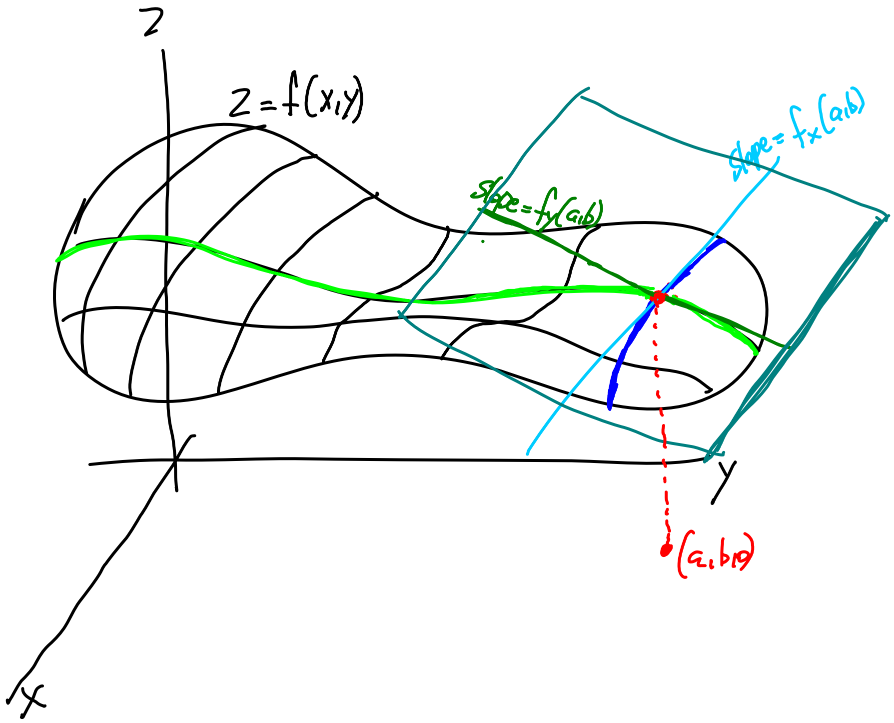
Section 4.4 Tangent Planes and linear approximations



The plane tangent to $z = f(x,y)$ at the point $(a,b,f(a,b))$ contains both lines with slopes $f_x(a,b)$ and $f_y(a,b)$.

The tangent line in the x-direction with $y=b$ constant has equation

$$z - f(a,b) = f_x(a,b)(x-a)$$

from single-variable calculus.

The tangent line in the y-direction with $x=a$ held constant has equation

$$z - f(a,b) = f_y(a,b)(y-b)$$

again from single-variable calculus.

Putting these together we have the following plane

$$z - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Remember, a plane has general equation

$$\vec{n}\cdot\langle x,y,z\rangle = \vec{n}\cdot\langle a,b,c\rangle$$

and this equation above can be rewritten as

$$z - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$$\underbrace{\langle -f_x(a,b), -f_y(a,b), 1\rangle}_{\vec{n}}\cdot\langle x,y,z\rangle = \underbrace{\langle -f_x(a,b), -f_y(a,b), 1\rangle}_{\vec{n}}\cdot\langle a,b,f(a,b)\rangle$$

So this plane $z - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b)$ is a plane containing both tangent lines so it is actually the tangent plane.

In addition to this, the line passing through $(a,b,f(a,b))$ with slope vector $\vec{n} = \langle -f_x, -f_y, 1\rangle$ is perpendicular to the tangent plane. This line is called the <u>normal line</u> to the surface $z=f(x,y)$ at $(x,y)=(a,b)$.

example Let $f(x,y)=\sqrt{20-x^2-7y^2}$

find the equation of the plane tangent to $z=f(x,y)$ at $(x,y)=(2,1)$ and the parametric equations of the normal line at $(x,y)=(2,1)$

tangent plane

$$f(x,y)=\sqrt{20-x^2-7y^2}=(20-x^2-7y^2)^{\frac{1}{2}}$$

$$f_x=\cancel{\tfrac{1}{2}}(20-x^2-7y^2)^{-\frac{1}{2}}(-\cancel{2}x)=\frac{-x}{\sqrt{20-x^2-7y^2}}$$

$$f_y=\cancel{\tfrac{1}{2}}(20-x^2-7y^2)^{-\frac{1}{2}}(-\cancel{14}^{7}y)=\frac{-7y}{\sqrt{20-x^2-7y^2}}$$

at $(x,y)=(2,1)$ $\quad z=f(2,1)=\sqrt{20-4-7}=\sqrt{9}=3$

$$f_x(2,1)=\frac{-2}{3} \qquad f_y(2,1)=\frac{-7}{3}$$

$$z-f(a,b)=f_x(a,b)(x-a)+f_y(a,b)(y-b)$$

$$\boxed{z-3=-\frac{2}{3}(x-2)-\frac{7}{3}(y-1)}$$

normal line

$\vec{n}=\langle -f_x,-f_y,1\rangle=\langle \frac{2}{3},\frac{7}{3},1\rangle$ passing through $(2,1,3)$

$$\boxed{\begin{aligned} x&=2+\tfrac{2}{3}t\\ y&=1+\tfrac{7}{3}t\\ z&=3+t\end{aligned}}$$

Differentials

The equation of the tangent plane to
$z = f(x, y)$ at $(x, y) = (a, b)$ is

$$z - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Shorten this to

$$\boxed{\Delta z = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y}$$

This form for the tangent plane is called the differential for $z = f(x,y)$

As with single-variable functions, we can approximate the actual change in the function for small variations in the input by using the differential.

example $z(x,y) = \sqrt{20 - x^2 - 7y^2}$ and $z(2,1) = \sqrt{20 - 4 - 7} = \sqrt{9} = 3$

Let's use the differential to approximate $z(1.95, 1.08)$

here $\Delta x = -.05$
$\Delta y = .08$

So now

$$\Delta z = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y$$

$$\Delta z = -\frac{2}{3}\Delta x - \frac{7}{3}\Delta y$$

$$\Delta z = -\frac{2}{3}(-.05) - \frac{7}{3}(.08) = -.15333$$

Therefore $z(1.95, 1.08) \approx z(2,1) + \Delta z = 3 - .15333 = \boxed{2.846666\ldots}$

no off by much

The actual value of $z(1.95, 1.08) = \boxed{2.83420\ldots}$