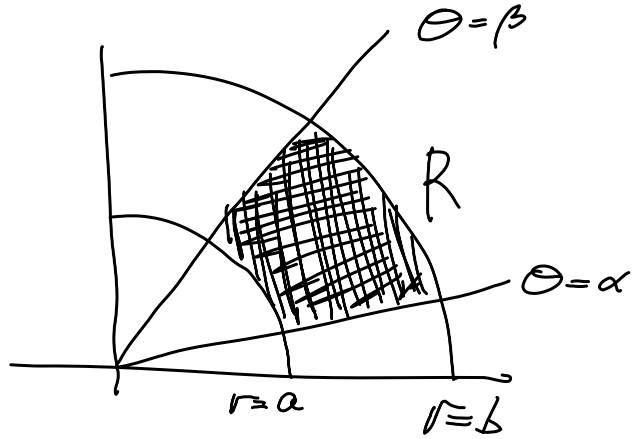
5.3

Double integrals in polar coordinates

A polar rectangle in the xy-plane is

$$R = \begin{matrix} a \le r \le b \\ \alpha \le \theta \le \beta \end{matrix}$$



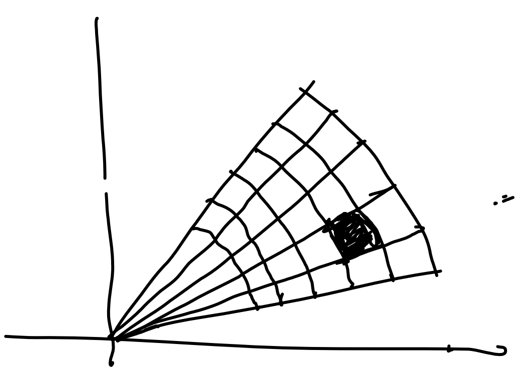
How can we calculate $\iint_R f(x,y)\,dA$ in terms of r and θ??

Divide $a \le r \le b$ into subintervals of length $\Delta r = \frac{b-a}{n}$

$r_i = a + i\Delta r$

and $\alpha \le \theta \le \beta$ into subintervals of length $\Delta\theta = \frac{\beta-\alpha}{m}$

$\theta_j = \alpha + j\Delta\theta$



what is the area of these little polar rectangles??

$$\Delta A = \tfrac{1}{2}(\Delta\theta R^2 - \Delta\theta r^2)$$

$$= \tfrac{1}{2}(\Delta\theta (R-r)(R+r))$$

$$\approx \tfrac{1}{2}(\Delta\theta\, \Delta r\, 2r)$$

$$= r\,\Delta\theta\,\Delta r$$

So $$\iint_R f(x,y)\,dA = \lim_{n\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f(r_i\cos(\theta_j), r_i\sin(\theta_j))\,\Delta A$$

$$\iint_R f(x,y)\,dA = \lim_{n\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f(r_i\cos(\theta_j), r_i\sin(\theta_j))\, r\,\Delta\theta\,\Delta r$$

$$\boxed{\begin{aligned} \iint_R f(x,y)\,dA &= \int_{\alpha}^{\beta}\int_{a}^{b} f(r\cos(\theta), r\sin(\theta))\, r\,dr\,d\theta \\ \iint_R f(x,y)\,dA &= \int_{a}^{b}\int_{\alpha}^{\beta} f(r\cos(\theta), r\sin(\theta))\, r\,d\theta\,dr \end{aligned}}$$

OR (between the two formulas above)

Remember

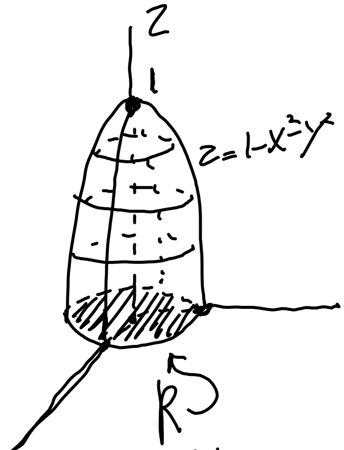
$x = r\cos(\theta)$

$y = r\sin(\theta)$

$x^2+y^2 = r^2$

$\frac{y}{x} = \tan(\theta)$

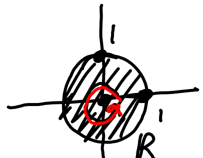
<u>example</u> Find the volume of the solid above the xy-plane and underneath the paraboloid $z = 1-x^2-y^2$



defined by

$0 = 1-x^2-y^2$

$x^2+y^2=1$



$$\boxed{\begin{aligned} 0 &\le r \le 1 \\ 0 &\le \theta \le 2\pi \end{aligned}}$$

$$\text{Volume} = \iint_R 1-(x^2+y^2)\,dA$$

$$= \int_0^{2\pi}\int_0^1 (1-r^2)\,r\,dr\,d\theta$$

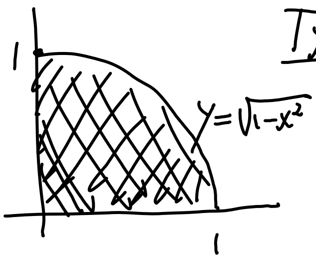
$$= \int_0^{2\pi}\int_0^1 r-r^3\,dr\,d\theta$$

$$= \int_0^{2\pi} \left.\tfrac{1}{2}r^2-\tfrac{1}{4}r^4\right]_{r=0}^{r=1} d\theta$$

$$= \int_0^{2\pi} \frac{1}{4}\, d\theta = \frac{1}{4} 2\pi = \frac{\pi}{2}$$

If we tried to do this in rectangular coordinates

R



Type I

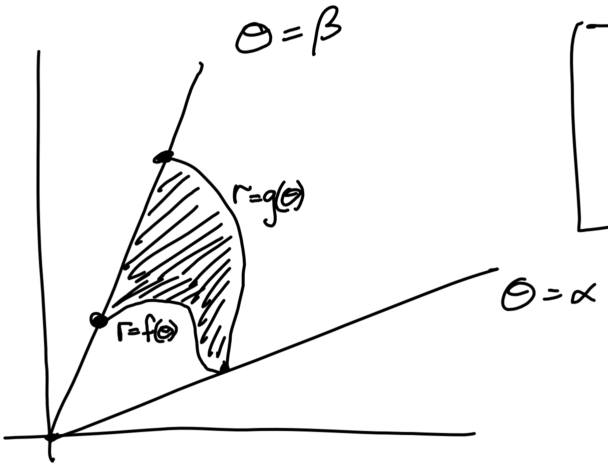
$$\boxed{\begin{array}{l} 0 \le x \le 1 \\ 0 \le y \le \sqrt{1-x^2} \end{array}}$$

$$\text{Volume} = 4\iint_R 1-(x^2+y^2)\, dA = 4\int_0^1\int_0^{\sqrt{1-x^2}} 1-x^2-y^2\, dy\, dx = 4\int_0^1 \left[(1-x^2)y - \frac{1}{3}y^3\right]_{y=0}^{y=\sqrt{1-x^2}} dx$$

$$= 4\int_0^1 (1-x^2)\sqrt{1-x^2} - \frac{1}{3}(1-x^2)^{3/2}\, dx = \frac{8}{3}\int_0^1 (1-x^2)^{3/2}\, dx$$

needs trig substitution
$x = \sin(\theta)$

More general Regions in Polar coordinates



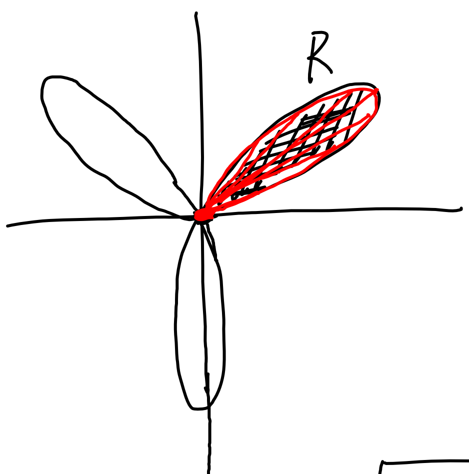
R

$$\boxed{\begin{array}{l} \alpha \le \theta \le \beta \\ f(\theta) \le r \le g(\theta) \end{array}}$$

Here we must calculate the integral in terms r first, then θ.

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_{f(\theta)}^{g(\theta)} f(r\cos(\theta), r\sin(\theta))\, r\,dr\,d\theta$$

<u>example</u> find the area of 1 petal of the 3-leaf rose $r = \sin(3\theta)$



θ	$r=\sin(3\theta)$
0	0
$\downarrow$	$\downarrow$
$\frac{\pi}{6}$	1
$\downarrow$	$\downarrow$
$\frac{\pi}{3}$	0

$$R \quad \boxed{\begin{array}{l} 0 \le \theta \le \frac{\pi}{3} \\ 0 \le r \le \sin(3\theta) \end{array}}$$

$$\text{Area} = \iint_R 1\,dA$$

$$= \int_0^{\frac{\pi}{3}} \int_0^{\sin(3\theta)} r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{3}} \frac{1}{2} r^2 \Big]_{r=0}^{r=\sin(3\theta)} d\theta$$

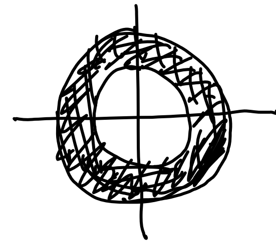
$$= \int_0^{\frac{\pi}{3}} \frac{1}{2}\sin^2(3\theta)\,d\theta = \frac{1}{2}\int_0^{\frac{\pi}{3}} \sin^2(3\theta)\,d\theta = \frac{1}{4}\int_0^{\frac{\pi}{3}} 1-\cos(6\theta)\,d\theta = \frac{1}{4}\left(\theta - \frac{1}{6}\sin(6\theta)\right)\Big]_0^{\frac{\pi}{3}}$$

$$\boxed{\sin^2(x) = \frac{1}{2}(1-\cos(2x))}$$

$$= \frac{1}{4}\left(\frac{\pi}{3} - 0 - 0\right)$$

$$= \frac{\pi}{12}$$

(134) Calculate $\iint_R x^2+y^2\,dA$ $\quad R =$



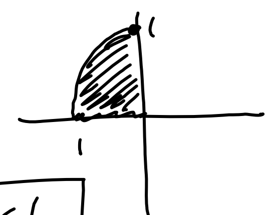
$0 \le \theta \le 2\pi$
$3 \le r \le 5$

$$\iint_R x^2+y^2\,dA = \int_0^{2\pi}\int_3^5 r^2\, r\,dr\,d\theta = \int_0^{2\pi}\int_3^5 r^3\,dr\,d\theta = \int_0^{2\pi} \frac{1}{4}r^4\Big]_{r=3}^{r=5}\,d\theta$$

$$\begin{array}{r} 625 \\ 81 \\ \hline 544 \end{array}$$

$$= \int_0^{2\pi} \frac{625-81}{4}\,d\theta = 136\int_0^{2\pi} d\theta = 272\pi$$

(138) Calculate $\iint_R (x^2+y^2)^{\frac{1}{3}}\,dA$ where $R =$



$$\boxed{\begin{array}{l} 0 \le r \le 1 \\ \frac{\pi}{2} \le \theta \le \pi \end{array}}$$

$$\iint_R (x^2+y^2)^{\frac{1}{3}}\,dA = \int_{\frac{\pi}{2}}^{\pi}\int_0^1 r^{\frac{2}{3}}\, r\,dr\,d\theta = \int_{\frac{\pi}{2}}^{\pi}\int_0^1 r^{5/3}\,dr\,d\theta = \int_{\frac{\pi}{2}}^{\pi} \frac{3}{8} r^{8/3}\Big]_{r=0}^{r=1}\,d\theta$$

$$\int_{\frac{\pi}{2}}^{\pi} \frac{3}{8}\,d\theta = \frac{3}{8}\left(\pi - \frac{\pi}{2}\right) = \frac{3\pi}{16}$$

(140) $\iint_R \sin\left(\arctan\left(\frac{y}{x}\right)\right) dA$ $\qquad R = \boxed{\begin{array}{l} 1 \le r \le 2 \\ \frac{\pi}{6} \le \theta \le \frac{\pi}{3} \end{array}}$

$$\iint_R \sin\left(\arctan\left(\frac{y}{x}\right)\right) dA = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\int_1^2 \sin(\arctan(\tan\theta))\, r\, dr\, d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\int_1^2 r\sin(\theta)\, dr\, d\theta$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{2}r^2\sin(\theta)\Big]_{r=1}^{r=2} d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{3}{2}\sin(\theta)\, d\theta = -\frac{3}{2}\cos(\theta)\Big]_{\theta=\frac{\pi}{6}}^{\theta=\frac{\pi}{3}} = -\frac{3}{2}\left(\frac{1}{2} - \frac{\sqrt{3}}{2}\right)$$

$$= \boxed{\frac{(\sqrt{3}-1)3}{2}}$$

(148) Convert from rectangular to polar coordinates

$$\int_0^3\int_0^{\sqrt{9-y^2}} x^2+y^2\, dx\, dy$$

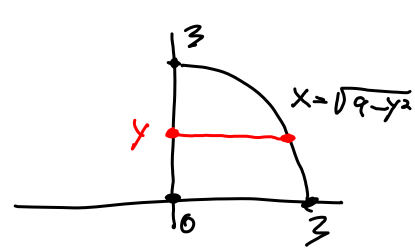
Type 2 iterated integral

$R = \boxed{\begin{array}{l} 0 \le y \le 3 \\ 0 \le x \le \sqrt{9-y^2} \end{array}}$

$x = \sqrt{9-y^2}$

$x^2+y^2=9$

R



in polar coordinates

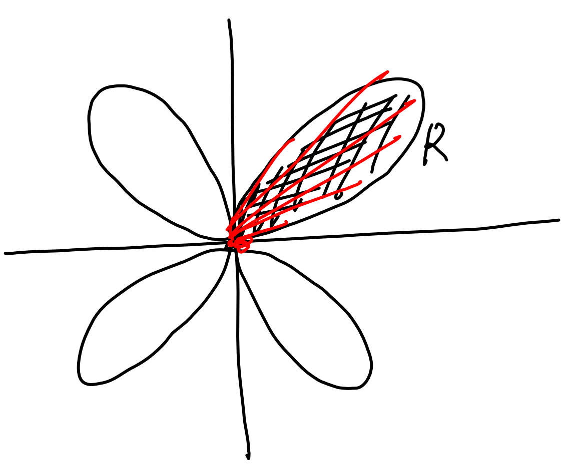
$$\boxed{\begin{array}{l} 0 \le r \le 3 \\ 0 \le \theta \le \frac{\pi}{2} \end{array}} R$$

$$\int_0^3 \int_0^{\sqrt{9-y^2}} x^2+y^2 \, dx\, dy = \iint_R x^2+y^2 \, dA = \int_0^{\frac{\pi}{2}} \int_0^3 r^2 \, r \, dr\, d\theta = \int_0^{\frac{\pi}{2}} \int_0^3 r^3 \, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{4} r^4 \Big]_{r=0}^{r=3} d\theta = \frac{81}{4} \int_0^{\frac{\pi}{2}} d\theta = \frac{81\pi}{8}$$

Try 149 on your own

(155) Find the total area enclosed by $r = \sin(2\theta)$



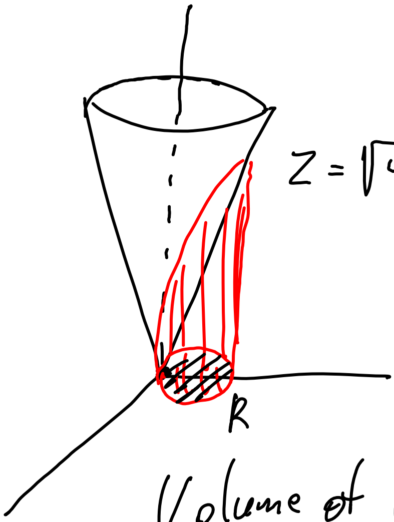
θ	$r = \sin(2\theta)$
0	0
$\frac{\pi}{4}$	1
$\frac{\pi}{2}$	0

$$R = \boxed{\begin{array}{l} 0 \le \theta \le \frac{\pi}{2} \\ 0 \le r \le \sin(2\theta) \end{array}}$$

$$\text{Total area} = 4 \iint_R 1 \, dA = 4 \int_0^{\frac{\pi}{2}} \int_0^{\sin(2\theta)} r \, dr\, d\theta = 4 \int_0^{\frac{\pi}{2}} \frac{1}{2} r^2 \Big]_0^{\sin(2\theta)} d\theta = 2 \int_0^{\frac{\pi}{2}} \sin^2(2\theta)\, d\theta =$$

$$= \int_0^{\frac{\pi}{2}} 1-\cos(4\theta)\,d\theta = \theta - \frac{1}{4}\sin(4\theta)\Big]_{\theta=0}^{\theta=\frac{\pi}{2}} = \left(\frac{\pi}{2} - 0 - 0\right) = \frac{\pi}{2}$$

166



$$Z = \sqrt{4x^2+4y^2} = 2\sqrt{x^2+y^2}$$

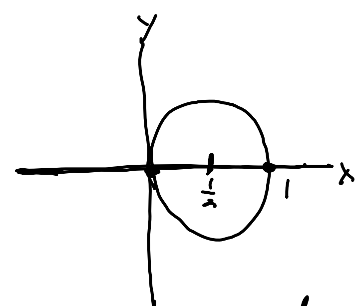
Volume of the solid under the cone inside the cylinder $(x-\frac{1}{2})^2+y^2=\frac{1}{4}$

$$x^2+y^2=x \rightarrow \boxed{\begin{aligned} r^2 &= r\cos(\theta) \\ r &= \cos\theta \end{aligned}}$$

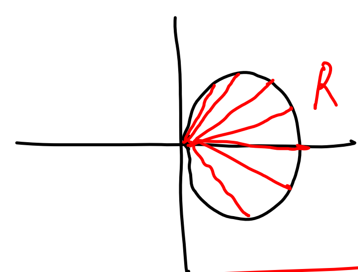
$$x^2-x+y^2=0$$

$$x^2-x+\frac{1}{4}+y^2=\frac{1}{4}$$

$$(x-\frac{1}{2})^2+y^2=\frac{1}{4}$$



Can be expressed as $r=\cos(\theta)$ for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$



$$R \quad \boxed{\begin{aligned} -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \\ 0 \le r \le \cos(\theta) \end{aligned}}$$

$$\text{Volume} = \iint_R 2\sqrt{x^2+y^2}\,dA$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\cos(\theta)} 2\sqrt{r^2}\; r\,dr\,d\theta$$

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\cos(\theta)} r^2\,dr\,d\theta$$

$$= \frac{2}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r^3\Big]_0^{\cos(\theta)}\,d\theta$$

$$= \frac{2}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3(\theta)\,d\theta = \frac{2}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(\theta)\cos(\theta)\,d\theta = \frac{2}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(1-\sin^2(\theta)\right)\cos(\theta)\,d\theta =$$

let
$u = \sin(\theta)$
$du = \cos(\theta)\,d\theta$

$$= \frac{2}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1-u^2\,du = \frac{2}{3}\left(u - \frac{1}{3}u^3\right)\Bigg]_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} = \frac{2}{3}\left(\sin(\theta) - \frac{1}{3}\sin^3(\theta)\right)\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{2}{3}\left(\frac{2}{3} - \left(-1 - \frac{1}{3}\right)\right) = \frac{2}{3}\left(\frac{2}{3} + \frac{2}{3}\right) = \frac{8}{9}$$