Section 2.4 Cross product.

The cross product $\vec{u} \times \vec{v}$ is for two 3-dimensional vectors only. The result is another 3-dimensional vector.

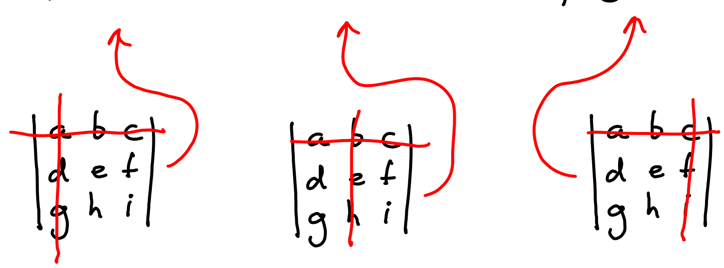
Determinants

2x2

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \qquad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

3x3

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$



Example

$$\begin{vmatrix} 4 & 2 & 1 \\ 1 & 0 & -1 \\ 0 & 2 & -3 \end{vmatrix} = 4 \begin{vmatrix} 0 & -1 \\ 2 & -3 \end{vmatrix} - 2 \begin{vmatrix} 1 & -1 \\ 0 & 3 \end{vmatrix} + 1 \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix}$$

$$= 4(0-(-3)) - 2(3-0) + (2-0)$$

$$= 8 + 6 + 2 = \boxed{16}$$

Def Given $\vec{u} = \langle a_1, b_1, c_1 \rangle$ $\vec{v} = \langle a_2, b_2, c_2 \rangle$

define
$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$
and
$$\vec{v} \times \vec{u} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix}$$

example

$$\langle 1,0,2 \rangle \times \langle -5,1,1 \rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2 \\ -5 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} \hat{i} - \begin{vmatrix} 1 & 2 \\ -5 & 1 \end{vmatrix} \hat{j} + \begin{vmatrix} 1 & 0 \\ -5 & 1 \end{vmatrix} \hat{k}$$

$$= (0-2)\hat{i} - (1-(-10))\hat{j} + (1-0)\hat{k}$$

$$= -2\hat{i} - 11\hat{j} + \hat{k} = \boxed{\langle -2, -11, 1 \rangle}$$

The cross product of two 3-dimensional vectors is another 3-dimensional vector.

Normally, I write the following

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2 \\ -5 & 1 & 1 \end{vmatrix} = \left\langle \begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix}, -\begin{vmatrix} 1 & 2 \\ -5 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ -5 & 1 \end{vmatrix} \right\rangle = \langle -2, -11, 1 \rangle$$

Some basic algebraic properties of the cross product

① $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$

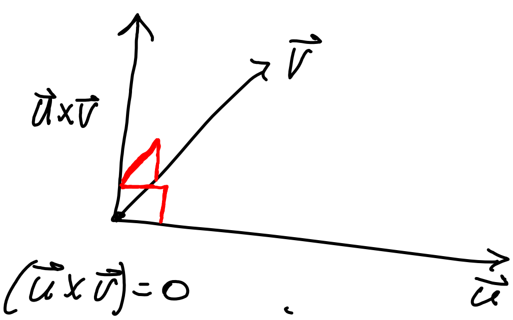
② $(k\vec{u}) \times \vec{v} = \vec{u} \times (k\vec{v}) = k(\vec{u} \times \vec{v})$

③ $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$

④ $\vec{u} \times \vec{0} = \vec{0}$

⑤ $\vec{u} \times \vec{v} = \vec{0}$ if and only if $\vec{u}$ and $\vec{v}$ are parallel.

⑥ $\vec{u} \times \vec{v}$ is perpendicular to both $\vec{u}$ and $\vec{v}$.



proof of ⑥

We'll show that $\vec{u} \cdot (\vec{u} \times \vec{v}) = 0$, the other $\vec{v} \cdot (\vec{u} \times \vec{v}) = 0$ is similar.

Let $\vec{u} = \langle a, b, c \rangle$ $\quad \vec{v} = \langle x, y, z \rangle$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & b & c \\ x & y & z \end{vmatrix} = \left\langle \begin{vmatrix} b & c \\ y & z \end{vmatrix}, -\begin{vmatrix} a & c \\ x & z \end{vmatrix}, \begin{vmatrix} a & b \\ x & y \end{vmatrix} \right\rangle$$

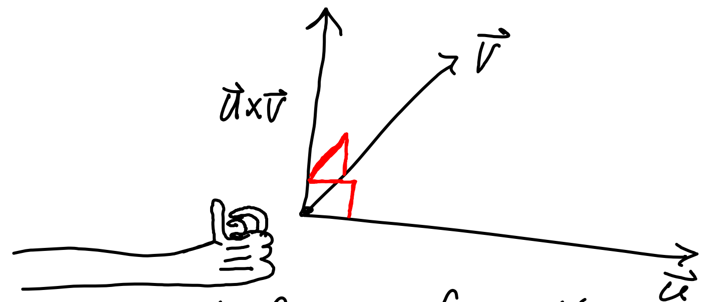
$$\vec{u} \cdot (\vec{u} \times \vec{v}) = \langle a, b, c \rangle \cdot \left\langle \begin{vmatrix} b & c \\ y & z \end{vmatrix}, -\begin{vmatrix} a & c \\ x & z \end{vmatrix}, \begin{vmatrix} a & b \\ x & y \end{vmatrix} \right\rangle$$

$$= a\begin{vmatrix} b & c \\ y & z \end{vmatrix} - b\begin{vmatrix} a & c \\ x & z \end{vmatrix} + c\begin{vmatrix} a & b \\ x & y \end{vmatrix} =$$

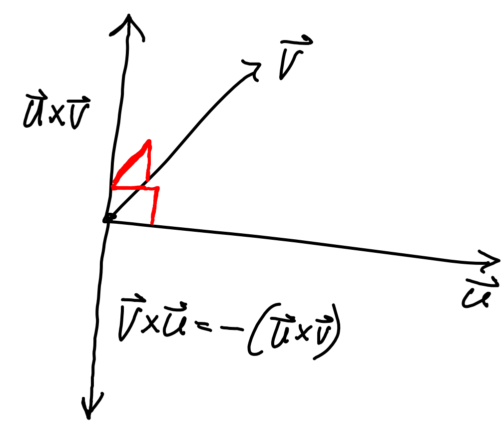
$$= a(bz - cy) - b(az - cx) + c(ay - bx) = 0.$$

Theorem ① $\vec{u}\times\vec{v}$ is perpendicular to both $\vec{u}$ and $\vec{v}$.

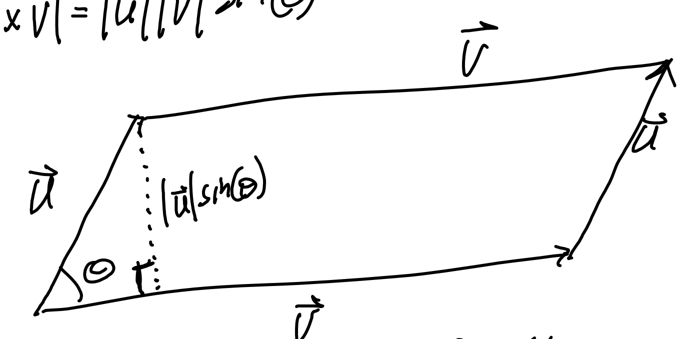
② The direction of $\vec{u}\times\vec{v}$ is given by the right-hand rule



curl fingers of right hand in the direction of $\vec{u}$ to $\vec{v}$ and thumb points in the direction for $\vec{u}\times\vec{v}$



③ $|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin(\theta)$



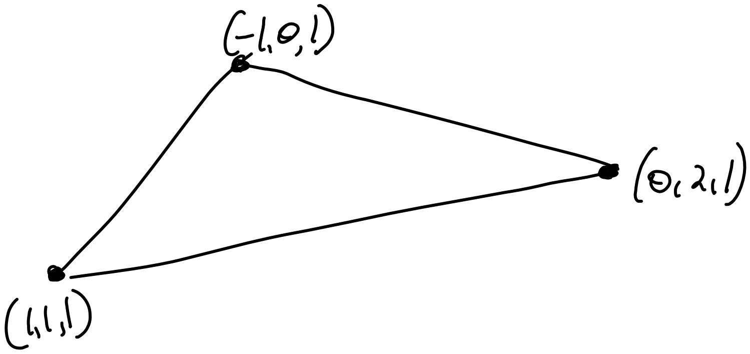
base = $|\vec{v}|$
height = $|\vec{u}|\sin(\theta)$

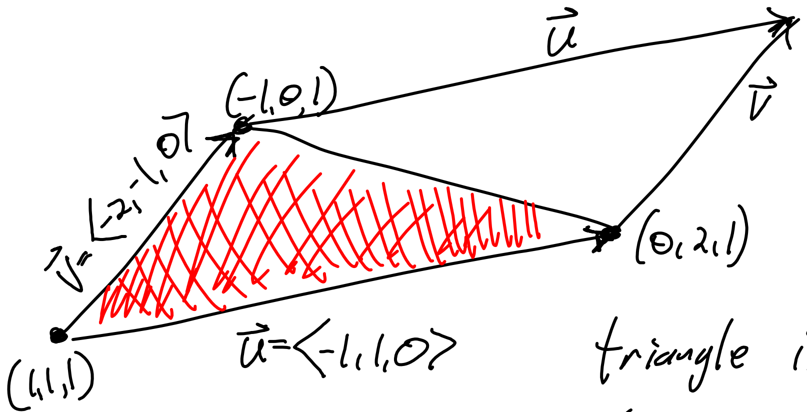
which is the area of this parallelogram.

~~proof~~ omitted.

Example

Find the area of the following triangle





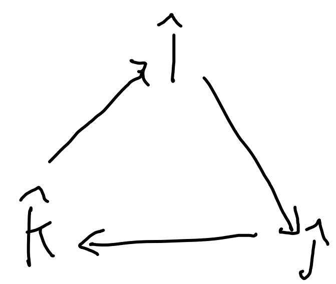
triangle is exactly half the parallelogram.

$$\text{Area of } \triangle = \frac{1}{2} \left| \vec{u} \times \vec{v} \right|$$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 0 \\ -2 & -1 & 0 \end{vmatrix} = \left\langle \begin{vmatrix} 1 & 0 \\ -1 & 0 \end{vmatrix}, - \begin{vmatrix} -1 & 0 \\ -2 & 0 \end{vmatrix}, \begin{vmatrix} -1 & 1 \\ -2 & -1 \end{vmatrix} \right\rangle = \langle 0, 0, 3 \rangle$$

so $\text{Area of } \triangle = \frac{1}{2} \left| \langle 0,0,3 \rangle \right| = \boxed{\frac{3}{2}}$

Unit vectors and cross products

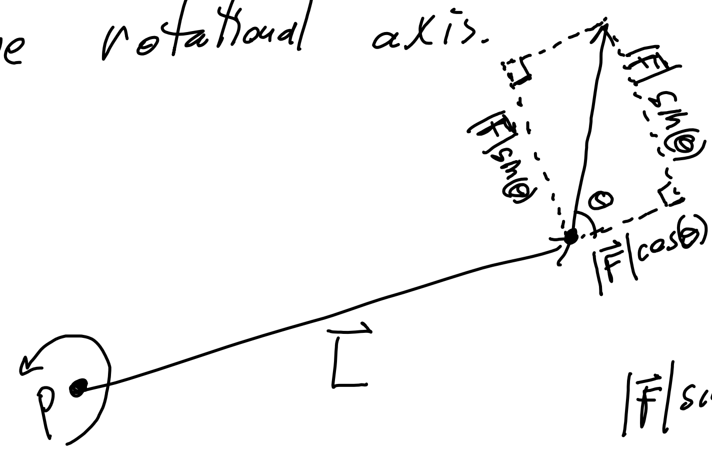


$\hat{i} \times \hat{j} = \hat{k}, \quad \hat{j} \times \hat{k} = \hat{i}, \quad \hat{k} \times \hat{i} = \hat{j}$

$\hat{j} \times \hat{i} = -\hat{k}, \quad \hat{k} \times \hat{j} = -\hat{i}, \quad \hat{i} \times \hat{k} = -\hat{j}$

Torque

In physics, <u>torque</u> is a force producing rotation about an axis. The torque vector is a vector parallel to the rotational axis.



$|\vec{F}|\sin(\theta)$ is the component of $\vec{F}$ which produces rotation for L around point P. ($|\vec{F}|\cos(\theta)$ is wasted force relative to rotation)

The torquing force around P is proportional to the length of L given perpendicular force $|F|\sin(\theta)$.

Thus we define $|\vec{Torque}| = |\vec{L}||\vec{F}|\sin(\theta) = |\vec{F} \times \vec{L}|$

The only possible direction for $\vec{T}$orque would be perpendicular to both $\vec{F}$ and $\vec{L}$ so we define

$$\vec{Torque} = \vec{F} \times \vec{L}$$