4.5 Chain Rule

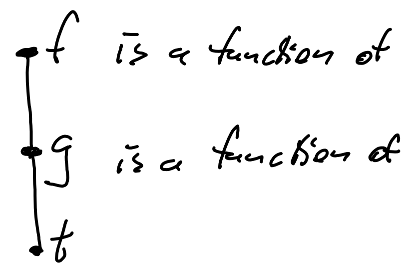
Recall the chain rule from single-variable calculus.

$$\frac{d}{dt} f(g(t)) = f'(g(t))\, g'(t)$$

Rewritten in differential notation this becomes

$$\frac{df}{dt} = \frac{df}{dg}\frac{dg}{dt}$$

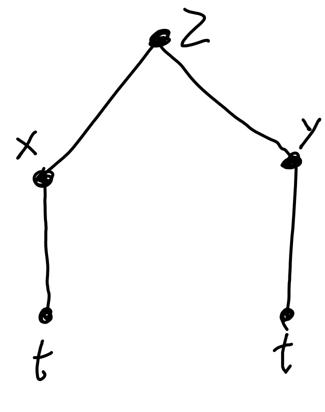
where



1st form of the multivariable chain Rule

Given $z = f(x,y)$ and $x = g(t)$ and $y = h(t)$

we have the following "tree" of dependencies among the variables.



So ultimately z is just a function of t.

The chain-rule states

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

proof

By definition, $\frac{dz}{dt} = \lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t}$

For a 2-variable function Δz is approximated by the differential $\Delta z = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y$ with the approximation getting better and better as Δx and Δy go to zero.

Therefore,
$$\frac{dz}{dt} = \lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t}$$

$$\frac{dz}{dt} = \lim_{\Delta t \to 0} \frac{\frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y}{\Delta t}$$

$$\frac{dz}{dt} = \lim_{\Delta t \to 0} \frac{\partial z}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial z}{\partial y}\frac{\Delta y}{\Delta t}$$

so as $\Delta t \to 0$ we get

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}}$$ which is what we want to show.

example $z = x^2 y^3$, $x = \sqrt{t}$, $y = t^2$

To write z as a function of t directly $z = (\sqrt{t})^2 (t^2)^3 = t^7$

$$\frac{dz}{dt} = 7t^6$$

Instead, let's calculate using the chain rule

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$\boxed{\frac{dz}{dt} = (2xy^3)\left(\frac{1}{2\sqrt{t}}\right) + (3x^2y^2)(2t)}$$ now plug in $x = \sqrt{t}$ and $y = t^2$

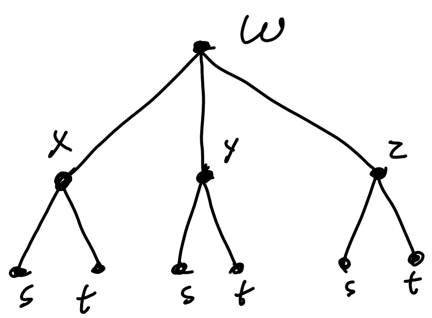
$$\frac{dz}{dt} = \cancel{2}\cancel{\sqrt{t}}\,t^6 \frac{1}{\cancel{2}\cancel{\sqrt{t}}} + 3(\sqrt{t})^2(t^2)^2(2t)$$

$$= t^6 + 6t^6 = 7t^6$$ Same answer as before

More general forms of the chain rule

Say $w = f(x,y,z)$, $x = g(s,t)$, $y = h(s,t)$, $z = k(s,t)$

This gives us the following tree of dependencies



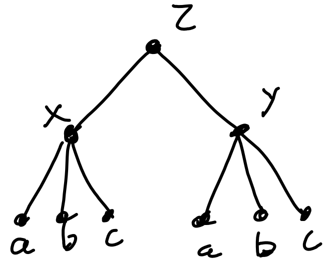
So w is ultimately a function of s and t.

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

and

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$

here's another example $z = f(x,y)$, $x = g(a,b,c)$, $y = h(a,b,c)$



so z is ultimately a function of a, b, c

The chain rule in this example is

$$\frac{\partial z}{\partial a} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial a} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial a} \quad \text{and}$$

$$\frac{\partial z}{\partial b} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial b} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial b} \quad \text{and}$$

$$\frac{\partial z}{\partial c} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial c} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial c}$$

Implicit differentiation for 2 variables x and y

Given a mixed equation of x and y we consider x as independent and $y = f(x)$ implicitly. To find $\frac{dy}{dx}$ we apply the differential operator $\frac{d}{dx}$ to each side of the mixed equation, apply the chain rule where appropriate, then solve for $\frac{dy}{dx}$.

This was the technique from Calculus I.

Here's a new way to calculate $\frac{dy}{dx}$ using the multivariable chain rule.

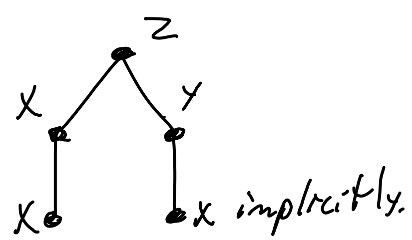
① Take the mixed equation and move all terms to the right-hand side to get

$$0 = f(x,y)$$

② This can be thought of as the level curve for the 2-variable function

$$z = f(x,y) \text{ where } z = 0.$$

Now we have the following tree of dependencies



③ The chain rule states

$$\frac{dz}{dx} = \frac{\partial z}{\partial x}\frac{dx}{dx} + \frac{\partial z}{\partial y}\frac{dy}{dx}$$

But $\frac{dx}{dx} = 1$ and $\frac{\partial z}{\partial x} = 0$ because $z = 0$ on this level curve.

$$0 = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx}$$

So now $$\boxed{\frac{dy}{dx} = -\frac{\frac{\partial z}{\partial x}}{\frac{\partial z}{\partial y}}}$$

example

$$x^2 + xy + y^2 - 1 = 0$$

So $z = x^2 + xy + y^2 - 1$

$$\frac{\partial z}{\partial x} = 2x + y$$

$$\frac{\partial z}{\partial y} = x + 2y$$

So $$\boxed{\frac{dy}{dx} = -\frac{2x+y}{x+2y}}$$

Here's the old technique

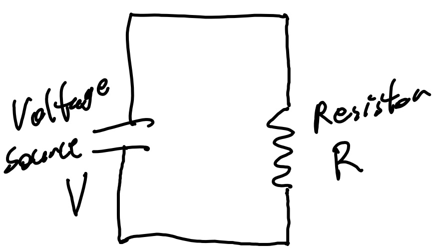
$$\frac{d}{dx}\left(x^2 + xy + y^2 - 1\right) = \frac{d}{dx} 0$$

$$2x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$2x + y = -(x+2y)\frac{dy}{dx}$$

$$\boxed{-\frac{2x+y}{x+2y} = \frac{dy}{dx}}$$

Here's an example for a related-rates problem



Ohm's law states

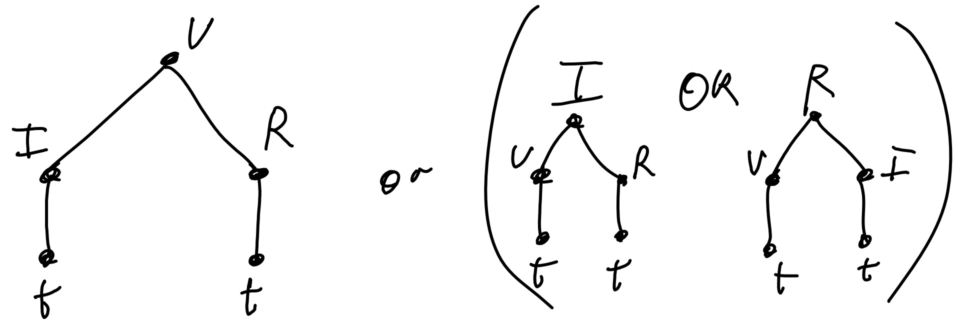
$$V = I\,R$$

voltage in volts

current in amps

resistance in Ohms.

Suppose V, I, and R are changing with respect to time t.



Now suppose that at a given point in time $t = a$.

$$R = 400, \quad I = .1, \quad \frac{dV}{dt} = -.01 \text{ V/sec}, \quad \frac{dR}{dt} = .03 \ \frac{\text{Ohms}}{\text{sec}}$$

Find $\frac{dI}{dt}$ in $\frac{\text{amps}}{\text{sec}}$ at this time t.

Chain Rule says
$$\frac{dV}{dt} = \frac{\partial V}{\partial I}\frac{dI}{dt} + \frac{\partial V}{\partial R}\frac{dR}{dt}$$

$$-.01 = R\frac{dI}{dt} + I(.03)$$

$$-.01 = 400\frac{dI}{dt} + (.1)(.03)$$

$$\boxed{-.000325 \ \frac{\text{amps}}{\text{sec}} = \frac{dI}{dt}}$$