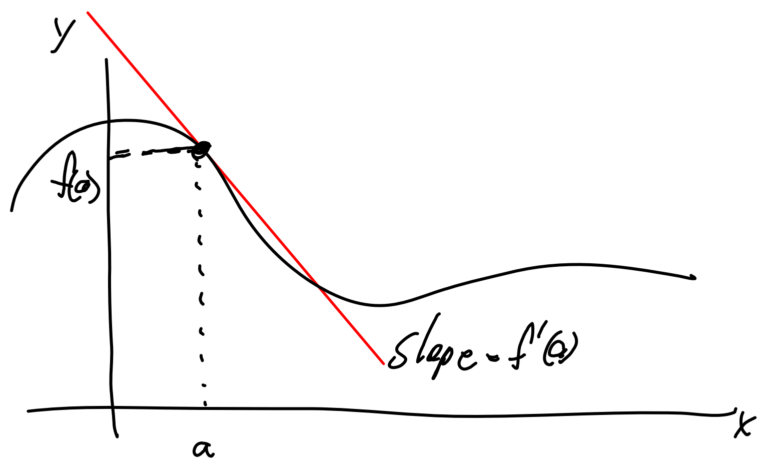
4.3 Partial derivatives

Recall the definition $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$ priming notation

or $\dfrac{d}{dx} f(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$

differential notation $\dfrac{d}{dx} f(x) = \dfrac{df}{dx}$

Geometrically, $f'(x)$ represents the slope of the line tangent to the graph $y=f(x)$ at any given fixed point $(a, f(a))$.



Now consider a 2-variable function $f(x,y)$

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1^{st} partial derivative of $f(x,y)$ with respect to x is defined as

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

In this definition, y is constant within the limit and x is variable.

Alternative notations to $f_x(x,y)$ are $\frac{\partial}{\partial x} f(x,y)$ and $\frac{\partial f}{\partial x}$

differential operator

differential notation.

<u>1^{st} partial derivative of $f(x,y)$ with respect to y</u> is defined as

$$f_y(x,y) = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

In this definition, x is constant within the limit and y is variable.

Alternative notations to $f_y(x,y)$ are $\frac{\partial}{\partial y} f(x,y)$ and $\frac{\partial f}{\partial y}$

differential operator

differential notation.

<u>example</u> Consider $f(x,y) = x^2y^3 + \cos(x)$

Then the 1^{st} partial derivatives for $f(x,y)$ are

$$f_x = 2xy^3 - \sin(x)$$

$$f_y = 3x^2y^2$$

Here is how the same example looks in differential notation

Given $z = x^2y^3 + \cos(x)$, the first partial derivatives are

$$\frac{\partial z}{\partial x} = 2xy^3 - \sin(x) \quad \text{and} \quad \frac{\partial z}{\partial y} = 3x^2y^2$$

Now, of course, we can take partial derivatives two times and we get 4 second partial derivatives.

Here is the same example again.

$$z = x^2y^3 + \cos(x)$$

1st partials

$$Z_x = 2xy^3 - \sin(x) \qquad Z_y = 3x^2y^2$$

<u>2nd partials</u>

$$Z_{xx} = 2y^3 - \cos(x) \qquad Z_{yy} = 6x^2y$$

$$Z_{xy} = 6xy^2 \qquad Z_{yx} = 6xy^2$$

Notice that $Z_{xy} = Z_{yx}$ in this example.

This is not a coincidence.

<u>Clairaut's Theorem</u> If $z = f(x,y)$ has continuous 1st partial derivatives, then $Z_{xy} = Z_{yx}$

For 3-variable functions $w = f(x, y, z)$

there are 1st partial derivatives with respect to the 3 variables

and 2nd partial derivatives as well.

example $\quad w = x^2 y^3 z^4 \quad$ (for $\frac{\partial}{\partial x}$ hold y and z constant)

1st partials $\quad w_x = 2xy^3z^4 \quad w_y = 3x^2y^2z^4 \quad w_z = 4x^2y^3z^3$

2nd partials

$w_{xx} = 2y^3z^4 \quad w_{yy} = 6x^2yz^4 \quad w_{zz} = 12x^2y^3z^2$

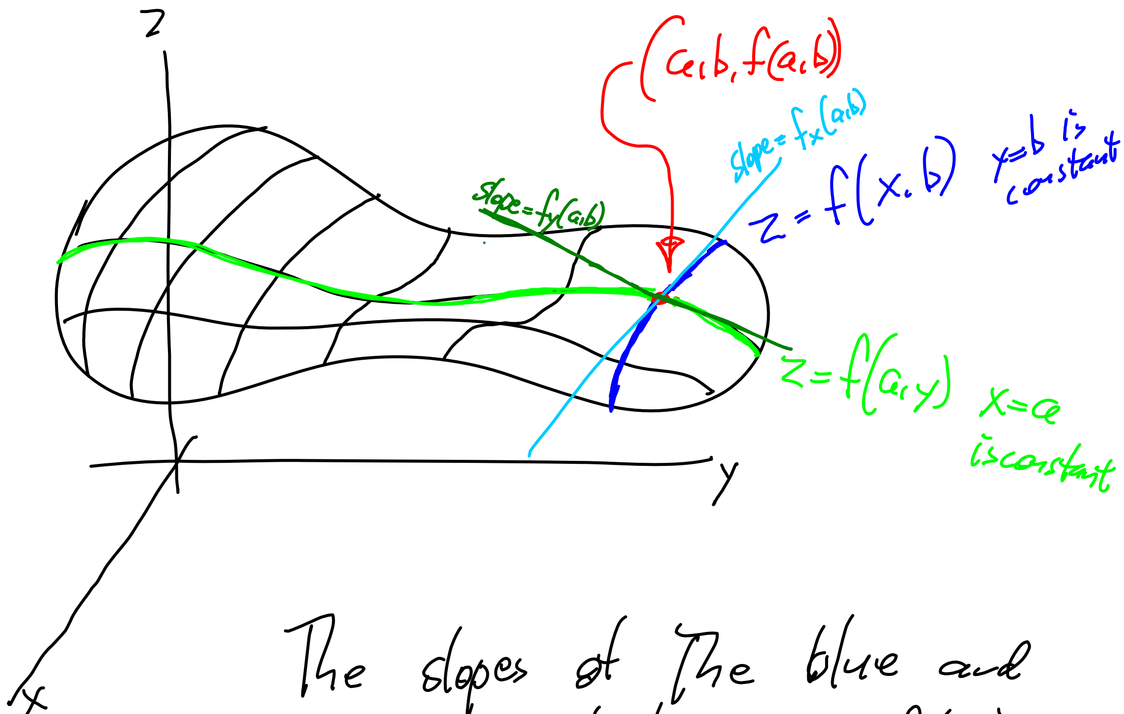
$w_{xy} = 6xy^2z^4 \quad w_{yx} = w_{xy} \quad w_{zx} = w_{xz}$

$w_{xz} = 8xy^3z^3 \quad w_{yz} = 12x^2y^2z^3 \quad w_{zy} = w_{yz}$

Geometric meaning of $f_x(x,y)$ and $f_y(x,y)$

Recall

$$f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b)}{h} \quad \text{and} \quad f_y(a,b) = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h}$$



The slopes of the blue and green tangent lines are $f_x(a,b)$ and $f_y(a,b)$ and together these lines define a "<u>tangent</u> <u>plane</u>" which we talk about in the next section