5.1 Integrals of 2-variable functions on rectangular regions.

Recall the definition of the integral of a single-variable function $f(x)$.

Say that $f(x)$ is continuous on $a \le x \le b$.

For any integer $n \ge 1$ define $\Delta x = \frac{b-a}{n}$

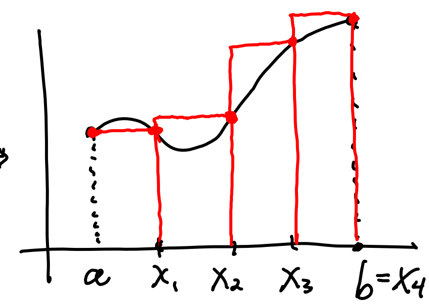
$$x_0 = a$$

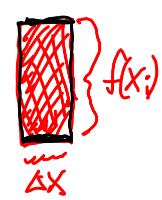
$$x_i = a + i\Delta x$$

Riemann Sum for a particular n

$$\sum_{i=1}^{n} f(x_i)\Delta x$$

and it measures the areas of these rectangles



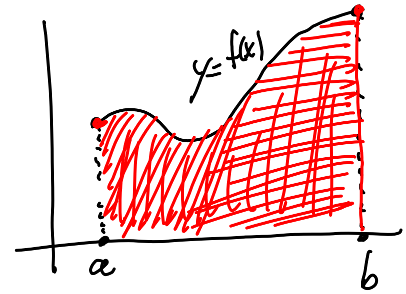


Area of rectangle is $f(x_i)\Delta x$

where area is allowed to be negative

Now The integral of $f(x)$ on $a \le x \le b$ is defined to be

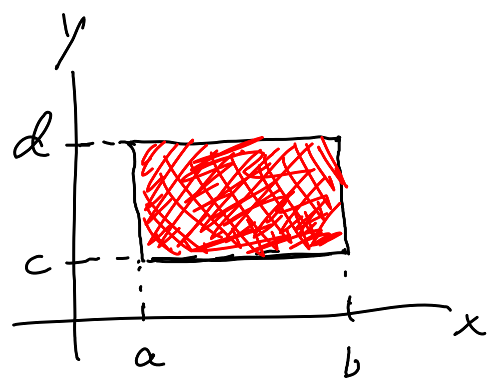
$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)\,dx$$



Geometrically $\int_a^b f(x)dx$ measures the total area under the curve $y=f(x)$.

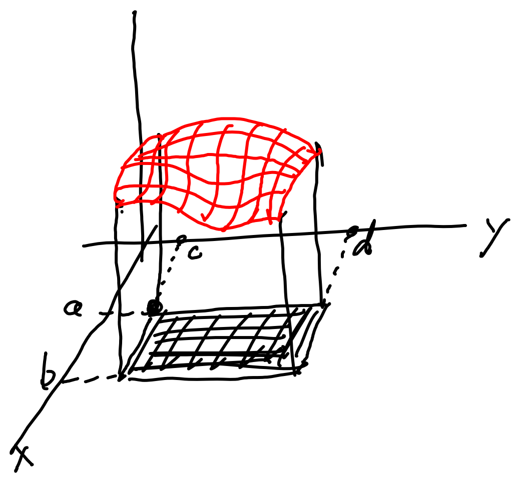
Now, consider a 2-variable function $f(x,y)$ and a rectangular Region

$$a \le x \le b$$
$$c \le y \le d$$



Consider the surface $z = f(x,y)$ above this rectangle. What is the volume of the solid between the surface and rectangle??

Negative volumes are allowed for cases where the surface is below the xy-plane.

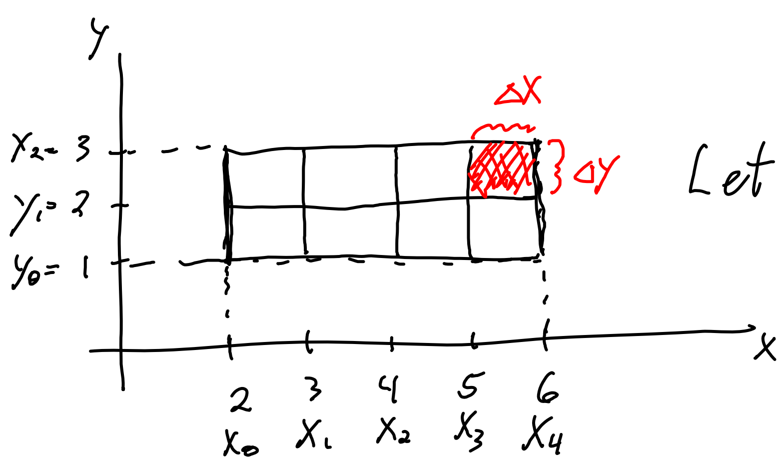


To measure this we will define a <u>double-integral</u>.

* <u>Riemann Sum which estimates the volume</u>

Let m, n be positive integers $\Delta x = \frac{b-a}{n}$ and $\Delta y = \frac{d-c}{m}$

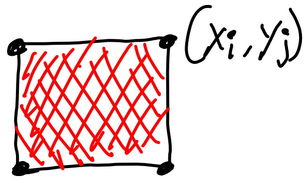
$x_i = a + i\Delta x \qquad y_j = c + j\Delta y$.



Let $m = 2$
$n = 4$

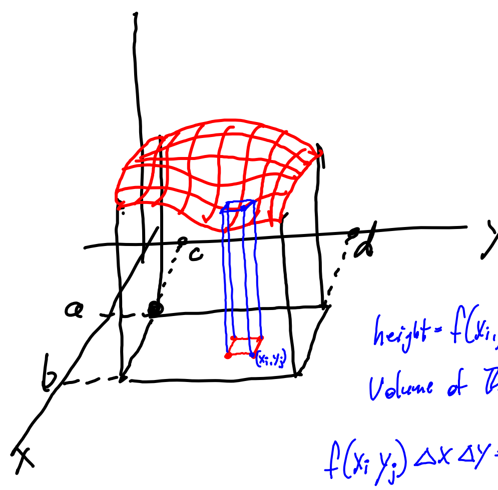
Area of each little rectangle is
$\Delta A = \Delta x \Delta y$

The upper right corner of each rectangle is (x_i, y_j) for some i and j.



Area $= \Delta A = \Delta x \, \Delta y$

Now consider the box with this rectangle as a base and having height $f(x_i, y_j)$



height $= f(x_i, y_j)$

Volume of the box is

$f(x_i, y_j)\,\Delta x\,\Delta y = \underline{f(x_i, y_j)\,\Delta A}$

Now the volume under the surface is approximately the following Riemann Sum.

$$\sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i, y_j)\,\Delta A$$

which is the sum of all of these boxes

The estimate gets better and better as

$m, n \to \infty$

* Define the double integral

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i, y_j)\,\Delta A$$

How might one actually calculate $\iint_R f(x,y)\,dA$??

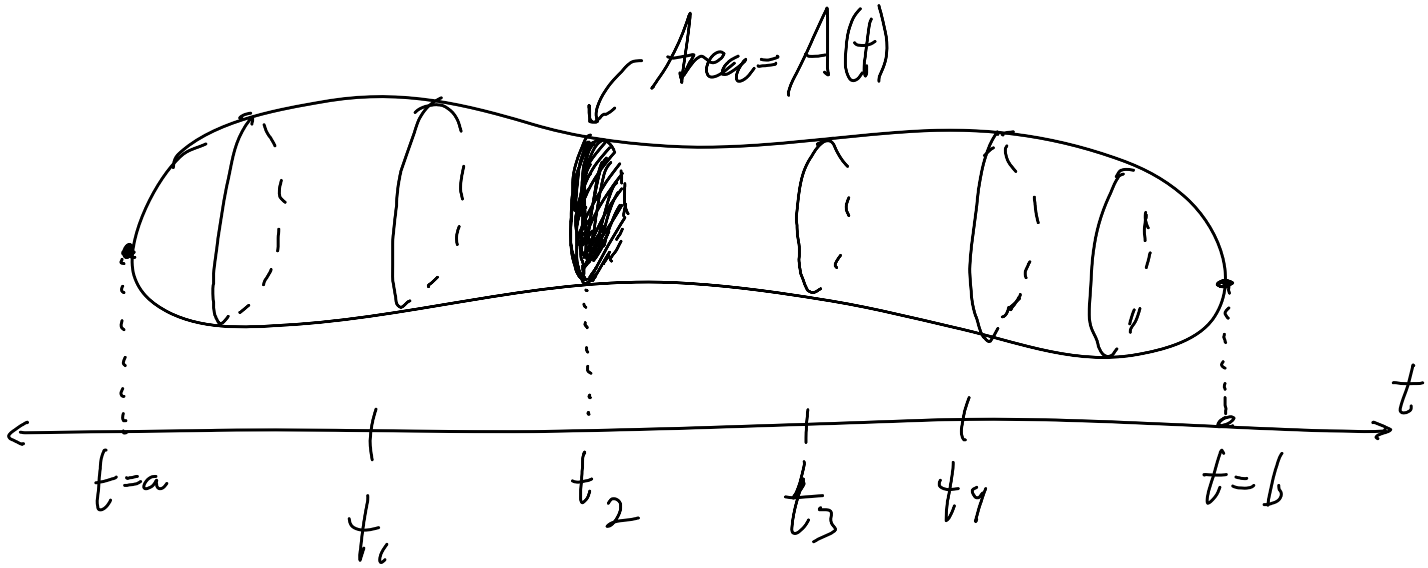
This can be done using "<u>iteration</u>" of integrals.

Recall from Calculus II.

Given a solid lined up along

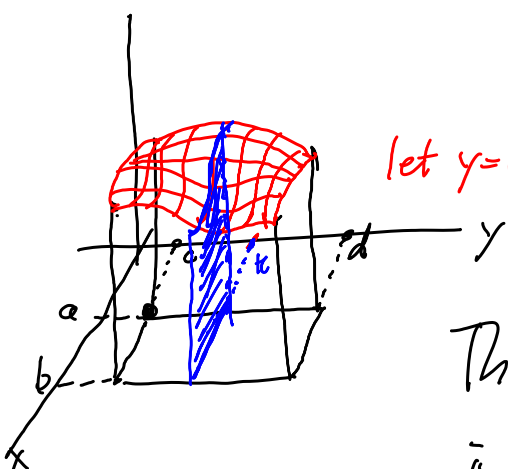
a t-axis whose cross-sectional

Area is $A(t)$ (as shown)



$$\text{Volume} = \int_a^b A(t)\,dt$$

Now for our volume under surface $z = f(x,y)$,



let $y=k$ be a constant between c and d

The area of this slice at $y=k$

is $\displaystyle\int_a^b f(x,k)\,dx$

$$\text{Now Volume} = \int_c^d \left[\int_a^b f(x,k)\,dx\right] dy$$

Or similarly

$$\text{Volume} = \int_a^b \left[\int_c^d f(x,y)\, dy \right] dx$$

Leaving out the brackets we simply write

$$\iint_R f(x,y)\, dA = \int_c^d \int_a^b f(x,y)\, dx\, dy$$

iterated integral

OR

$$\iint_R f(x,y)\, dA = \int_a^b \int_c^d f(x,y)\, dy\, dx$$

Fubini's Theorem

$$\int_c^d \int_a^b f(x,y)\, dx\, dy = \int_a^b \int_c^d f(x,y)\, dy\, dx$$

integrate with respect to x first, holding y as a constant.

integrate with respect to y first, holding x as a constant.

example Calculate $\iint_R 6-xy^2\,dA$ where $R = \boxed{\begin{array}{c} 0\le x\le 2 \\ 0\le y\le 1\end{array}}$

in 2 different ways.

1st calculation

$$\iint_R 6-xy^2\,dA = \int_0^1\int_0^2 6-xy^2\,dx\,dy = \int_0^1\left(6x-\tfrac{1}{2}x^2y^2\Big]_{x=0}^{x=2}\right)dy$$

antiderivative with respect to x holding y as a constant. That is a "partial antiderivative"

$$= \int_0^1 \left(12-2y^2-0\right)dy = \underbrace{\int_0^1 12-2y^2\,dy}_{} = 12y-\tfrac{2}{3}y^3\Big]_{y=0}^{y=1} = 12-\tfrac{2}{3}-0$$

x's are gone now we are left with just y.

$$= \frac{34}{3}$$

2nd calculation

$$\iint_R 6-xy^2\,dA = \int_0^2\int_0^1 6-xy^2\,dy\,dx = \int_0^2\left(6y-\tfrac{1}{3}xy^3\Big]_{y=0}^{y=1}\right)dx$$

antiderivate with respect to y holding x as a constant

$$= \int_0^2 6-\tfrac{1}{3}x-0\,dx = \int_0^2 6-\tfrac{1}{3}x\,dx = 6x-\tfrac{1}{6}x^2\Big]_{x=0}^{x=2}$$

integral in terms of x only. The y's have disappeared

$$= 12 - \frac{4}{6} - 0$$

$$= 12 - \frac{2}{3} = \frac{34}{3}$$

Same answer as before.