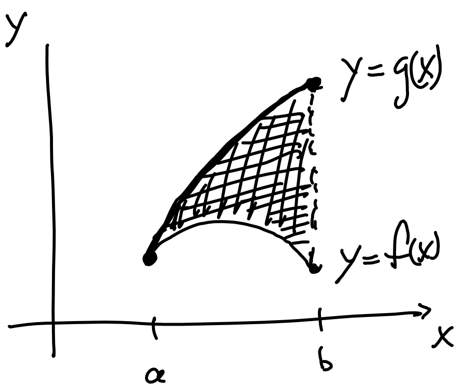
5.2 Double integrals over more general regions

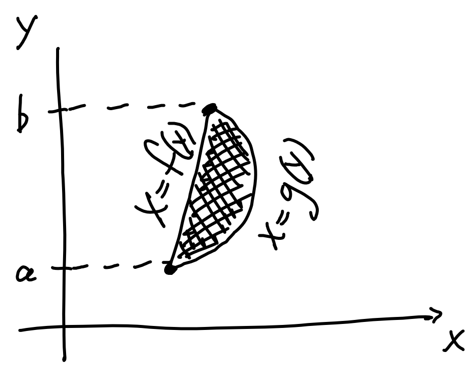
Type 1

$$\boxed{\begin{array}{l} a \le x \le b \\ f(x) \le y \le g(x) \end{array}}$$

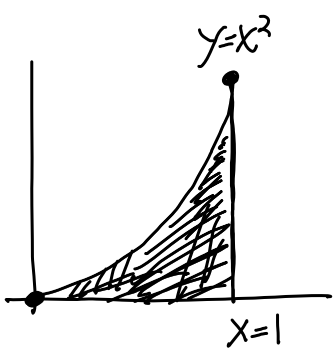


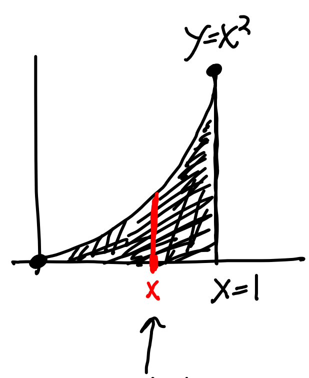
Type 2

$$\boxed{\begin{array}{l} a \le y \le b \\ f(y) \le x \le g(y) \end{array}}$$



example Describe the region shown as a Type 1 and as a Type 2 region.

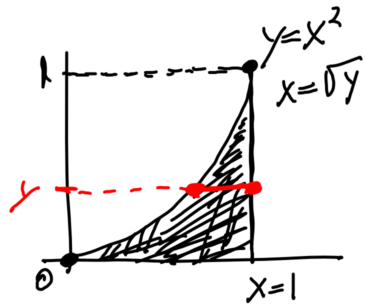




red line bounded above by $y=x^2$ and below by $y=0$

Type 1

$$\boxed{\begin{array}{l} 0 \le x \le 1 \\ 0 \le y \le x^2 \end{array}}$$



Type 2

$$\boxed{\begin{array}{c} 0 \le y \le 1 \\ \sqrt{y} \le x \le 1 \end{array}}$$

The red line is bounded on the left by $x=\sqrt{y}$ and on the right by $x=1$

Now given a type 1 region $R = \boxed{\begin{array}{c} a \le x \le b \\ f(x) \le y \le g(x) \end{array}}$

$$\iint_R f(x,y)\,dA = \int_a^b \int_{f(x)}^{g(x)} f(x,y)\,dy\,dx$$

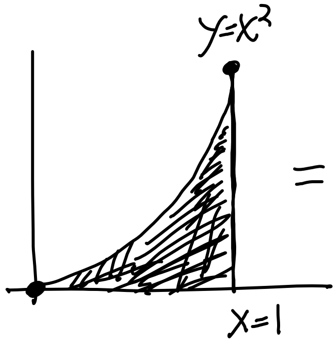
y must be integrated before x for Type 1.

Now given a type 2 region $R = \boxed{\begin{array}{c} a \le y \le b \\ f(y) \le x \le g(y) \end{array}}$

$$\iint_R f(x,y)\,dA = \int_a^b \int_{f(y)}^{g(y)} f(x,y)\,dx\,dy$$

x must be integrated before y for Type 2.

example



$= R$

Calculate $\iint_R x\cos(y)\,dA$ as both Type 1 and Type 2.

Type 1 $\boxed{\begin{array}{l} 0 \le x \le 1 \\ 0 \le y \le x^2 \end{array}}$

$$\iint_R x\cos(y)\,dA = \int_0^1\int_0^{x^2} x\cos(y)\,dy\,dx = \int_0^1 x\sin(y)\Big]_{y=0}^{y=x^2} dx = \int_0^1 x\sin(x^2) - 0\,dx$$

$$= \underbrace{\int_0^1 x\sin(x^2)\,dx}_{\text{y's dissappear and left with just x's.}} = \int_{x=0}^{x=1} \cancel{x}\sin(u)\,\frac{du}{2\cancel{x}} = \frac{1}{2}\int_{x=0}^{x=1} \sin(u)\,du$$

let $u = x^2$

$du = 2x\,dx$

$\frac{du}{2x} = dx$

$$= -\frac{1}{2}\cos(u)\Big]_{x=0}^{x=1} = -\frac{1}{2}\cos(x^2)\Big]_{x=0}^{x=1} = -\frac{1}{2}\left(\cos(1) - \cos(0)\right)$$

$$= \boxed{\frac{1-\cos(1)}{2}}$$

Type 2 $\boxed{\begin{array}{l} 0 \le y \le 1 \\ \sqrt{y} \le x \le 1 \end{array}}$

$$\iint_R x\cos(y)\,dA = \int_0^1\int_{\sqrt{y}}^1 x\cos(y)\,dx\,dy = \int_0^1 \frac{1}{2}x^2\cos(y)\Big]_{x=\sqrt{y}}^{x=1} dy$$

$$= \int_0^1 \frac{1}{2}\cos(y) - \frac{1}{2}y\cos(y)\,dy = \int_0^1 \frac{1}{2}\cos(y)\,dy - \frac{1}{2}\int_0^1 y\cos(y)\,dy$$

$$= \frac{1}{2}\sin(y)\Big]_0^1 - \frac{1}{2}\int_0^1 y\cos(y)\,dy = \frac{1}{2}\sin(1) - \frac{1}{2}\int_0^1 y\cos(y)\,dy$$

$u = y \qquad du = dy$

$dv = \cos(y)dy \quad v = \sin(y)$

$$= \frac{1}{2}\sin(1) - \frac{1}{2}\left(y\sin(y)\Big]_{y=0}^{y=1} - \int_0^1 \sin(y)\,dy \right)$$

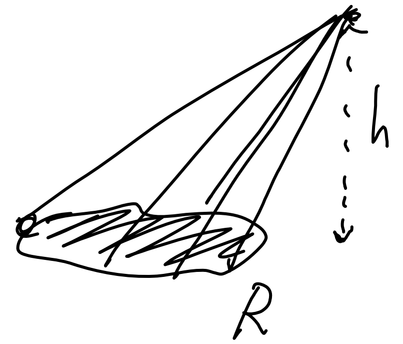
$$= \frac{1}{2}\sin(1) - \frac{1}{2}\left(\sin(1) + \cos(y)\Big]_{y=0}^{y=1} \right) = \cancel{\frac{1}{2}\sin(1)} - \frac{1}{2}\left(\cancel{\sin(1)} + \cos(1) - \cos(0)\right)$$

$$= \boxed{\frac{1-\cos(1)}{2}}$$

Same answer as before.

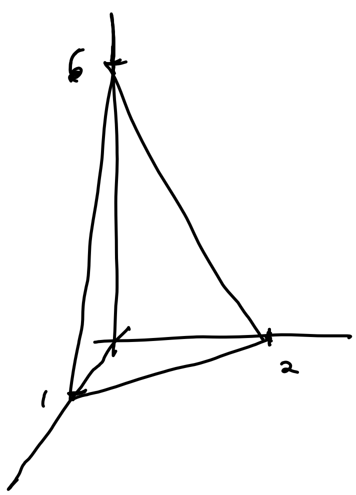
example

The volume of any "cone"



is $\text{Area}(R)\cdot\frac{h}{3}$.

The volume of the following pyramid is



$$\text{Area}(R)\frac{h}{3} = \frac{1}{2}(1)(2)\frac{6}{3} = 2$$

Lets calculate this volume with a double integral instead.

The slanted triangular face is part of the plane

$$z = 6 + Ax + By$$

$$z = 6 - 6x - 3y$$

$\underline{x=0,\ z=0}$

$y=2$

$0 = 6 + B2$

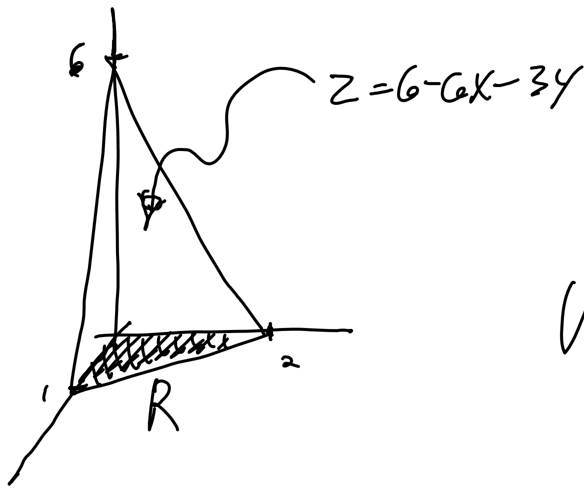
$-3 = B$

$\underline{y=0,\ z=0}$

$x=1$

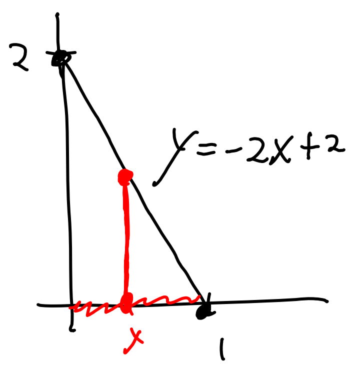
$0 = 6 + A$

$-6 = A$



$$\text{Volume} = \iint_R (6-6x-3y)\,dA$$

In order to calculate this double integral we need a description of R as Type 1 or Type 2 region.



Type 1

$$\boxed{\begin{array}{l} 0 \le x \le 1 \\ 0 \le y \le 2-2x \end{array}}$$

$$\text{Volume} = \iint_R 6-6x-3y\,dA = \int_0^1\int_0^{2-2x} 6-6x-3y\,dy\,dx$$

$$= \int_0^1 \left[(6-6x)y - \frac{3}{2}y^2\right]_{y=0}^{y=2-2x} dx = \int_0^1 6(1-x)(2-2x) - \frac{3}{2}(2-2x)^2 - 0\,dx$$

$$= \int_0^1 12(1-x)^2 - 6(1-x)^2\,dx = \int_0^1 6(1-x)^2\,dx \underset{\uparrow}{=} -\frac{6}{3}(1-x)^3\Big]_{x=0}^{x=1}$$

let $u=1-x$

$= -2(0)^3 + 2(1)^3 = 2$

Same answer as the general "cone construction"