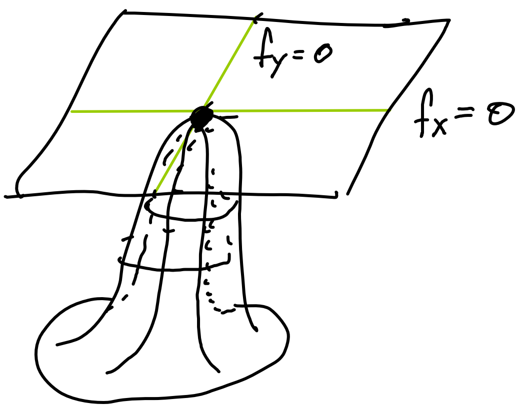
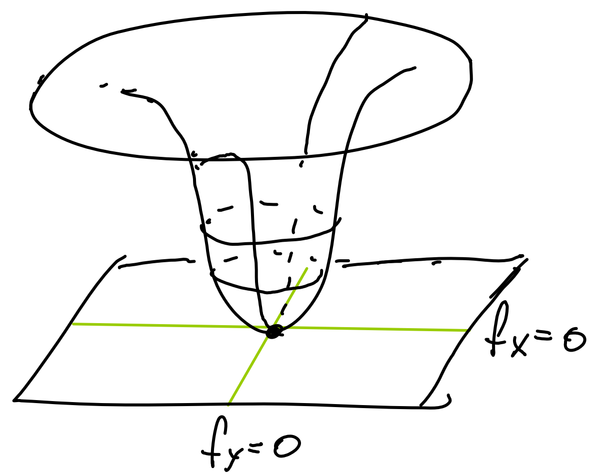
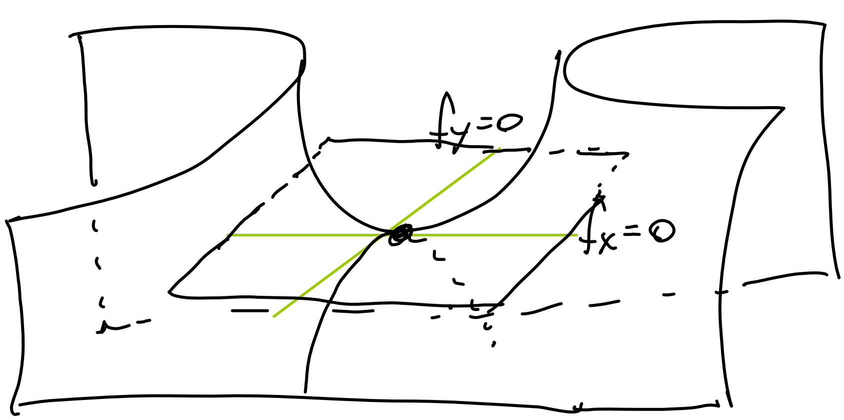
Maximum / minimum Values of 2-variable functions

Suppose that $z = f(x,y)$ is a smooth and continuous surface. Relative maximums and relative minimums for $z = f(x,y)$ are points that must have horizontal tangent planes.





Not that having a horizontal tangent plane necessarily means that you must have a relative maximum or minimum.



One can calculate when a given function $z = f(x,y)$ has a horizontal tangent plane. Just solve the equations $f_x = 0$ and $f_y = 0$ simultaneously.

Points (a,b) where $f_x(a,b) = f_y(a,b) = 0$ are called <u>critical points</u> just like in Calculus I.

<u>example</u> find the critical points for

$$f(x,y) = xy + \frac{1}{x} + \frac{1}{y}$$

$$f_x = y - \frac{1}{x^2} = 0 \longrightarrow \quad y = \frac{1}{x^2}$$

$$f_y = x - \frac{1}{y^2} = 0 \longrightarrow \quad x = \frac{1}{y^2}$$

Substituting second equation into the first equation, we obtain

$$y = \frac{1}{\left(\frac{1}{y^2}\right)^2}$$

$$y = y^4$$

$$0 = y^4 - y = y(y^3 - 1)$$

$$y = 0 \text{ or } y^3 - 1 = 0$$

$$y = 0 \text{ and } y = 1.$$

$\underline{y=0}$

$y=\frac{1}{x^2}$

$0=\frac{1}{x^2}$

no solution

$\underline{y=1}$

$y=\frac{1}{x^2}$

$1=\frac{1}{x^2}$

$x^2=1$

$x=1,-1$

potential critical points are $(1,1)$ and $(-1,1)$

but $(-1,1)$ does not satisfy both equations $\boxed{\begin{matrix} y=\frac{1}{x^2} \\ x=\frac{1}{y^2} \ \checkmark \end{matrix}}$

So the only critical point is $(1,1)$.

Second Derivative Test

If (a,b) is a critical point for $z=f(x,y)$, then

let $D(a,b)=\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy}-f_{xy}f_{yx}$

usually $f_{xy}=f_{yx}$ when 1st partials are continuous. (Clairaut's Theorem)

$= f_{xx}f_{yy}-(f_{xy})^2$.

① If $D>0$ and $f_{xx}>0$, then (a,b) is a local min.

② If $D>0$ and $f_{xx}<0$, then (a,b) is a local max

③ If $D<0$, then (a,b) is neither a local max nor local min.

④ If $D=0$, then 2nd derivative test says nothing.

example Is critical point $(1,1)$ for $f(x,y) = xy + \frac{1}{x} + \frac{1}{y}$

a local max, local min, or neither??

$f_x = y - \frac{1}{x^2}$ $\quad f_y = x - \frac{1}{y^2}$ $\quad f_{xy} = 1$

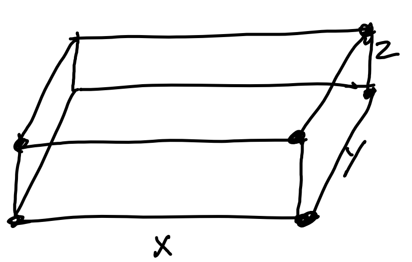
$f_{xx} = \frac{2}{x^3}$ $\quad f_{yy} = \frac{2}{y^3}$

$$D(1,1) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = (2)(2)-(1)(1) = 3 > 0$$

$D > 0$ and $f_{xx}(1,1) = 2 > 0$ so $(1,1)$ is a local minimum.

Applied Max/Min Problems

example A rectangular box with an open top is to be made so as to have volume 1000 ft^3.



what dimensions minimize the surface area??

Surface Area

$A = xy + 2yz + 2xz$

This is a 3-variable function.

Need to reduce it to a 2-variable function.

$$1000 = xyz = \text{volume}$$

$$\frac{1000}{xy} = z$$

So now $A(x,y) = xy + 2y\left(\frac{1000}{xy}\right) + 2x\left(\frac{1000}{xy}\right)$

$$A(x,y) = xy + \frac{2000}{x} + \frac{2000}{y}$$

minimize this function. So we start looking for critical points.

$$A_x = y - \frac{2000}{x^2} = 0 \longrightarrow y = \frac{2000}{x^2}$$

$$A_y = x - \frac{2000}{y^2} = 0 \longrightarrow x = \frac{2000}{y^2}$$

$$x = \frac{2000}{\left(\frac{2000}{x^2}\right)^2}$$

$$x = \frac{x^4}{2000}$$

$$2000x = x^4$$

$$0 = x^4 - 2000x$$

$$0 = x(x^3 - 2000)$$

$\underline{x = 0}$ omit this because $x > 0$ is necessary

$x^3 = 2000$

$x = \sqrt[3]{2000} \approx 12.6$

$$y = \frac{2000}{x^2} = \frac{2000}{2000^{2/3}} = (2000)^{1/3} \approx 12.6$$

So $(x,y) = (12.6, 12.6)$ is the only critical point. So this must be where the minimum surface area occurs.

$$z = \frac{1000}{xy} = \frac{1000}{(2000)^{2/3}}$$

Dimensions are

$12.6 \times 12.6 \times 6.3$

<u>example</u> Consider the following lines.

L_1	L_2
$x = 1+t$	$x = 1+2s$
$y = 1-t$	$y = -s$
$z = 1$	$z = 1-s$

What is the shortest distance between a point on L_1 to a point on L_2 ??

$$d(s,t) = \sqrt{(t-2s)^2 + (1-t+s)^2 + (s)^2}$$

minimize $f(s,t) = (t-2s)^2 + (1-t+s)^2 + s^2$

Find critical points

$$f_s = 2(t-2s)(-2) + 2(1-t+s)(1) + 2s = \boxed{12s - 6t + 2 = 0}$$

$$f_t = 2(t-2s)(1) + 2(1-t+s)(-1) + 0 = \boxed{4t - 6s - 2 = 0}$$

$$\boxed{\begin{aligned} 12s - 6t &= -2 \\ -6s + 4t &= 2 \end{aligned}}$$

$s = \frac{1}{3}, \quad t = 1$

This is the only critical point so this is where the min must occur.

$$d(s,t) = \sqrt{(t-2s)^2 + (1-t+s)^2 + (s)^2}$$

minimum distance $= d\left(\frac{1}{3}, 1\right) = \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{-1}{3}\right)^2 + \left(\frac{1}{3}\right)^2} = \frac{\sqrt{3}}{3}$