} dy\, dx = \int_0^1\int_0^1 \frac{1}{2}xy\, dy\, dx = \int_0^1 \frac{1}{2}\frac{1}{2}xy^2\Big]_{y=0}^{y=1} dx = \int_0^1 \frac{1}{4}x\, dx$$

$$= \frac{1}{8}x^2\Big]_{x=0}^{x=1} = \frac{1}{8}$$

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