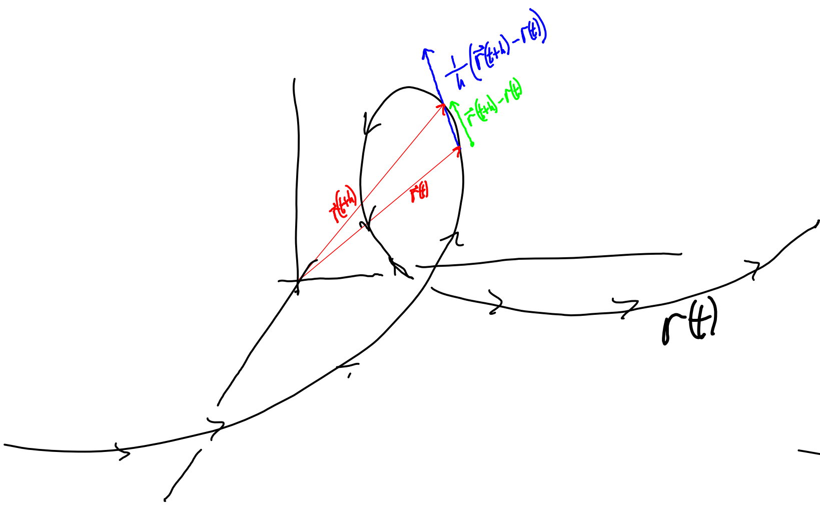
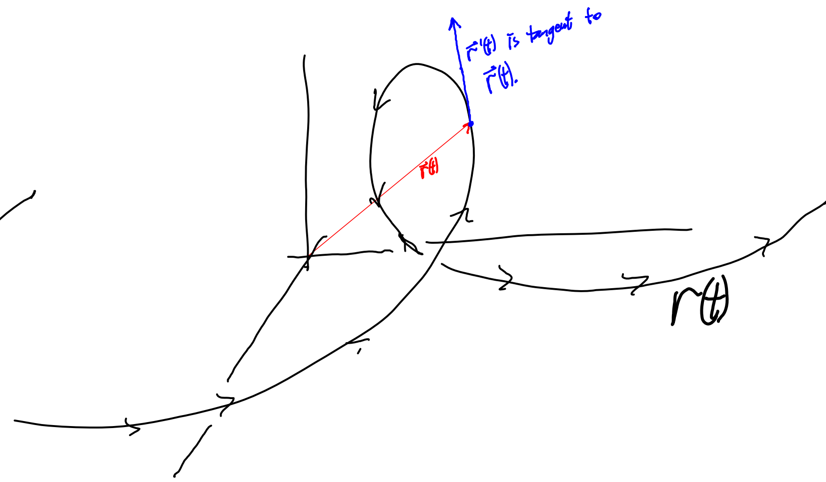
Section 3.2 Derivatives and integrals of vector functions

Given $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ define

$$\vec{r}\,'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - r(t)}{h}$$





Geometrically $\vec{r}\,'(t)$ is tangent to $\vec{r}(t)$ at any coordinate given by a fixed value for t.

How do we compute $\vec{r}\,'(t)$?

$$\vec{r}\,'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - r(t)}{h} = \lim_{h \to 0} \frac{1}{h}\Big(\langle x(t+h), y(t+h), z(t+h) \rangle - \langle x(t), y(t), z(t) \rangle \Big)$$

$$= \lim_{h\to 0} \left\langle \frac{x(t+h)-x(t)}{h}, \frac{y(t+h)-y(t)}{h}, \frac{z(t+h)-z(t)}{h} \right\rangle$$

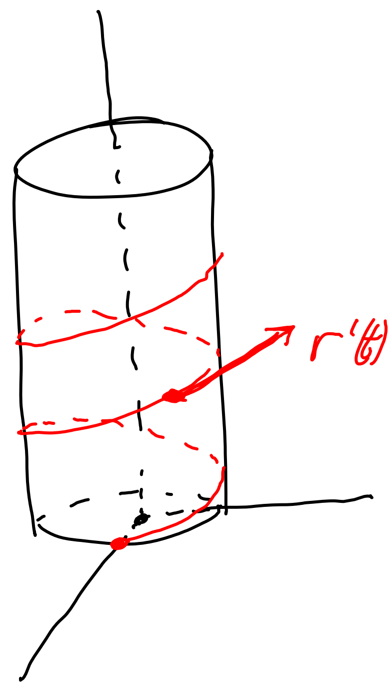
$$= \left\langle \lim_{h\to 0} \frac{x(t+h)-x(t)}{h}, \lim_{h\to 0} \frac{y(t+h)-y(t)}{h}, \lim_{t\to 0} \frac{z(t+h)-z(t)}{h} \right\rangle$$

$$= \langle x'(t), y'(t), z'(t) \rangle = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle.$$

<u>example</u>

$$\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$$

$$\vec{r}\,'(t) = \langle -\sin(t), \cos(t), 1 \rangle$$



Algebraic Properties of $\vec{r}\,'(t)$ (we can also write $\vec{r}\,'(t)$ as $\frac{d}{dt}\vec{r}(t)$)

① $\frac{d}{dt}\left(\vec{r}(t)+\vec{s}(t)\right) = \frac{d}{dt}\vec{r}(t) + \frac{d}{dt}\vec{s}(t) = \vec{r}\,'(t) + \vec{s}\,'(t)$

② $\frac{d}{dt}\left(k\vec{r}(t)\right) = k\frac{d}{dt}\left(\vec{r}(t)\right) = k\,\vec{r}\,'(t)$

③ $\frac{d}{dt}\left(\vec{r}(t)\cdot\vec{s}(t)\right) = \vec{r}\,'(t)\cdot\vec{s}(t) + \vec{r}(t)\cdot\vec{s}\,'(t)$

④ $\frac{d}{dt}\left(\vec{r}(t)\times\vec{s}(t)\right) = \vec{r}\,'(t)\times\vec{s}(t) + \vec{r}(t)\times\vec{s}\,'(t)$

⑤ $\frac{d}{dt}\vec{r}\left(g(t)\right) = g'(t)\,\vec{r}\,'\left(g(t)\right)$.

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proof for #3

Let $\vec{r}(t) = \langle x(t), y(t)\rangle$ and $s(t) = \langle a(t), b(t)\rangle$

So $\vec{r}(t)\cdot\vec{s}(t) = x(t)a(t) + y(t)b(t)$

So $\frac{d}{dt}\left(\vec{r}(t)\cdot\vec{s}(t)\right) = \frac{d}{dt}\left(x(t)a(t) + y(t)b(t)\right)$

$= \frac{d}{dt}x(t)a(t) + \frac{d}{dt}y(t)b(t)$ by ordinary product rule

$= x'(t)a(t) + x(t)a'(t) + y'(t)b(t) + y(t)b'(t)$

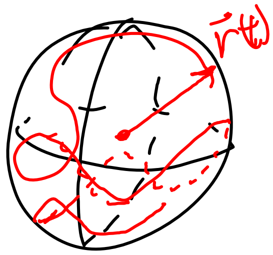
$= \left(x'(t)a(t) + y'(t)b(t)\right) + \left(x(t)a'(t) + y(t)b'(t)\right)$

$$= \langle x'(t), y'(t) \rangle \cdot \langle a(t), b(t) \rangle + \langle x(t), y(t) \rangle \cdot \langle a'(t), b'(t) \rangle$$

$$= \vec{r}\,'(t) \cdot \vec{s}(t) \quad + \quad \vec{r}(t) \cdot \vec{s}\,'(t).$$

Done

example Suppose that $\vec{r}(t)$ lives on the sphere of radius K.



In other words, for any t,

$$K = |\vec{r}(t)| = \sqrt{x^2(t) + y^2(t) + z^2(t)} \quad \text{Thus}$$

$$K^2 = |\vec{r}(t)|^2 = x^2(t) + y^2(t) + z^2(t) \quad \text{Thus}$$

$$K^2 = \vec{r}(t) \cdot \vec{r}(t).$$

Now let's show that $\vec{r}\,'(t)$ is orthogonal to $\vec{r}(t)$ for all t.

$$K^2 = \vec{r}(t) \cdot \vec{r}(t)$$

$$\frac{d}{dt}(K^2) = \frac{d}{dt} \vec{r}(t) \cdot \vec{r}(t)$$

$$0 = \vec{r}\,'(t) \cdot \vec{r}(t) + \vec{r}(t) \cdot \vec{r}\,'(t)$$

$$0 = \vec{r}(t) \cdot r'(t) + \vec{r}(t) \cdot \vec{r}\,'(t)$$

$$0 = 2\vec{r}(t) \cdot \vec{r}\,'(t)$$

$$0 = \vec{r}(t) \cdot \vec{r}\,'(t)$$

Thus $\vec{r}(t)$ and $\vec{r}\,'(t)$ are orthogonal.

Antiderivatives

Just as derivatives are done coordinate by coordinate, we define antiderivates coordinate by coordinate.

Def

$$\int \langle x(t), y(t), z(t) \rangle \, dt = \left\langle \int x(t)\,dt, \int y(t)\,dt, \int z(t)\,dt \right\rangle$$

example Given $\vec{r}\,'(t) = \langle \sin(t), 1, 0 \rangle$ and $\vec{r}(0) = \langle 2, -1, 2 \rangle$ find $\vec{r}(t)$.

$$\vec{r}(t) = \int \vec{r}\,'(t)\,dt = \int \langle \sin(t), 1, 0 \rangle \, dt = \langle -\cos(t), t, 0 \rangle + \vec{C}.$$

arbitrary constant in each coordinate.

To find $\vec{C}$ use $\vec{r}(0)$.

$$\langle 2, -1, 2 \rangle = \vec{r}(0) = \langle -1, 0, 0 \rangle + \vec{C}$$

$$\langle 3, -1, 2 \rangle = \vec{C}$$

$$\boxed{\vec{r}(t) = \langle -\cos(t) + 3, t - 1, 2 \rangle}$$