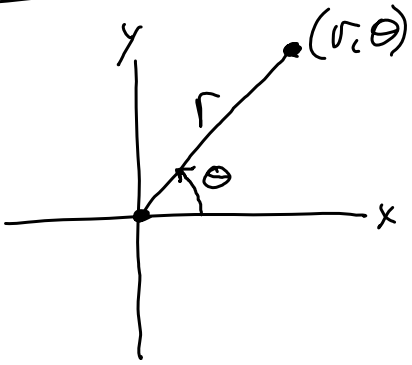
2.7 Cylindrical and spherical coordinates in 3D

Points in 3D can be represented by Cartesian coordinates (x,y,z)

as we have been discussing. Two other popular coordinate systems are Spherical and Cyclindrical.

Cylindrical

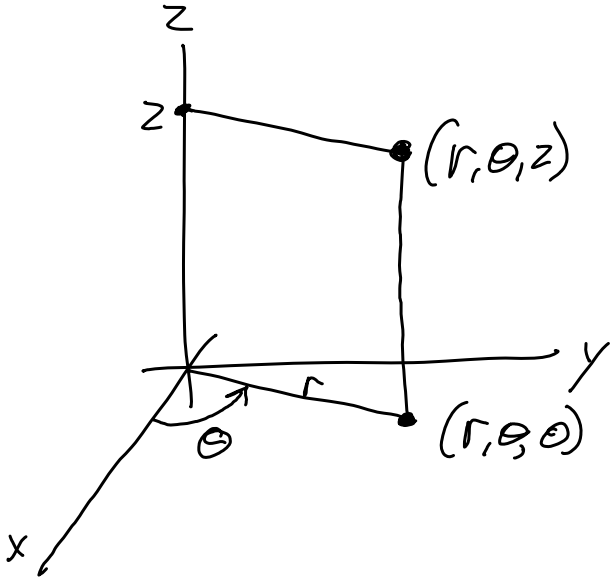


Recall the Polar-coordinate system in 2D.

Where

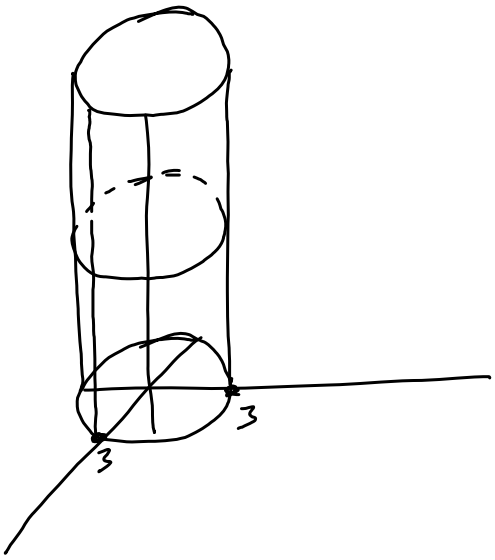
$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$
$$x^2+y^2 = r^2$$
$$\frac{y}{x} = \tan(\theta)$$

This is extended to 3D by just adding in the z-coordinate as is.



$$x = r\cos(\theta) \qquad x^2+y^2 = r^2$$
$$y = r\sin(\theta) \qquad \frac{y}{x} = \tan(\theta)$$
$$z = z$$

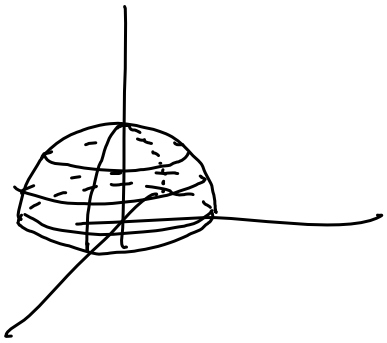
example The cylindrical-coordinate equation $r = 3$ defines the same surface as $r^2 = 9$ or $x^2 + y^2 = 9$ in Cartesian coordinates.



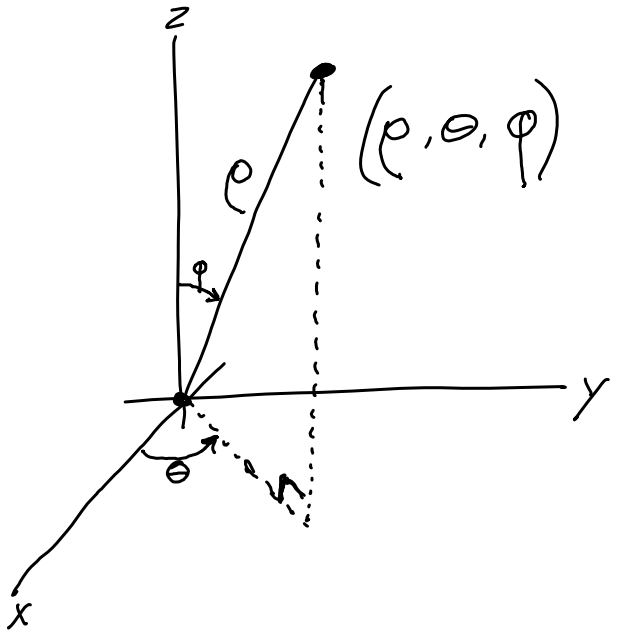
example The top half of the unit sphere $x^2 + y^2 + z^2 = 1$ is given by the cyclindrical-coordinate equation

$$r^2 + z^2 = 1$$

$$z = \sqrt{1 - r^2}$$



Spherical Coordinates



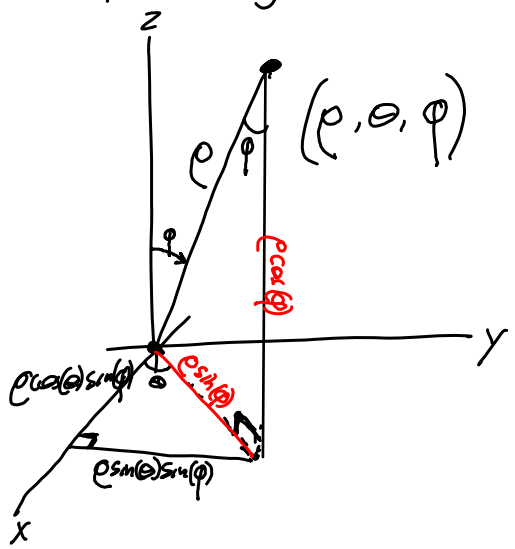
Normally we restrict

$0 \le \rho$

$0 \le \theta < 2\pi$

$0 \le \varphi \le \pi$

Transforming from spherical to cartesian coordinates



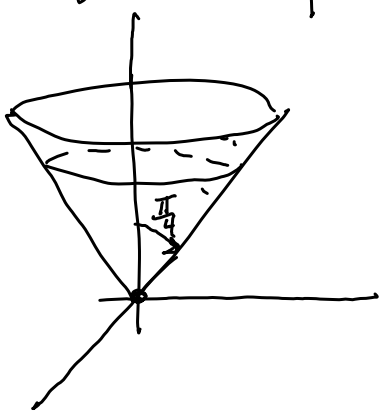
$x = \rho\cos(\theta)\sin(\varphi)$

$y = \rho\sin(\theta)\sin(\varphi)$

$z = \rho\cos(\varphi)$

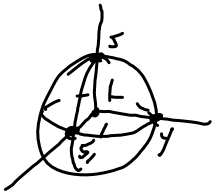
$x^2+y^2+z^2 = \rho^2$

Example what surface is given by the spherical-coordinate equation $\varphi = \frac{\pi}{4}$?? (ρ and θ are anything)



we also know that this surface is given by $z = \sqrt{x^2+y^2}$

example The spherical-coordinate equation $\rho = 5$ is the sphere of radius 5 centered at the origin.

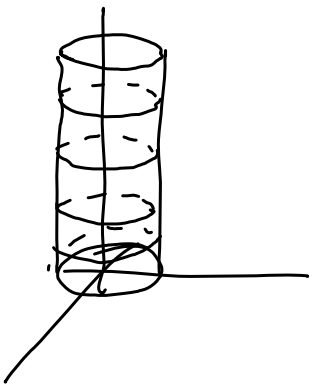


$$\rho = 5$$
$$\rho^2 = 25$$
$$x^2+y^2+z^2 = 25$$

example What surface is given by the spherical-coordinate equation

$$\rho = \frac{4}{\sin(\varphi)} \quad ?$$

$$\rho\sin(\varphi) = 4$$
$$\rho^2\sin^2(\varphi) = 16$$
$$\rho^2(1-\cos^2(\varphi)) = 16$$
$$\rho^2 - \rho^2\cos^2(\varphi) = 16$$
$$\rho^2 - (\rho\cos(\varphi))^2 = 16$$
$$\rho^2 - z^2 = 16$$
$$x^2+y^2+z^2-z^2 = 16$$
$$\boxed{x^2+y^2=16}$$ cylinder of radius 4 centered along the z-axis



A fun application

radius of Earth is $\rho = 3,959$ miles on average

Dayton, OH

39.75° N Lat

84.19° W Long

$\rho = 3,959$

$\theta = 360 - 84.19 = 275.81°$

$\varphi = 90 - 39.75 = 50.25°$

Greenwich, England

51.49° N Lat

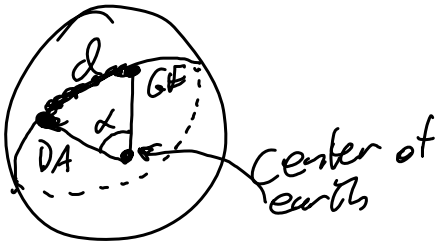
0° Long

$\rho = 3,959$

$\theta = 0°$

$\varphi = 90 - 51.49 = 38.51°$

What is the flying distance from Dayton to Greenwich ??



Google says 3,923 miles.

Let's confirm this.

Dayton

$x = 3959 \cos(275.81)\sin(50.25) = 308.12$

$y = 3959 \sin(275.81)\sin(50.25) = -3028.21$

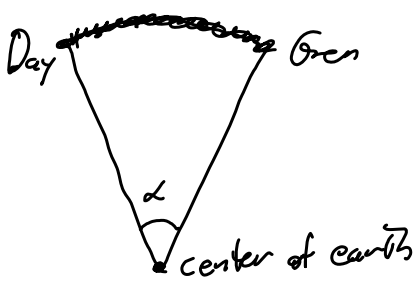
$z = 3959 \cos(50.25) = 2531.54$

Greenwich

$x = 2465.08$

$y = 0$

$z = 3097.92$



$$\cos(\alpha) = \frac{\langle 308.12, -3028.21, 2531.54\rangle \cdot \langle 2465.68, 0, 3087.82\rangle}{(3959)^2}$$

$$\cos(\alpha) \approx .54872$$

$$\alpha \approx \arccos(.54872) \approx .9899 \text{ radians}$$

arclength from Dayton to Greenwich $\approx (.9899)(3959) \approx 3919$ miles

very close to the google number.