4.6 Directional Derivatives.

Recall that for $f(x,y)$ we define

$$f_x = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} \quad \text{and} \quad f_y = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

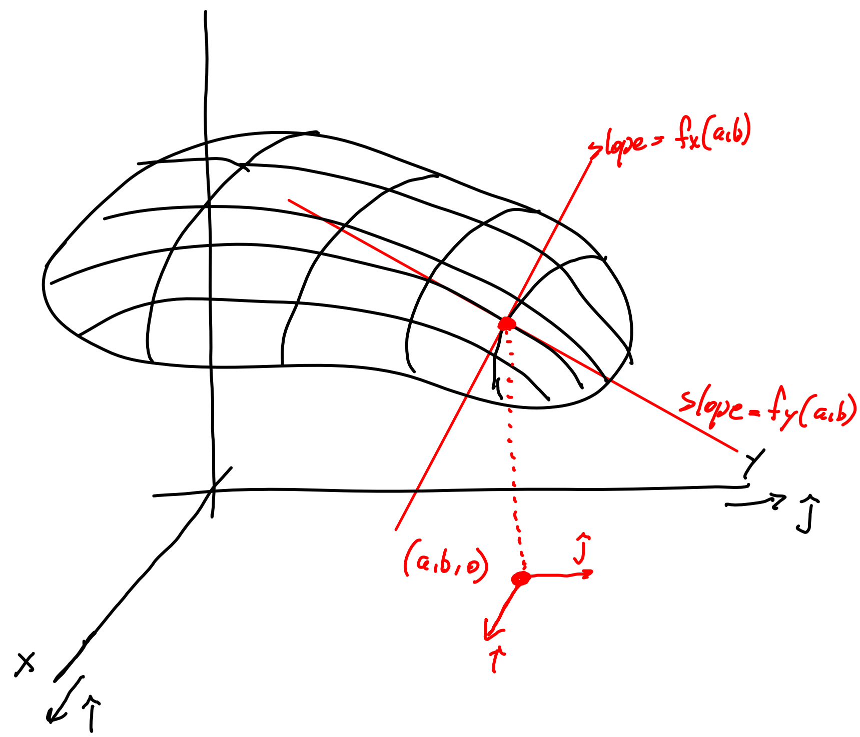
Note $\langle x+h, y\rangle = \langle x,y\rangle + h\langle 1,0\rangle = \langle x,y\rangle + h\,\hat{\imath}$

$\langle x, y+h\rangle = \langle x,y\rangle + h\langle 0,1\rangle = \langle x,y\rangle + h\,\hat{\jmath}$

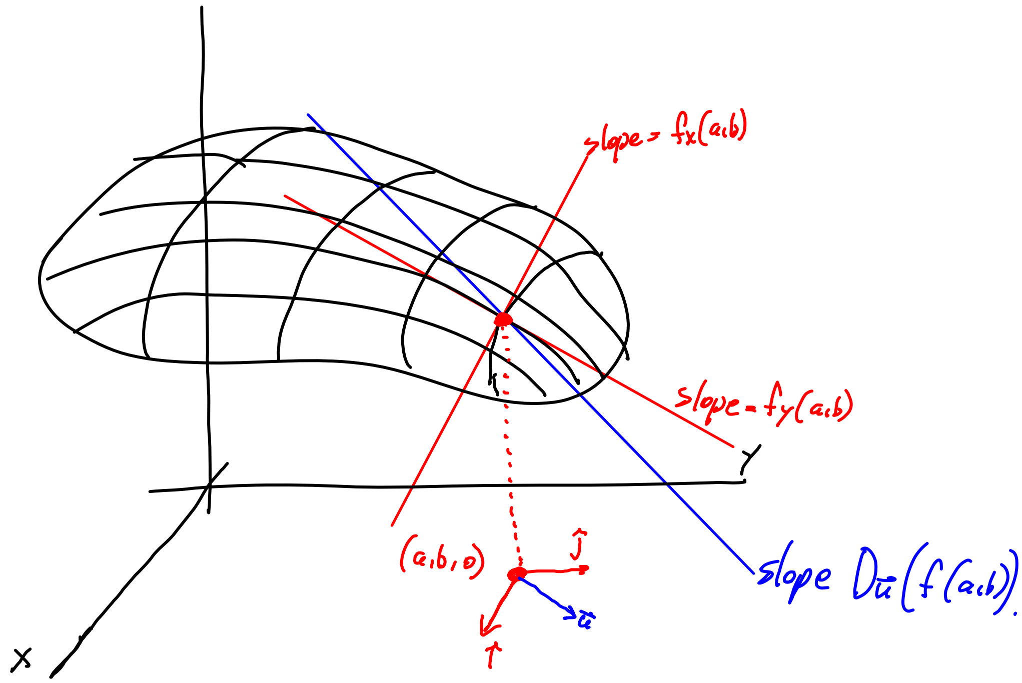
Now let $\vec{u} = \langle a,b\rangle$ be some other unit vector other than $\hat{\imath}$ and $\hat{\jmath}$. This means $|\vec{u}| = \sqrt{a^2+b^2} = 1$ or just $a^2+b^2=1$.

* The partial derivative $f_x(a,b)$ is the slope of surface $z=f(x,y)$ at $(a,b,f(a,b))$ in the direction of $\hat{i}$ (i.e, the direction of the x-axis)

* The partial derivative $f_y(a,b)$ is the slope of surface $z=f(x,y)$ at $(a,b,f(a,b))$ in the direction of $\hat{j}$ (i.e, the direction of the y-axis)



For a given unit vector $\vec{u}=\langle p,q\rangle$, the slope of the surface $z=f(x,y)$ at $(a,b,f(a,b))$ in the direction of $\vec{u}$ is called the <u>directional derivative</u> $D_{\vec{u}}f(a,b)$.



How do we calculate $D_{\vec u}(f(a,b))$

We define the same way as with f_x and f_y

$$D_{\vec u}(f(a,b)) = \lim_{h \to 0} \frac{f(a+ph,\, b+qh) - f(a,b)}{h}$$

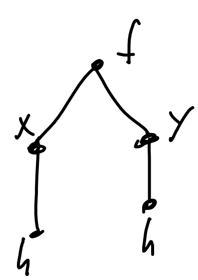
where $\vec u = \langle p, q \rangle$

So now given a fixed value for $(a,b) = (x,y)$

f now becomes a function in terms of x and y

but x and y are now functions of h.

$x = a+ph$, $y = b+qh$



So now the directional derivative is just the derivative of this function dependent on h, $\frac{df}{dh}$

$$\frac{df}{dh} = \frac{\partial f}{\partial x}\frac{dx}{dh} + \frac{\partial f}{\partial y}\frac{dy}{dh}$$

$$D_{\vec{u}}\big(f(a,b)\big) = f_x(a,b)\,p + f_y(a,b)\,q$$

$$D_{\vec{u}}\big(f(a,b)\big) = \langle f_x(a,b), f_y(a,b)\rangle \cdot \langle p,q\rangle$$

This is called the gradient vector

$$\nabla f(a,b) = \langle f_x(a,b), f_y(a,b)\rangle$$

This was $\vec{u}$

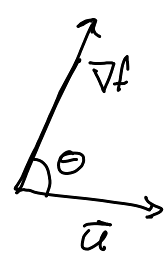
Therefore

$$\boxed{D_{\vec{u}}\big(f(a,b)\big) = \nabla f(a,b) \cdot \vec{u}}$$

$$D_{\vec{u}}\big(f(a,b)\big) = |\nabla f(a,b)|\,|\vec{u}|\cos(\theta)$$

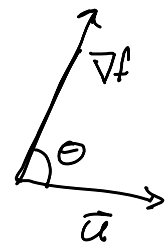
$$\boxed{D_{\vec{u}}\big(f(a,b)\big) = |\nabla f(a,b)|\cos(\theta)}$$

because $|\vec{u}|=1$.



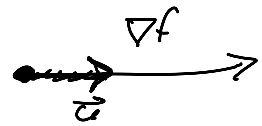
This second formulation has an interesting consequence.

$$D_{\vec{u}}\left(f(a,b)\right) = \left|\nabla f(a,b)\right| \cos(\theta)$$



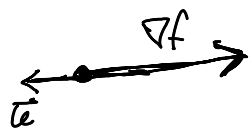
* Max value of $D_{\vec{u}}\left(f(a,b)\right)$ is $|\nabla f|$ when $\cos(\theta) = 1$

which occurs when $\theta = 0^\circ$



* Min value of $D_{\vec{u}}\left(f(a,b)\right)$ is $-|\nabla f|$ when $\cos(\theta) = -1$

which occurs when $\theta = 180^\circ$



* $D_{\vec{u}}\left(f(a,b)\right) = 0$ occurs when $\vec{u}$ and ∇f are perpendicular

<u>example</u> Let $z = x^2 - y^2$ so

$$\nabla z(x,y) = \langle z_x , z_y \rangle = \langle 2x , -2y \rangle$$

So at various points we get various gradients

$$\nabla z(1,2) = \langle 2, -4 \rangle \quad \text{and} \quad \nabla z(2,0) = \langle 4, 0 \rangle$$

Consider some various unit vectors $\vec{u} = \langle \frac{3}{5}, \frac{-4}{5} \rangle$

$\vec{v} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

$$D_{\vec{u}}\left(z(1,2)\right) = \nabla z(1,2) \cdot \vec{u} = \langle 2, -4 \rangle \cdot \langle \frac{3}{5}, \frac{-4}{5} \rangle = \frac{6}{5} + \frac{16}{5} = \frac{22}{5}$$

$$D_{\vec{v}}\left(z(2,0)\right) = \nabla z(2,0) \cdot \vec{v} = \langle 4, 0 \rangle \cdot \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle = \frac{4}{\sqrt{2}}$$

The max value of $D_{\vec{u}}\left(z(1,2)\right) = |\nabla z(1,2)| = |\langle 2, -4 \rangle| = \sqrt{4+16} = 2\sqrt{5}$

The direction is $\frac{1}{|\nabla z|} \nabla z = \frac{1}{2\sqrt{5}} \langle 2, -4 \rangle = \langle \frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}} \rangle$

negation

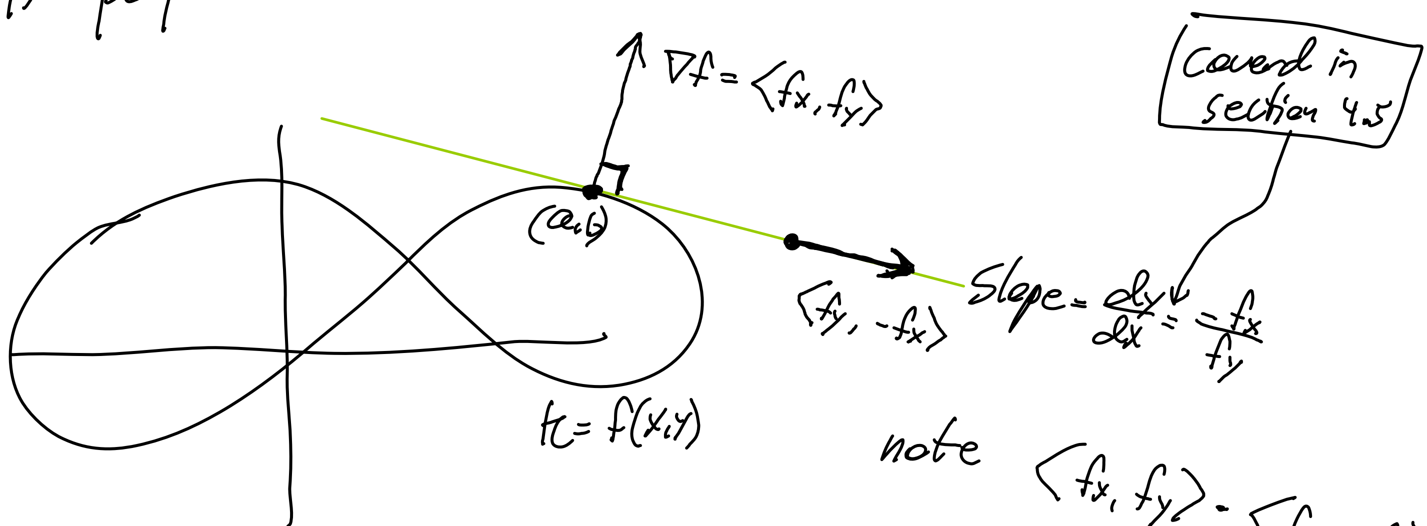
The min value of $D_{\vec{u}}\left(z(1,2)\right) = -|\nabla z(1,2)| = -2\sqrt{5}$

The direction is $\frac{-1}{|\nabla z|} \nabla z = \langle \frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle$

$D_{\vec{u}}\left(z(1,2)\right) = 0$ when $\vec{u} = \langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \rangle$ or $\langle \frac{-2}{\sqrt{5}}, \frac{-1}{\sqrt{5}} \rangle$

Gradient $\nabla f = \langle f_x, f_y \rangle$ and level Curves

Given a level curve $k = f(x,y)$ and any point (a,b) for which $k = f(a,b)$. The gradient $\nabla f(a,b)$ is perpendicular to the level curve.



note $\langle f_x, f_y \rangle \cdot \langle f_y, -f_x \rangle = 0$

An interesting application. Suppose $z = f(x,y)$ represents elevation or air-pressure at a given point (x,y) on a map of some region.

In this case $-\nabla f$ represents which is perpendicular to the level curves points down the path of steepest descent. So water running down elevation or wind following a path from greater to lesser pressure will travel along the gradient and perpendicular to level curves.

Level surfaces and the gradient

given a level surface $k = f(x,y,z)$
and a point (a,b,c) for which $k = f(a,b,c)$,
the gradient $\nabla f(a,b,c) = \langle f_x, f_y, f_z \rangle$ is
a normal vector to the tangent plane
of the level surface, in other words
$\nabla f(a,b,c)$ is perpendicular to the surface
at any point.