Section 3.4

In section 3.3 (and in Section 6.2) it is shown that

the length of a continuous curve $\vec{r}(t)$ for $a \le t \le b$

is
$$\text{length} = \int_a^b |\vec{r}\,'(t)|\,dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + z'(t)^2}\,dt \quad 3D$$

or

$$\int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\,dt \quad 2D$$

So now say that $\vec{r}(t)$ represents the position of a point in motion (in 2D or 3D) at time t.

Thus the curve $\vec{r}(t)$ traces the path that point follows over time.

So now

$$\begin{matrix}\text{distance traveled} \\ \text{from } t=a \text{ to } t=b\end{matrix} = \text{length} = \int_a^b |\vec{r}\,'(t)|\,dt$$

The function which when integrated over time yields the linear distance traveled must be the "speed" of travel at time t.

So if $\vec{r}(t)=$ position at time t and

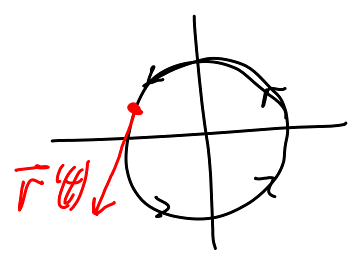
$|\vec{r}\,'(t)| =$ speed, then because $\vec{r}\,'(t)$ is tangent to $\vec{r}(t)$,

$\vec{r}\,'(t)$ is rightly called velocity.

So also $\vec{r}\,''(t)$ is rightly called acceleration.

Interesting example

① $\vec{r}(t) = \langle \cos(t), \sin(t) \rangle$



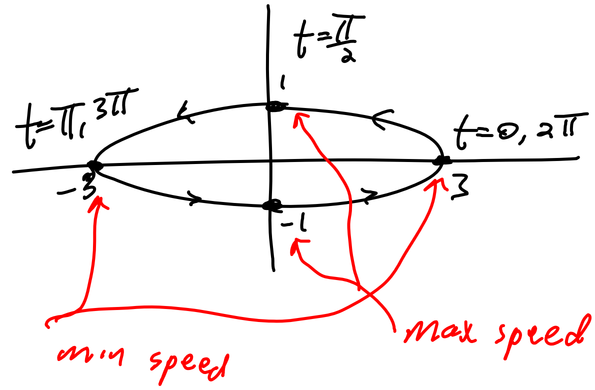
$\vec{r}\,'(t) = \langle -\sin(t), \cos(t) \rangle$

$|\vec{r}\,'(t)| = \sqrt{\sin^2(t)+\cos^2(t)} = 1$

So speed is constant.

(2)

$$\vec{r}(t) = \langle 3\cos(t), \sin(t) \rangle$$



$$\vec{r}\,'(t) = \langle -3\sin(t), \cos(t) \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{9\sin^2(t) + \cos^2(t)}$$

$$= \sqrt{8\sin^2(t) + \sin^2(t) + \cos^2(t)}$$

$$= \sqrt{8\sin^2(t) + 1}$$

min speed $\sqrt{0+1} = 1$ when $t = 0, \pi, 2\pi, 3\pi, 4\pi, \ldots$

max speed $\sqrt{8+1} = 3$ when $t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2},$

Acceleration and Newton's 2nd law of motion

Given a point traveling along $\vec{r}(t)$.

Acceleration $\vec{a}(t) = \vec{r}\,''(t)$.

Newton's 2nd law of motion states that a force $\vec{F}(t)$ acting on the traveling object satisfies

$$\vec{F}(t) = m\vec{a}(t) \qquad m = \text{mass}.$$

So $\frac{1}{m}\vec{F}(t) = \vec{a}(t)$

Since force is something that could be measured via instrumentation, this yields a method of finding position just by measuring the force for acceleration. Because antiderivative of acceleration is velocity and antiderivative of velocity is position.

<u>example</u> Suppose $\vec{a}(t) = \langle \sin(t), 1, 0 \rangle$ and that

$$\vec{V}(0) = \langle 0, 0, 0 \rangle \quad \text{and}$$

$$\vec{r}(0) = \langle 0, 0, 0 \rangle$$

Now find $\vec{v}(t)$ and $\vec{r}(t)$.

$$\vec{V}(t) = \int \vec{a}(t)\,dt = \langle -\cos(t), t, 0 \rangle + \vec{C}$$

$$\langle 0\ 0\ 0 \rangle = \vec{V}(0) = \langle -1, 0, 0 \rangle + \vec{C}$$

$$\langle 1\ 0\ 0 \rangle = \vec{C}$$

$$\boxed{\vec{V}(t) = \langle 1 - \cos(t), t, 0 \rangle}$$

$$\vec{r}(t) = \int \vec{v}(t)\,dt = \left\langle t - \sin(t),\ \frac{1}{2}t^2,\ 0 \right\rangle + \vec{C}$$

$$\langle 0\,0\,0 \rangle = \vec{r}(0) = \langle 0\,0\,0 \rangle + \vec{C}$$

$$\langle 0\,0\,0 \rangle = \vec{C}$$

$$\boxed{\vec{r}(t) = \left\langle t - \sin(t),\ \frac{1}{2}t^2,\ 0 \right\rangle}$$