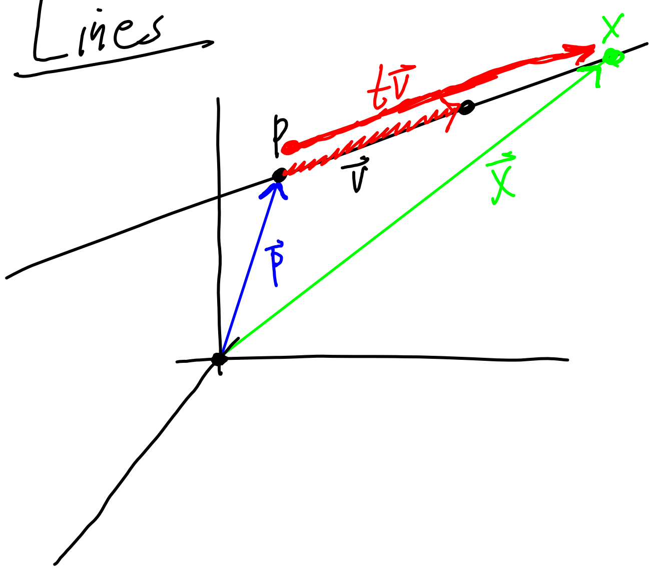
2.5 Lines and Planes in 3 dimensions

Lines



Line L containing point P and vector $\vec{v}$

Stretch or compress $\vec{v}$ by scalar t to get

$$\boxed{\vec{X} = \vec{P} + t\vec{v}}$$

every scalar t gives a different $\vec{X}$

vector equation of a line in 3D.

Sometimes we call $\vec{v}$ a <u>slope vector</u> for line L.

Given $P = (a, b, c)$ and $\vec{v} = \langle p, q, r \rangle$ and $\vec{X} = \langle x, y, z \rangle$

The vector equation $\vec{X} = \vec{P} + t\vec{v}$ becomes

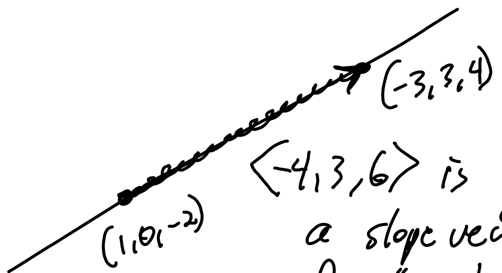
$$\langle x, y, z \rangle = \langle a, b, c \rangle + t\langle p, q, r \rangle \quad \text{so}$$

$$\langle x, y, z \rangle = \langle a + tp, b + tq, c + tr \rangle \quad \text{so}$$

$$\boxed{\begin{aligned} x &= a + tp \\ y &= b + tq \\ z &= c + tr \end{aligned}}$$

parametric equations for line L

example Find two different sets of parametric equations for the line in 3D containing points $(1,0,-2)$ and $(-3,3,4)$



$\langle -4,3,6\rangle$ is a slope vector for the line.

$$\boxed{\begin{aligned} x &= 1-4t \\ y &= 3t \\ z &= -2+6t \end{aligned}}$$

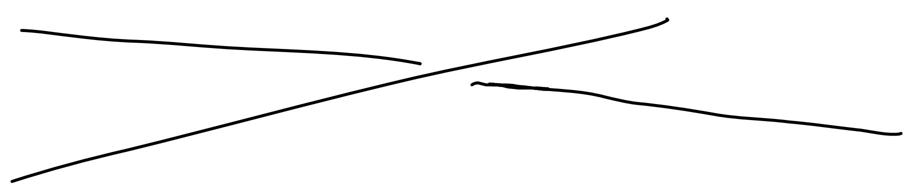
using $(1,0,-2)$ as a basepoint

$$\boxed{\begin{aligned} x &= -3-4t \\ y &= 3+3t \\ z &= 4+6t \end{aligned}}$$

using $(-3,3,4)$ as a basepoint.

* Two lines in 3D are parallel when their slope vectors are scalar multiples of each other, that is, the slope vectors are parallel.

* Two lines in 3D which are not parallel need not intersect, however.



example Do these two lines intersect or not?

$$\boxed{\begin{aligned} x &= t \\ y &= 1-t \\ z &= 2+3t \end{aligned}}\ L_1 \qquad \boxed{\begin{aligned} x &= 9-2s \\ y &= 1+2s \\ z &= 3+s \end{aligned}}\ L_2$$

First, note L_1 and L_2 are not parallel because

$$\langle 1,-1,3\rangle \neq k\langle -2,2,1\rangle \text{ for any } k.$$

If L_1 and L_2 did intersect then there must be s and t

such that

$$t = x = 9-2s \longrightarrow t+2s=9$$
$$1-t = y = 1+2s \longrightarrow -t-2s=0$$
$$2+3t = z = 3+s \longrightarrow 3t-s=1$$

$$-t-2s=0$$
$$t=-2s$$

$$3t-s=1$$
$$3(-2s)-s=1$$
$$-7s=1$$
$$s=-\frac{1}{7}$$
$$t=\frac{2}{7}$$

but now

$$9 = t+2s = \frac{2}{7}-\frac{2}{7}=0 \text{ contradiction}$$

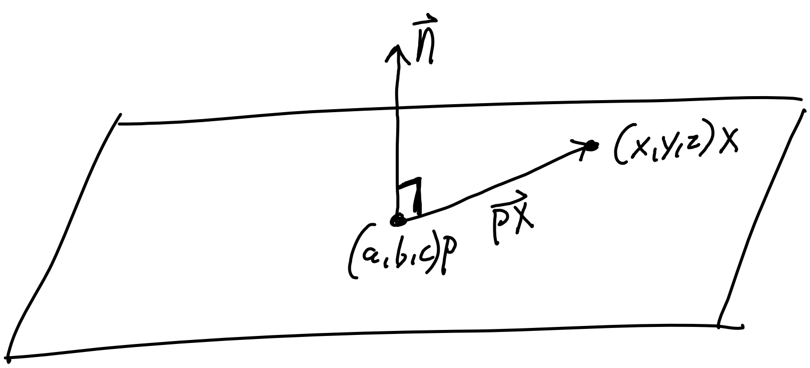
Planes in 3D

To define a plane in 3D you need

① any point (a,b,c) on the plane

and

② A normal vector (a perpendicular vector) to the plane.



Now any other point $X = \langle x,y,z\rangle$ on the plane satisfies

$$\vec{PX} \cdot \vec{n} = 0$$

That is

$$\left(\langle x,y,z\rangle - \langle a,b,c\rangle\right)\cdot\vec{n} = 0$$

That is

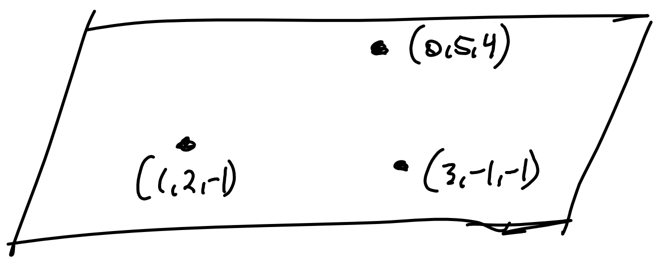
$$\langle x,y,z\rangle \cdot \vec{n} = \langle a,b,c\rangle \cdot \vec{n}$$

So now letting $\vec{n} = \langle p,q,r\rangle$ we get

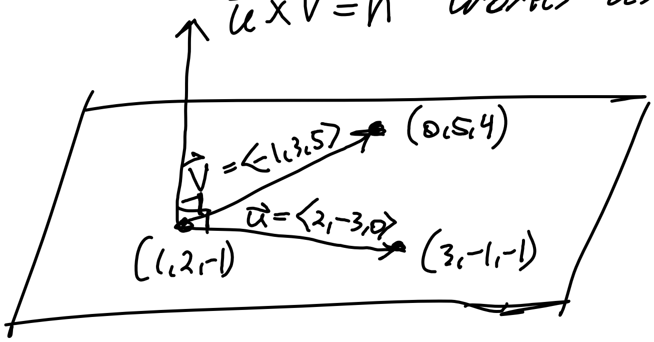
$$\boxed{px + qy + rz = pa + qb + rc}$$

Cartesian equation of the plane containing point (a,b,c) and having normal vector $\vec{n} = \langle p,q,r \rangle$

<u>example</u> Find the cartesian equation of the plane containing the 3 points shown,



$\vec{u} \times \vec{v} = \vec{n}$ works as a normal vector



$$\vec{n} = \vec{u} \times \vec{v} = \langle -15,-10,3 \rangle$$

$$\langle -15,-10,3 \rangle \cdot \langle x,y,z \rangle = \langle -15,-10,3 \rangle \cdot \langle 0,5,4 \rangle$$

$$\boxed{-15x - 10y + 3z = -38}$$

<u>notice</u> $\langle -15,-10,3 \rangle \cdot \langle 0,5,4 \rangle = -38$
$\langle -15,-10,3 \rangle \cdot \langle 1,2,-1 \rangle = -38$
$\langle -15,-10,3 \rangle \cdot \langle 3,-1,-1 \rangle = -38$

so it doesn't matter which of the 3 points are used.